International Journal of Bifurcation and Chaos, Vol. 15, No. 4 (2005) 1467–1483
© World Scientific Publishing Company

CLOSED-FORM CONDITIONS OF BIFURCATION POINTS FOR GENERAL DIFFERENTIAL EQUATIONS

P. YU
Department of Applied Mathematics, The University of Western Ontario,
London, Ontario, Canada N6A 5B7
pyu@pyu1.apmaths.uwo.ca

Received October 13, 2003; Revised April 13, 2004

This note presents closed-form formulas for determining the critical points of general n-dimensional differential equations. The formulas do not require commutating the eigenvalues of the Jacobian of a system. Based on the Hurwitz criterion, explicit necessary and sufficient conditions are obtained. Particular attention is focused on Hopf and double Hopf bifurcations. A model of induction machine is presented to show the application of the results.

Keywords: Differential equation; fixed point; stability; bifurcation; Hopf bifurcation; double Hopf bifurcation; Hurwitz criterion.

1. Introduction

In the study of instability and bifurcation problems, one often needs to determine the critical (bifurcation) points for a given system (e.g. see [Takens, 1974; Marsden & McCracken, 1976; Chua *et al.*, 1987; Guckenheimer & Holmes, 1993; Chen *et al.*, 2000; Yu, 1998, 2000, 2001, 2002, 2003]). Such conditions are usually obtained from the Jacobian of the system evaluated at a fixed point (an equlibrium solution), by using a numerical approach to slove a ststem of nonlinear equations. However, when sloving real engineering problems which may have very large dimensions, numerical computations might be difficult to handle and could result in errors. Moreover, in most cases, we need to find the conditions in terms of system parameters, in order to consider the stability changes with respect to the parameters. Numerical methods are not applicable in these cases, and one has to find closed-form expressions of the critical conditions.

The well-known Hurwitz criterion (e.g. see [Porter, 1967]), which is based on the characteristic function of the Jacobian of a system, can be used to determine the stability of a fixed point. The stability boundary determines the conditions of critical points. Hopf bifurcation condition has been briefly discussed in [Porter, 1967], where an outline of a proof was given based on a theorem of Orlando [1911]. However, other singularities such as double Hopf, k-Hopf, Hopf-zero, etc. were not discussed. In this note, we use a different method to establish necessary and sufficient conditions for determining the critical points of general nonlinear systems. Particular attention will be focused on Hopf, double Hopf and k-Hopf bifurcations. Based on the Hurwitz criterion, explicit necessary and sufficient conditions are obtained, in terms of system parameters. Other types of singularities such as double zero, Hopf-zero, etc. are also discussed.

In the next section, the problem under consideration is described. Proofs are given in Sec. 3 for the necessary and sufficient conditions of Hopf, double Hopf and k-Hopf bifurcations. The results for other types of singularities are also summarized in this section. An example of induction machine is presented in Sec. 4 to demonstrate the application of the results. Conclusion is drawn in Sec. 5.

2. Problem Statement

Consider the system described by the following general nonlinear differential equation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\mu}), \quad \mathbf{x} \in \mathbf{R}^n, \ \ \boldsymbol{\mu} \in \mathbf{R}^m, \quad \mathbf{f} : \mathbf{R}^{n+m} \to \mathbf{R}^n, \tag{1}$$

where the dot denotes differentiation with respect to time, t; $\mathbf{x}$ and $\boldsymbol{\mu}$ are the n-dimensional state variable and m-dimensional parameter variable, respectively. It is assumed that the nonlinear function $\mathbf{f}(\mathbf{x}, \boldsymbol{\mu})$ is analytic with respect to $\mathbf{x}$ and $\boldsymbol{\mu}$. In general, the first step in analyzing the nonlinear system (1) is to find the fixed points of the system and determine their stabilities. Then, consider possible bifurcations from a fixed point as the parameter $\boldsymbol{\mu}$ is varied. The fixed points are determined from the algebraic equation $d\mathbf{x}/dt = \mathbf{0}$, i.e. $\mathbf{f}(\mathbf{x}, \boldsymbol{\mu}) = \mathbf{0}$. Suppose that the fixed points of Eq. (1) are given in the form of

$$\mathbf{x}_e = \mathbf{x}_e(\boldsymbol{\mu}), \tag{2}$$

which are usually represent multiple solutions, then the Jacobian of system (1) at $\mathbf{x} = \mathbf{x}_e(\boldsymbol{\mu})$ yields

$$J(\boldsymbol{\mu}) = D\mathbf{f}|_{\mathbf{x}=\mathbf{x}_e(\boldsymbol{\mu})} = \frac{\partial f_i(\mathbf{x}_e(\boldsymbol{\mu}), \boldsymbol{\mu})}{\partial x_j}. \tag{3}$$

If all the eigenvalues of $J(\boldsymbol{\mu})$ have nonzero real parts, then the system is called hyperbolic and no complex dynamics exists in the vicinity of the fixed point. If at some point $\boldsymbol{\mu} = \boldsymbol{\mu}_c$, at least one of the eigenvalues of $J(\boldsymbol{\mu})$ has zero real part, then $\boldsymbol{\mu}_c$ is called a critical point, and bifurcations may occur from $\mathbf{x}_e$.

To determine the stability of the fixed points, we need to find the eigenvalues of the Jacobian $J(\boldsymbol{\mu})$, which are given by the following polynomial equation:

$$\begin{aligned} P_n(\lambda) &= \det[\lambda I - J(\boldsymbol{\mu})] \\ &= \lambda^n + a_1(\boldsymbol{\mu})\lambda^{n-1} \\ &\quad + a_2(\boldsymbol{\mu})\lambda^{n-2} + \cdots + a_{n-2}(\boldsymbol{\mu})\lambda^2 \\ &\quad + a_{n-1}(\boldsymbol{\mu})\lambda + a_n(\boldsymbol{\mu}) \\ &= 0. \end{aligned} \tag{4}$$

If for a value of $\boldsymbol{\mu}$, all the roots of the polynomial $P_n(\lambda)$ have negative real parts, then the fixed point is (asymptotically) stable for this value of $\boldsymbol{\mu}$. If at least one of the eigenvalues has zero real part as $\boldsymbol{\mu}$ reaches a critical point, $\boldsymbol{\mu}_c$, then the fixed point becomes unstable and bifurcations occur from the critical point. When all the roots of $P_n(\lambda)$ have negative real parts, we call $P_n(\lambda)$ a *stable polynomial*, otherwise an *unstable polynomial*.

According to the well-known Hurwitz criterion, the necessary and sufficient conditions, under which all the roots of the polynomial $P_n(\lambda)$ have negative real parts, are given by

$$\Delta_i(\boldsymbol{\mu}) > 0, \quad i = 1, 2, \ldots, n, \tag{5}$$

where $\Delta_i(\boldsymbol{\mu})$ are called principal minors of the Hurwitz arrangement of *order* n, defined as follows (here, *order* n means that there are n coefficients, a_i $(i = 1, 2, \ldots, n)$, which construct the Hurwitz principal minors):

$$\Delta_1 = |a_1| = a_1,$$

$$\Delta_2 = \det \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix} = a_1 a_2 - a_3,$$

$$\Delta_3 = \det \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix} = (a_1 a_2 - a_3)a_3 - (a_1 a_4 - a_5)a_1,$$

$$\vdots$$

$$\Delta_{n-1} = \det \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-2} & a_{n-3} & a_{n-4} & a_{n-5} \\ 0 & 0 & 0 & 0 & \cdots & a_n & a_{n-1} & a_{n-2} & a_{n-3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_n & a_{n-1} \end{bmatrix}$$

$$\Delta_n = \det \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_n & a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_n & a_{n-1} & a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & a_n \end{bmatrix} = a_n \Delta_{n-1}. \tag{6}$$

Suppose as $\boldsymbol{\mu}$ is varied to reach a point, $\boldsymbol{\mu} = \boldsymbol{\mu}_c$, at least one of Δ_i's becomes zero, then the fixed point $\mathbf{x}_e$ becomes unstable at $\boldsymbol{\mu} = \boldsymbol{\mu}_c$, called critical point. It is easy to see from Eq. (6) that if $a_n(\boldsymbol{\mu}_c) = 0, P_n(\lambda)$ has one zero root, indicating that system (1) has a simple zero singularity and static bifurcation occurs from $\mathbf{x}_e$. When $a_n(\boldsymbol{\mu}_c) = a_{n-1}(\boldsymbol{\mu}_c) = 0$, system (1) exhibits a double zero singularity and more complex dynamical behavior exists in the vicinity of the fixed point. This shows that it is straightforward to determine the purely zero singularities. However, it becomes much more difficult to determine the singularities which involve purely imaginary pairs. For example, Hopf bifurcation occurs at a critical point at which $P_n(\lambda)$ has a pair of purely imaginary eigenvalues, $\pm i\omega$ $(\omega > 0)$. For lower dimensional systems, it is not difficult to find the conditions. For example, the conditions under which there exists a Hopf bifurcation for the lower dimensional cases: $n = 2, 3, 4$ and 5, are listed in Table 1. Note that the conditions given in this table actually yield $\Delta_{n-1} = 0$, but $a_n > 0$. If, in addition, one wants the remaining eigenvalues of $P_n(\lambda)$ to still have negative real parts, then except Δ_{n-1}, other Hurwitz inequalities should still hold.

It is easy to verify that the conditions given in Table 1 for $n = 2, 3$ and 4 are necessary and sufficient. For example, consider $n = 4$. The sufficiency can be shown by a direct calculation. From the condition, one has

$$a_4 = \frac{(a_1 a_2 - a_3) a_3}{a_1^2},$$

and then one can rewrite the polynomial $P_4(\lambda)$ as

$$\begin{aligned} P_4(\lambda) &= \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + \frac{(a_1 a_2 - a_3) a_3}{a_1^2} \\ &= \left(\lambda^2 + \frac{a_3}{a_1}\right)\left(\lambda^2 + a_1\lambda + a_2 - \frac{a_3}{a_1}\right), \end{aligned}$$

which indicates that $P_4(\lambda)$ has a pair of purely eigenvalues $\lambda_{1,2} = \pm i\sqrt{a_3/a_1}$. If the other two Hurwitz inequalities $a_1 > 0$ and $a_1 a_2 - a_3 = a_1(a_2 - (a_3/a_1)) > 0$ are still satisfied, then the remaining two eigenvalues $\lambda_{3,4} = (1/2)[-a_1 \pm \sqrt{a_1^2 - 4(a_2 - (a_3/a_1))}]$ would still have negative real parts.

For necessity, suppose that $P_4(\lambda)$ has a pair of purely imaginary eigenvalues, $\pm i\omega$ $(\omega > 0)$, and the remaining two eigenvalues have negative real parts, then $P_4(\lambda)$ can be rewritten as

$$\begin{aligned} P_4(\lambda) &= (\lambda^2 + \omega^2)(\lambda^2 + b_1\lambda + b_2) \\ &= \lambda^4 + b_1\lambda^3 + (b_2 + \omega^2)\lambda^2 + \omega^2 b_1\lambda + \omega^2 b_2, \end{aligned}$$

Table 1. Critical condition for Hopf bifurcation.

n	$P_n(\lambda)$	Condition	ω^2
2	$\lambda^2 + a_1\lambda + a_2$	$a_1 = 0$	a_2
3	$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$	$a_1a_2 - a_3 = 0$	a_2
4	$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$	$(a_1a_2 - a_3)a_3 - a_4a_1^2 = 0$	$\dfrac{a_3}{a_1}$
5	$\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5$	$(a_1a_2 - a_3)(a_3a_4 - a_2a_5) - (a_1a_4 - a_5)^2 = 0$	$\dfrac{a_5 - a_1a_4}{a_3 - a_1a_2}$

where $b_1 > 0$ and $b_2 > 0$. Thus,

$$\Delta_3 = \begin{vmatrix} b_1 & 1 & 0 \\ \omega^2 b_1 & b_2 + \omega^2 & b_1 \\ 0 & \omega^2 b_2 & \omega^2 b_1 \end{vmatrix}$$

$$= \omega^2 \begin{vmatrix} b_1 & 1 & 0 \\ \omega^2 b_1 & b_2 + \omega^2 & b_1 \\ 0 & b_2 & b_1 \end{vmatrix} = \omega^2 \begin{vmatrix} b_1 & 1 & 0 \\ \omega^2 b_1 & \omega^2 & 0 \\ 0 & b_2 & b_1 \end{vmatrix}$$

$$= 0.$$

For $n = 5$, one can similarly prove the necessity, but not easy to use factorization to prove the sufficiency.

The trend shown in Table 1 suggests that the necessary and sufficient condition for system (1) to have a Hopf bifurcation is $\Delta_{n-1} = 0$ for a general system. In fact, it has been shown using a theorem obtained by Orlando [1911] that the necessary and sufficient condition for a general n-dimensional system to have Hopf singularity is $\Delta_{n-1} = 0$. The Orlando's theorem established the relation between Δ_{n-1} and the eigenvalues of the characteristic polynomial, given by

$$\Delta_{n-1} = (-1)^{\frac{n(n-1)}{2}} \prod_{j<k}^{1,\ldots,n} (\lambda_j + \lambda_k) \quad (n \geq 2), \qquad (7)$$

where λ_i's are the eigenvalues of $P_n(\lambda)$. A brief discussion about the proof can be found in [Porter, 1967]. In this note, we will use a different approach, based on matrix theory, to prove the necessary and sufficient condition for Hopf bifurcation, and extend the proof to consider other singularities like double Hopf, k-Hopf, as well as a combination of Hopf and zeros.

It should be pointed out that for a large dimensional system, it is usually not possible to find the explicit expression of the fixed point from the nonlinear algebraic equation $\mathbf{f}(\mathbf{x}, \boldsymbol{\mu}) = \mathbf{0}$, and one has to employ a numerical approach. So why is it then important to find closed-form formulas for determining the critical points? The reasons are as follows. Take Hopf bifurcation as an example. In this case, μ is a scalar (i.e. $m = 1$), and thus $\mathbf{f}(\mathbf{x}, \mu) = \mathbf{0}$ has n nonlinear algebraic equations with $n+1$ variables. If we can find an analytical equation, say, $f_{n+1}(\mathbf{x}, \mu) = 0$, from the critical condition of Hopf bifurcation, then we have a total of $n+1$ equations for $n+1$ variables. This makes it much easier to apply a numerical approach such as Newton's method to solve the $n+1$ equations. Without this additional explicit equation, one has to start by taking a trial value of μ and then solve the equation $\mathbf{f}(\mathbf{x}, \mu) = \mathbf{0}$ to find $\mathbf{x}$, which is then substituted into the Jacobian to obtain $P_n(\lambda)$ and check if $P_n(\lambda)$ has a pair of purely imaginary eigenvalues; if not, try again. This can be very computationally demanding.

Similarly, we can discuss double Hopf bifurcation [Yu, 2001, 2002] for which the characteristic polynomial $P_n(\lambda)$ has two purely imaginary pairs, $\pm i\omega_1$ and $\pm i\omega_2$ $(\omega_1, \omega_2 > 0)$. In this case, the dimension of $\boldsymbol{\mu}$ must be at least two (i.e. $m \geq 2$). In the next section, we shall present the results for several singularities, and pay particular attention to Hopf, double Hopf and k-Hopf bifurcations.

3. Main Results

We first summarize the results for zero singularities, then particularly prove the necessary and sufficient critical conditions for Hopf, double Hopf and k-Hopf singularities, and finally discuss other types of singularities.

3.1. *Zero singularity*

As discussed in the previous section, it is easy to obtain the critical condition for a simple zero singularity: $a_n = 0$; and that for a double zero singularity: $a_n = a_{n-1} = 0$. In general, we have

Theorem 1. *The necessary and sufficient conditions for system (1) to have a k-zero singularity at a fixed point of the system are given by*

$$a_n = a_{n-1} = \cdots = a_{n+1-k} = 0, \tag{8}$$

where a_i's are the coefficients of the characteristic polynomial (4). Further, if the remaining coefficients $a_1, a_2, \ldots, a_{n-k}$ still obey the Hurwitz conditions of order $n-k$, then all the remaining eigenvalues of the Jacobian have negative real parts.

Proof. The proof is straightforward. Under the condition (8), we can rewrite the characteristic polynomial $P_n(\lambda)$ as

$$\begin{aligned} P_n(\lambda) &= (\lambda^{n-k} + a_1\lambda^{n-k-1} + a_2\lambda^{n-k-2} \\ &\quad + \cdots + a_{n-k-1}\lambda + a_{n-k})\lambda^k \\ &\equiv p_{n-k}(\lambda)\lambda^k. \end{aligned} \tag{9}$$

Then k eigenvalues of $P_n(\lambda)$ are zero and the remaining $(n-k)$ eigenvalues are given by $P_{n-k}(\lambda)$. The necessary and sufficient conditions for all the eigenvalues of $P_{n-k}(\lambda)$ having negative real parts are: the $(n-k)$ coefficients $a_1, a_2, \ldots, a_{n-k}$ yield $(n-k)$ positive Hurwitz inequalities. ■

3.2. *Hopf bifurcation*

For Hopf bifurcation, we have the following result.

Theorem 2. *The necessary and sufficient condition for system (1) to have a Hopf bifurcation at a fixed point of the system is*

$$\Delta_{n-1} = 0. \tag{10}$$

Further, if other Hurwitz conditions are still held, i.e. $\Delta_i > 0$ for $i = 1, 2, \ldots, n-2$ and $a_n > 0$, then all the remaining eigenvalues of the Jacobian have negative real parts.

Proof. Suppose that the characteristic polynomial of system (1) is given by Eq. (4). If all the Hurwitz conditions are satisfied, i.e. $\Delta_i > 0,\ i = 1, 2, \ldots, n$, then all the eigenvalues of the characteristic polynomial $P(\lambda)$ have negative real parts. Then, we consider as the parameter $\boldsymbol{\mu}$ varies, two eigenvalues first across the imaginary axis on the complex plane, while the remaining eigenvalues are still located on the left half-plane. Thus, some of Δ_i's become zero at the crossing. In this case, we can rewrite $P(\lambda)$ as

$$\begin{aligned} P_n(\lambda) &= P_{n-2}(\lambda)(\lambda-\lambda_1)(\lambda-\lambda_2) \\ &\equiv (\lambda^{n-2} + b_1\lambda^{n-3} + b_2\lambda^{n-4} \\ &\quad + \cdots + b_{n-3}\lambda + b_{n-2})(\lambda-\lambda_1)(\lambda-\lambda_2) \\ &= \lambda^n + [b_1 - (\lambda_1+\lambda_2)]\lambda^{n-1} \\ &\quad + [b_2 - (\lambda_1+\lambda_2)b_1 + \lambda_1\lambda_2]\lambda^{n-2} \\ &\quad + [b_3 - (\lambda_1+\lambda_2)b_2 + \lambda_1\lambda_2 b_1]\lambda^{n-3} \\ &\quad + \cdots + [b_{n-3} - (\lambda_1+\lambda_2)b_{n-4} \\ &\quad + \lambda_1\lambda_2 b_{n-5}]\lambda^2 + [b_{n-2} - (\lambda_1+\lambda_2)b_{n-3} \\ &\quad + \lambda_1\lambda_2 b_{n-4}]\lambda^2 + [-(\lambda_1+\lambda_2)b_{n-2} \\ &\quad + \lambda_1\lambda_2 b_{n-3}]\lambda + \lambda_1\lambda_2 b_{n-2} \\ &\equiv \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} \\ &\quad + \cdots + a_{n-2}\lambda^2 + a_{n-1}\lambda + a_n, \end{aligned} \tag{11}$$

where λ_1 and λ_2 are two roots of $P_n(\lambda)$. The $(n-2)$ coefficients of the polynomial $P_{n-2}(\lambda)$, b_i's obey the Hurwitz inequalities of order $(n-2)$, which can be obtained from the following $(n-2)\times(n-2)$ matrix:

$$B = \begin{bmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ b_5 & b_4 & b_3 & b_2 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-2} & b_{n-3} & b_{n-4} & b_{n-5} & b_{n-6} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-2} & b_{n-3} & b_{n-4} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & b_{n-2} \end{bmatrix}. \tag{12}$$

For example, $\Delta^B_{n-2} = \det(B)$, where the superscript B denotes the matrix B.

It follows from Eq. (11) that

$$\begin{aligned} a_1 &= b_1 - (\lambda_1 + \lambda_2), \\ a_2 &= b_2 - (\lambda_1 + \lambda_2) b_1 + \lambda_1 \lambda_2, \\ a_3 &= b_3 - (\lambda_1 + \lambda_2) b_2 + \lambda_1 \lambda_2 b_1, \\ &\vdots \\ a_{n-2} &= b_{n-2} - (\lambda_1 + \lambda_2) b_{n-3} + \lambda_1 \lambda_2 b_{n-4}, \\ a_{n-1} &= -(\lambda_1 + \lambda_2) b_{n-2} + \lambda_1 \lambda_2 b_{n-3}, \\ a_n &= \lambda_1 \lambda_2 b_{n-2}. \end{aligned} \tag{13}$$

Since we only consider the possibility that two eigenvalues cross the imaginary axis from left to right half-plane, there are only two possibilities for λ_1 and λ_2 before the crossing, either

(i) both λ_1 and λ_2 are real, and $\lambda_1 < 0, \lambda_2 < 0$; or
(ii) λ_1 and λ_2 are complex conjugates with negative real part, i.e. $\lambda_1 = \overline{\lambda}_2 = \alpha + i\beta$, with $\alpha < 0$.

Note that both cases yield $\lambda_1 + \lambda_2 < 0$ and $\lambda_1 \lambda_2 > 0$.

It is clear that in order to have $a_n = \lambda_1 \lambda_2 b_{n-2} = 0$, where $b_{n-2} \neq 0$ according to the assumption, one must choose $\lambda_1 \lambda_2 = 0$, resulting either in a simple zero or a double zero singularity. This has been discussed in Sec. 3.1. So we now assume that λ_1 and λ_2 are complex conjugate before they cross the imaginary axis. Thus, $\lambda_1 + \lambda_2 = 2\alpha$ and $\lambda_1 \lambda_2 = \alpha^2 + \beta^2$, giving rise to a Hopf bifurcation at $\alpha = 0$ if $\partial \alpha / \partial \boldsymbol{\mu} \neq 0$. Applying the formula Δ_{n-1} given in Eq. (6) with the aid of Eq. (12) yields

$$\Delta_{n-1} = \det \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-2} & a_{n-3} & a_{n-4} & a_{n-5} \\ 0 & 0 & 0 & 0 & \cdots & a_n & a_{n-1} & a_{n-2} & a_{n-3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_n & a_{n-1} \end{bmatrix}$$

$$= \det \left\{ \begin{bmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ b_5 & b_4 & b_3 & b_2 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-2} & b_{n-3} & b_{n-4} & b_{n-5} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-2} & b_{n-3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \right.$$

$$- (\lambda_1 + \lambda_2) \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_2 & b_1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_4 & b_3 & b_2 & b_1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-3} & b_{n-4} & b_{n-5} & b_{n-6} \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & b_{n-3} & b_{n-4} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_{n-2} \end{bmatrix}$$

$$
+\lambda_1\lambda_2 \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-4} & b_{n-5} & b_{n-6} & b_{n-7} \\ 0 & 0 & 0 & 0 & \cdots & b_{n-2} & b_{n-3} & b_{n-4} & b_{n-5} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-2} & b_{n-3} \end{bmatrix} \Bigg\}
$$

$$
\equiv \det\left[A_1 - 2\alpha A_2 + (\alpha^2+\beta^2)A_3\right]. \tag{14}
$$

First notice that the kth row of the matrix A_3 is the $(k-1)$th row of the matrix A_1 for $k = 2, 3, \ldots, n-1$ and the first row of A_3 is zero, so the matrix A_3 contributes zero to the determinant due to the property of basic matrix operations. Next, it can be observed that the kth column of the matrix A_2 is the $(k+1)$th column of the matrix A_1 for $k = 1, 2, \ldots, n-2$. This indicates that these $(n-2)$ columns of the matrix A_2 contributes zero to the determinant, and only the last column of A_2 has nonzero contribution. Therefore, Δ_{n-1} given in Eq. (14) can be rewritten as

$$
\Delta_{n-1} = \det \begin{bmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ b_5 & b_4 & b_3 & b_2 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-2} & b_{n-3} & b_{n-4} & b_{n-5} - 2\alpha b_{n-6} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-2} & b_{n-3} - 2\alpha b_{n-4} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -2\alpha b_{n-2} \end{bmatrix}
$$

$$
= -2\alpha b_{n-2}\det(B), \tag{15}
$$

where B is given in Eq. (12). According to the assumption, $b_{n-2} > 0$ and $\det(B) = b_{n-2}\Delta^B_{n-3} > 0$, thus $\Delta_{n-1} \geq 0$ as long as $\alpha \leq 0$, and $\Delta_{n-1} = 0$ if only if $\alpha = 0$.

Further, we need to prove that all the other Hurwitz inequalities, $\Delta_i > 0$, $i = 1, 2, \ldots, n-2$, still hold as long as $P_{n-2}(\lambda)$ is a *stable* polynomial. For example, we show $\Delta_{n-2} > 0$.

$$
\Delta_{n-2} = \det \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-2} & a_{n-3} & a_{n-4} \\ 0 & 0 & 0 & 0 & \cdots & a_n & a_{n-1} & a_{n-2} \end{bmatrix}
$$

$$
\begin{aligned}
&= \det \left\{ \begin{bmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 \\ b_5 & b_4 & b_3 & b_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-2} & b_{n-3} & b_{n-4} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-2} \end{bmatrix} \right. \\
&\qquad - (\lambda_1 + \lambda_2) \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_2 & b_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ b_4 & b_3 & b_2 & b_1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-3} & b_{n-4} & b_{n-5} \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & b_{n-3} \end{bmatrix} \\
&\qquad \left. + \lambda_1\lambda_2 \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-4} & b_{n-5} & b_{n-6} \\ 0 & 0 & 0 & 0 & \cdots & b_{n-2} & b_{n-3} & b_{n-4} \end{bmatrix} \right\} \\
&= \det \begin{bmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 \\ b_5 & b_4 & b_3 & b_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-2} & b_{n-3} & b_{n-4} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-2} \end{bmatrix} \\
&\qquad - 2\alpha \det \begin{bmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 \\ b_5 & b_4 & b_3 & b_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-2} & b_{n-3} & b_{n-5} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-3} \end{bmatrix} \\
&= (b_{n-2} - 2\alpha b_{n-3})\Delta_{n-3}^B \\
&\geq b_{n-2}\ \Delta_{n-3}^B \\
&> 0,
\end{aligned}
$$

since $\alpha \leq 0$, and $b_{n-2} > 0, b_{n-3} > 0$, $\Delta_{n-3}^B > 0$ due to $P_{n-2}(\lambda)$ being a stable polynomial. Similarly, we can prove that $\Delta_i > 0,\ i = 1, 2, \ldots, n-3$.

Summarizing the above results shows that

$$\Delta_{n-1} = 0 \quad \Leftrightarrow \quad \alpha = 0$$
$$\Leftrightarrow \quad \text{Hopf singularity} \left(\text{if } \frac{\partial \alpha}{\partial \boldsymbol{\mu}} \neq 0\right). \quad (16)$$

This completes the proof of Theorem 2. ■

Note: $\Delta_{n-1} = 0$ implies that $\Delta_n = a_n \Delta_{n-1} = 0$.

3.3. *k-Hopf bifurcation*

k-Hopf bifurcation means that the characteristic polynomial $P_n(\lambda)$ has k ($k \leq [n/2]$) purely imaginary pairs. In other words, as the parameter $\boldsymbol{\mu}$ is varied, there are k complex conjugates simultaneously crossing the imaginary axis from the left to the right half complex plane. We first consider double Hopf bifurcation ($k = 2$), and then extend the result to k-Hopf bifurcation.

Based on the proof given in the previous subsection for Hopf bifurcation, we can obtain the necessary and sufficient conditions for double Hopf bifurcation. The critical point of system (1) at which a double Hopf bifurcation occurs is defined when the characteristic polynomial $P_n(\lambda)$ has two pairs of purely imaginary eigenvalues $\pm i\omega_1$ and $\pm i\omega_2$ ($\omega_1 > 0, \omega_2 > 0$). In other words, on the complex plane, as the parameter $\boldsymbol{\mu}$ is varied, there exist two complex conjugates which cross the imaginary axis simultaneously. If $\omega_1/\omega_2 = p/q$, where p and q are positive integers, the case is called *resonance*, otherwise it is *nonresonance*.

For double Hopf bifurcation, we have the following result.

Theorem 3. *The necessary and sufficient conditions for system (1) to have a double Hopf bifurcation at a fixed point of the system are given by*

$$\Delta_{n-1} = \Delta_{n-3} = 0 \quad (\Delta_{n-2} = 0). \quad (17)$$

Further, if other Hurwitz conditions are still held, i.e. $\Delta_i > 0$ for $i = 1, 2, \ldots, n-4$ and $a_n > 0$, then all the remaining eigenvalues of the Jacobian have negative real parts.

Proof. Note that $\Delta_{n-2} = 0$ is not required, but due to $\Delta_{n-1} = \Delta_{n-3} = 0$. Since we can consider a double Hopf bifurcation as two Hopf bifurcations which occur simultaneously, we may apply the results obtained in the previous subsection for Hopf bifurcation to consider double Hopf bifurcation. For a Hopf bifurcation, i.e. the characteristic polynomial (4) has a pair of purely imaginary eigenvalues, $\pm i\omega_1$, the necessary and sufficient condition is $\Delta_{n-1} = 0$. In other words, $P_n(\lambda)$ having a purely imaginary pair and the remaining ones having negative parts is equivalent to $\Delta_{n-1} = 0$, but $\Delta_i > 0$, $i = 1, 2, \ldots, n-2$ and $a_n > 0$. Thus, we may write $P_n(\lambda)$ as

$$\begin{aligned} P_n(\lambda) &= P_{n-2}(\lambda)(\lambda^2 + \omega^2) \\ &\equiv (\lambda^{n-2} + b_1\lambda^{n-3} + b_2\lambda^{n-4} \\ &\quad + \cdots + b_{n-3}\lambda + b_{n-2})(\lambda^2 + \omega^2) \\ &= \lambda^n + b_1\lambda^{n-1} + (b_2 + \omega^2)\lambda^{n-2} \\ &\quad + (b_3 + \omega^2 b_1)\lambda^{n-3} + \cdots (b_{n-2} + \omega^2 b_{n-4})\lambda^2 \\ &\quad + \omega^2 b_{n-3}\lambda + \omega^2 b_{n-2} \\ &\equiv \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} \\ &\quad + \cdots + a_{n-2}\lambda^2 + a_{n-1}\lambda + a_n \end{aligned} \quad (18)$$

which gives $\Delta_{n-1} = 0$, but $\Delta_i > 0$, $i = 1, 2, \ldots, n-2$ and $a_n > 0$. Here, $\omega > 0$ is one of the two frequencies.

Now, we consider the second Hopf bifurcation based on the characteristic polynomial of $P_{n-2}(\lambda)$. Applying the results of Hopf bifurcation to $P_{n-2}(\lambda)$ suggests that the necessary and sufficient condition for $P_{n-2}(\lambda)$ to have a purely imaginary pair is

$$\Delta_{n-3}^B = \det \begin{bmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ b_5 & b_4 & b_3 & b_2 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-2} & b_{n-3} & b_{n-4} & b_{n-5} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-2} & b_{n-3} \end{bmatrix} = 0, \quad (19)$$

and $b_{n-2} > 0$. The remaining eigenvalues still have negative real parts if $\Delta_i^B > 0$, $i = 1, 2, \ldots, n-4$. Based on the above results, we derive the necessary and sufficient conditions for system (1) to have a double Hopf bifurcation, based on the original characteristic polynomial $P_n(\lambda)$.

First, note from Eq. (18) that $a_n = \omega^2 b_{n-2} > 0$ since $b_{n-2} > 0$. We already know that $\Delta_{n-1} = 0$. Next, we show $\Delta_{n-3} = 0$ as follows.

$$
\begin{aligned}
\Delta_{n-3} &= \det \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-3} & a_{n-4} & a_{n-5} \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-2} & a_{n-3} \end{bmatrix} \\
&= \det \left\{ \begin{bmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 \\ b_5 & b_4 & b_3 & b_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-3} & b_{n-4} & b_{n-5} \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & b_{n-3} \end{bmatrix} \right. \\
&\qquad \left. + \omega^2 \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-5} & b_{n-6} & b_{n-7} \\ 0 & 0 & 0 & 0 & \cdots & b_{n-3} & b_{n-4} & b_{n-5} \end{bmatrix} \right\} \\
&= \det \begin{bmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 \\ b_5 & b_4 & b_3 & b_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-3} & b_{n-4} & b_{n-5} \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & b_{n-3} \end{bmatrix} \\
&= \Delta_{n-3}^B \\
&= 0,
\end{aligned} \tag{20}
$$

where the result $\Delta_{n-3}^B = 0$, given in Eq. (19), has been used.

However, we can show that $\Delta_{n-4} > 0$ in a similar procedure.

$$
\Delta_{n-4} = \det \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-4} & a_{n-5} & a_{n-6} \\ 0 & 0 & 0 & 0 & \cdots & a_{n-2} & a_{n-3} & a_{n-4} \end{bmatrix}
$$

$$
\begin{aligned}
&= \det\left\{\begin{bmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 \\ b_5 & b_4 & b_3 & b_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-4} & b_{n-5} & b_{n-6} \\ 0 & 0 & 0 & 0 & \cdots & b_{n-2} & b_{n-3} & b_{n-4} \end{bmatrix}\right. \\
&\qquad \left. + \,\omega^2 \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-6} & b_{n-7} & b_{n-8} \\ 0 & 0 & 0 & 0 & \cdots & b_{n-4} & b_{n-5} & b_{n-6} \end{bmatrix}\right\} \\
&= \det\begin{bmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 \\ b_5 & b_4 & b_3 & b_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-4} & b_{n-5} & b_{n-6} \\ 0 & 0 & 0 & 0 & \cdots & b_{n-2} & b_{n-3} & b_{n-4} \end{bmatrix} \\
&= \Delta_{n-4}^B \\
&> 0. \qquad (21)
\end{aligned}
$$

In fact, following the above formula, we can show that $\Delta_i = \Delta_i^B > 0,\ i = 1, 2, \ldots, n-4$.

For Δ_{n-2}, we find

$$
\Delta_{n-2} = \det\begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-3} & a_{n-4} & a_{n-5} & a_{n-6} \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n & a_{n-1} & a_{n-2} \end{bmatrix}
$$

$$
= \det\left\{\begin{bmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ b_5 & b_4 & b_3 & b_2 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-3} & b_{n-4} & b_{n-5} & b_{n-6} \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & b_{n-3} & b_{n-4} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_{n-2} \end{bmatrix}\right.
$$

$$+\omega^2 \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-5} & b_{n-6} & b_{n-7} & b_{n-8} \\ 0 & 0 & 0 & 0 & \cdots & b_{n-3} & b_{n-4} & b_{n-5} & b_{n-6} \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & b_{n-3} & b_{n-4} \end{bmatrix} \Bigg\}$$

$$= \det \begin{bmatrix} b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ b_5 & b_4 & b_3 & b_2 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-3} & b_{n-4} & b_{n-5} & b_{n-6} \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & b_{n-3} & b_{n-4} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_{n-2} \end{bmatrix}$$

$$= b_{n-2}\Delta^B_{n-3}$$
$$= 0, \tag{22}$$

which is also due to $\Delta^B_{n-3} = 0$, implicitly because of $\Delta_{n-1} = 0$.

Thus, the proof of Theorem 3 is Completed. ■

With a similar proof to that of Theorem 3, we can prove the following theorem.

Theorem 4. *The necessary and sufficient conditions for system (1) to have a k-Hopf bifurcation at a fixed point of the system are given by*

$$\Delta_{n-1} = \Delta_{n-3} = \cdots = \Delta_{n-(2k-1)} = 0 \quad \left(k \leq \left[\frac{n}{2}\right]\right). \tag{23}$$

Further, if other Hurwitz conditions are still held, i.e. $\Delta_i > 0$ *for* $i = 1, 2, \ldots, n - 2k$ *and* $a_n > 0$, *then all the remaining eigenvalues of the Jacobian have negative real parts.*

3.4. *Other types of singularities*

Finally, we list the results of some combined singularities of zero and purely imaginary pairs in Table 2. The proof can follow the procedure given for Hopf and double Hopf bifurcations. The remaining eigenvalues still have negative real parts if and only if the other Hurwitz conditions still hold. Other possible singularities can be similarly obtained.

Table 2. Critical conditions for other singularities.

Singularity	Necessary and Sufficient Conditions
Hopf-Zero	$a_n = \Delta_{n-1} = 0$
Double-Zero-Hopf	$a_n = a_{n-1} = \Delta_{n-3} = 0$
Double-Hopf-Zero	$a_n = \Delta_{n-1} = \Delta_{n-3} = 0$

4. An Example

In this section, we present a model of induction machine to demonstrate the application of the theorems obtained in the previous section. The model is based on the one discussed in [Krause *et al.*, 2002] and the same notations are adopted here. Since in this paper we are mainly interested in the application of the critical conditions, we will not give the detailed derivation of the model.

An induction machine (or asynchronous machine) is one of the electrical machines which is widely used in industrial applications. The behavior of induction machine is studied for years, but the main attention has been focused on steady state solutions due to the complexity of the model (even with simplifying assumptions). In order to study dynamical behavior of the model such as instability and bifurcations, the conditions of the bifurcation (critical) points are to be determined.

The model is described by a system of seven ordinary differential equations, given as follows:

$$\begin{aligned}
\dot{\phi}_{qs} &= \omega_b\left\{u_q - \phi_{ds} + \frac{r_s}{X_{1s}}\left[X_{aq}\left(\frac{\phi_{qs}}{X_{1s}} + \frac{\phi'_{qr}}{X'_{1r}}\right) - \phi_{qs}\right]\right\},\\
\dot{\phi}_{ds} &= \omega_b\left\{u_d + \phi_{qs} + \frac{r_s}{X_{1s}}\left[X_{aq}\left(\frac{\phi_{ds}}{X_{1s}} + \frac{\phi'_{dr}}{X'_{1r}}\right) - \phi_{ds}\right]\right\},\\
\dot{\phi}_{0s} &= \omega_b\left\{\frac{r_s}{X_{1s}}(-\phi_{0s})\right\},\\
\dot{\phi}'_{qr} &= \omega_b\left\{-(1-\omega_r)\phi'_{dr} + \frac{r'_r}{X'_{1r}}\left[X_{aq}\left(\frac{\phi_{qs}}{X_{1s}} + \frac{\phi'_{qr}}{X'_{1r}}\right) - \phi'_{qr}\right]\right\},\\
\dot{\phi}'_{dr} &= \omega_b\left\{(1-\omega_r)\phi'_{qr} + \frac{r'_r}{X'_{1r}}\left[X_{aq}\left(\frac{\phi_{ds}}{X_{1s}} + \frac{\phi'_{dr}}{X'_{1r}}\right) - \phi'_{dr}\right]\right\},\\
\dot{\phi}_{0r} &= \omega_b\left\{\frac{r'_r}{X'_{1r}}(-\phi'_{0r})\right\},\\
\dot{\omega}'_r &= \frac{1}{2H}\left\{\frac{X_{ad}}{X_{1s}X'_{1r}}(\phi_{qs}\phi'_{dr} - \phi_{ds}\phi'_{qr}) - T_L\right\},
\end{aligned} \tag{24}$$

where, except the state variables, all the other variables are system parameters. Letting

$$x_1 = \phi_{qs}, \quad x_2 = \phi_{ds}, \quad x_3 = \phi_{0s}, \quad x_4 = \phi'_{qr},$$
$$x_5 = \phi'_{dr}, \quad x_6 = \phi'_{0r}, \quad x_7 = \omega_r,$$

and substituting proper parameter values to Eq. (24) finally yields a model of a 3hp induction machine as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, V, T_L)$:

$$\begin{aligned}
\dot{x}_1 &= -\frac{3}{10}x_1 - x_2 + \frac{3}{10}x_4 + V,\\
\dot{x}_2 &= x_1 - \frac{3}{10}x_2 + \frac{3}{10}x_5,\\
\dot{x}_3 &= -\frac{3}{5}x_3,\\
\dot{x}_4 &= \frac{1}{2}x_1 - \frac{1}{2}x_4 - x_5 + x_5x_7,\\
\dot{x}_5 &= \frac{1}{2}x_2 + x_4 - \frac{1}{2}x_5 - x_4x_7,\\
\dot{x}_6 &= -x_6,\\
\dot{x}_7 &= \frac{7}{120\pi^3}(14x_1x_5 - 14x_2x_4 - T_L),
\end{aligned} \tag{25}$$

where $V > 0$ and $T_L > 0$ are parameters, representing the input voltage of the motor and the load torque, respectively.

Setting $\dot{x}_i = 0$, $i = 1, 2, \ldots, 7$, results in two equilibrium solutions (fixed points):

$$\begin{aligned}
x_1^{\pm} &= -\frac{3(-350V^2 + 15T_L \pm 10S)}{7630V},\\
x_2^{\pm} &= \frac{7315V^2 - 150T_L \pm 9S}{7360V},\\
x_3^{\pm} &= 0,\\
x_4^{\pm} &= -\frac{T_L}{14V},\\
x_5^{\pm} &= \frac{35V^2 \pm S}{70V},\\
x_6^{\pm} &= 0,\\
x_7^{\pm} &= \frac{-350V^2 + 124T_L \pm 10S}{109T_L},
\end{aligned} \tag{26}$$

where $S = \sqrt{1225V^4 - 105V^2T_L - 25T_L^2}$, indicating that the equilibrium solutions exist when $V^2 \geq (3 + \sqrt{109})/70T_L$ ($T_L > 0$). The Jacobian of

Eq. (25) is given by

$$J(\mathbf{x}) = \begin{bmatrix} -\frac{3}{10} & -1 & 0 & \frac{3}{10} & 0 & 0 & 0 \\ 1 & -\frac{3}{10} & 0 & 0 & \frac{3}{10} & 0 & 0 \\ 0 & 0 & -\frac{3}{5} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} & -1+x_7 & 0 & x_5 \\ 0 & \frac{1}{2} & 0 & 1-x_7 & -\frac{1}{2} & 0 & -x_4 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ \frac{49x_5}{60\pi^3} & -\frac{49x_4}{60\pi^3} & 0 & -\frac{49x_2}{60\pi^3} & \frac{49x_1}{60\pi^3} & 0 & 0 \end{bmatrix}. \tag{27}$$

(Note that in general J is a function of the state variable $\mathbf{x}$ and the parameter variable $\boldsymbol{\mu}$, $J(\mathbf{x}, \boldsymbol{\mu})$.) The characteristic polynomial of $J(\mathbf{x})$ is obtained as $P_7(\lambda) = \lambda^7 + \sum_{i=1}^{7} a_i \lambda^{7-i}$, where

$$\begin{aligned}
a_1 &= \frac{16}{5}, \\
a_2 &= \frac{1}{60\pi^3}\left[49(x_1x_4 + x_2x_5) + 12(29 - 10x_7 + 5x_7^2)\pi^3\right], \\
a_3 &= -\frac{1}{3000\pi^3}\left[735(x_4^2 + x_5^2) - 6615(x_1x_4 + x_2x_5) - 2450(x_1x_5 - x_2x_4)(x_7 - 1) - (20352 - 13200x_7 + 6600x_7^2)\pi^3\right], \\
a_4 &= -\frac{1}{1500\pi^3}\left[882(x_4^2 + x_5^2) - 4410(x_1x_4 + x_2x_5) - 2695(x_1x_5 - x_2x_4)(x_7 - 1) - (8676 - 8400x_7 + 3975x_7^2)\pi^3\right], \\
a_5 &= -\frac{1}{30000\pi^3}\left[(6468 + 7350x_7)(x_4^2 + x_5^2) + (68600 - 64925x_7)(x_1x_5 - x_2x_4) - 77028(x_1x_4 + x_2x_5) - (107520 - 140640x_7 + 63120x_7^2)\pi^3\right], \\
a_6 &= -\frac{1}{7500\pi^3}\left[-(2058 - 2940x_7)(x_4^2 + x_5^2) + (14357 - 12887x_7)(x_1x_5 - x_2x_4) - 9457(x_1x_4 + x_2x_5) - (7380 - 11160x_7 + 4905x_7^2)\pi^3\right], \\
a_7 &= -\frac{49}{10000\pi^3}\left[30(x_4^2 + x_5^2)(x_7 - 1) + (124 - 109x_7)(x_1x_5 - x_2x_4) - 50(x_1x_4 + x_2x_5)\right].
\end{aligned} \tag{28}$$

In the following, we consider several singularities. Since the system has only two parameters, the singularities can only be codimension one or two, which include simple zero, double zero, Hopf bifurcation, double Hopf bifurcation and Hopf-zero singularity.

4.1. *Simple zero eigenvalue*

Letting $a_7 = 0$ and using Eq. (26) yields

$$a_7(\mathbf{x}^{\pm}) = \pm\frac{7}{10000\pi^3}\sqrt{1225V^4 - 105V^2T_L - 25T_L^2}. \tag{29}$$

Since the existence of the equilibrium solutions $\mathbf{x}^{\pm}$ requires $1225V^4 - 105V^2T_L - 25T_L^2 \geq 0$, $\mathbf{x}^-$ is always unstable as long as $1225V^4 - 105V^2T_L - 25T_L^2 > 0$. So we consider $\mathbf{x}^+$. Suppose V and T_L have been chosen such that $\mathbf{x}^+$ is stable, then when $1225V^4 - 105V^2T_L - 25T_L^2 = 0$, i.e.

$$V^2 = \left(\frac{3 + \sqrt{109}}{70}\right) T_L \quad (T_L > 0), \tag{30}$$

$\mathbf{x}^+$ becomes unstable, and static bifurcation (jumping) occurs from $\mathbf{x}^+$. Since this paper is focused on the study of the conditions of critical points, no further discussion will be held on the bifurcation solutions. Simple zero singularity only needs one parameter, we may fix one of them. For example, choose $T_L = 1$, then it can be shown that $\Delta_i > 0$, $i = 2, 3, 4, 5$ for any values of $V > 0$ (noting that $a_1 = 16/5 > 0$); $\Delta_6 > 0$ when $V < 6.2395593195$ or $V > 7.3536924239$; and $a_7 > 0$ when $V > 0.4381830425$. Therefore, $\mathbf{x}^+$ is stable when $0.4381830425 < V < 6.2395593195$ or $V > 7.35369242394$; unstable when $6.2395593195 < V < 7.35369242394$. The point, defined by

$$V_0 = \left(\frac{3+\sqrt{109}}{70}\right)^{1/2} \approx 0.4381830425, \tag{31}$$

is a static critical point form where a static bifurcation occurs.

4.2. *Double zero eigenvalue*

For this case, consider $\mathbf{x}^+$. In addition to the condition $a_7(\mathbf{x}^+) = 0$, we also need $a_6(\mathbf{x}^+) = 0$. Using Eqs. (26) and (28) results in

$$\begin{aligned} a_6(\mathbf{x}^+) = \frac{7}{65400\pi^3 T_L^2}\Big[&(70V^2 - 3T_L) \\ &\times (18000\pi^3 V^2 + 237T_L^2) \\ &+ 2(1289T_L^2 - 18000\pi^3 V^2) \\ &\times \sqrt{1225V^4 - 105V^2 T_L - 25T_L^2}\Big]. \end{aligned} \tag{32}$$

Thus, setting $a_7(\mathbf{x}^+) = a_6(\mathbf{x}^+) = 0$ gives

$$\begin{aligned} 1225V^4 - 105V^2 T_L - 25T_L^2 = 0, \\ 70V^2 - 3T_L = 0, \end{aligned} \tag{33}$$

which do not have positive solutions. Therefore, double zero singularity does not exist in this example.

4.3. *Hopf bifurcation*

For Hopf bifurcation, we only need one parameter, and again choose $T_L = 1$. Then, the existence of the two fixed points requires that $V \geq V_0 = ((3+\sqrt{109})/70)^{1/2} \approx 0.4381830425$. Since $\mathbf{x}^-$ is always unstable when $V > V_0$, we consider $\mathbf{x}^+$. Under the choice of $T_L = 1$, it has been shown in Sec. 4.1 that $\mathbf{x}^+$ is stable when $0.4381830425 < V < 0.62395593195$ or $V > 7.35369242394$; unstable when $0.62395593195 < V < 7.35369242394$. The point $V_0 = 0.4381830425$ is a static critical point. In this section, we shall show that the two points $V = 0.62395593195$ and $V = 7.35369242394$ are actually Hopf bifurcation points.

To analytically obtain the critical conditions under which a Hopf bifurcation may occur, apply the formula Δ_6 given in Eq. (6) to obtain the equation:

$$\begin{aligned} \Delta_6 &= Q_1(V^2) - Q_2(V^2)\sqrt{1225V^4 - 105V^2 - 25} \\ &= 0, \end{aligned} \tag{34}$$

where $Q_1(V^2)$ and $Q_2(V^2)$ are polynomials of V^2. Equation (34) yields a polynomial in the form

$$Q_1^2(V^2) - Q_2^2(V^2)(1225V^4 - 105V^2 - 25) = 0,$$

which is a 21st-degree polynomial of V^2. One may apply the built-in program *fsolve* in Maple to solve this polynomial to find all the three real solutions:

$$V^2 = 38.9321005020, \quad 54.0767922650 \quad \text{and} \quad -26.6127343788.$$

Taking positive values yields

$$V_h = 6.2395593195 \quad \text{or} \quad 7.3536924239. \tag{35}$$

These two solutions satisfy Eq. (34) and thus are true Hopf critical points, and are only Hopf critical points. From the discussion in Sec. 4.1 we know that at these two critical points, except $\Delta_6 = 0$, all other Hurwitz equalities still remain positive. Thus, at these two Hopf critical points, the characteristic polynomial has a purely imaginary pair and the remaining five eigenvalues still have negative parts. For example, we substitute $V = 6.2395593195$ into Eq. (26) to find the eigenvalues of $J(\mathbf{x})$ as follows:

$$\pm 0.7905733366i, \quad -1, \quad -0.6, \quad -0.5630004665, \quad -0.5184997667 \pm 1.0893171380i.$$

At the critical point $V = V_h = 6.2395593195$, the equilibrium solution $\mathbf{x}^+$ becomes [see Eq. (26)]

$$\begin{aligned} x_1^+ = 0.0000063079, \quad x_2^+ = 6.2361231187, \\ x_4^+ = -0.0114476949, \quad x_5^+ = 6.2361020924, \\ x_7^+ = 0.9990816376, \quad x_3^+ = x_6^+ = 0; \end{aligned} \tag{36}$$

and at $V = V_h = 7.3536924239$, $\mathbf{x}^+$ is given by

$$\begin{aligned} x_1^+ = 0.0000038521, \quad x_2^+ = 7.3507772803, \\ x_4^+ = -0.0097132933, \quad x_5^+ = 7.3507644401, \\ x_7^+ = 0.9993390384, \quad x_3^+ = x_6^+ = 0. \end{aligned} \tag{37}$$

The above results are obtained under the choice $T_L = 1$. For different values of T_L we shall obtain a different V_h. In general, $V_h = V_h(T_L)$, representing a curve in the V–T_L plane.

4.4. *Double Hopf bifurcation*

Now, we consider double Hopf bifurcation, which requires two parameters V and T_L. The condition for the existence of the equilibrium solutions $\mathbf{x}^{\pm}$ is $V^2 \geq ((3+\sqrt{109})/70)T_L$ $(T_L > 0)$. It is seen from Eq. (29) that $a_7(\mathbf{x}^-) < 0$ as long as $V^2 > ((3+\sqrt{109})/70)T_L$. Thus, we assume $V^2 > ((3+\sqrt{109})/70)T_L$ and consider possible double Hopf bifurcation from the equilibrium solution $\mathbf{x}^+$.

The necessary and sufficient conditions for the existence of a double Hopf bifurcation are $\Delta_4 = \Delta_6 = 0$. Applying the formulas given in Eq. (6) with the aid of Eq. (28) yields

$$\begin{aligned} \Delta_4 &= G_1(V^2, T_L) - G_2(V^2, T_L) \\ &\quad \times \sqrt{1225V^4 - 105V^2T_L - 25T_L^2} = 0, \\ \Delta_6 &= Q_1(V^2, T_L) - Q_2(V^2, T_L) \\ &\quad \times \sqrt{1225V^4 - 105V^2T_L - 25T_L^2} = 0, \end{aligned} \tag{38}$$

which can be rewritten as two polynomial equations:

$$\begin{aligned} &G_1^2(V^2, T_L) - G_2^2(V^2, T_L)(1225V^4 - 105V^2T_L \\ &\qquad - 25T_L^2) = G_{9,12} = 0, \\ &Q_1^2(V^2, T_L) - Q_2^2(V^2, T_L)(1225V^4 - 105V^2T_L \\ &\qquad - 25T_L^2) = A_{3,4}B_{3,4}C_{9,12} = 0, \end{aligned} \tag{39}$$

where $G_{9,12}, A_{3,4}, B_{3,4}$ and $C_{9,12}$ are all polynomials of V^2 and T_L. The subscripts m and n denote the degrees of the polynomials with respect to V^2 and T_L, respectively. Therefore, we need to solve three sets of equations:

$$\begin{aligned} \text{I. } & G_{9,12} = A_{3,4} = 0, \\ \text{II. } & G_{9,12} = B_{3,4} = 0, \\ \text{III. } & G_{9,12} = C_{9,12} = 0. \end{aligned}$$

For each set of the equations, we first eliminate V^2 to find a polynomial of single variable T_L, and then solve all the real solutions of T_L from that polynomial. It can be shown that the three sets only have the following real solutions:

$$\begin{aligned} \text{I. } (T_L, V^2) &= (1614.581037, -171.790184), \\ &\quad (2869.546219, -315.371235); \\ \text{II. } (T_L, V^2) &= (1418.010870, -152.195957), \\ &\quad (2869.546219, -315.371235); \\ \text{III. } (T_L, V^2) &= (955.334730, -128.759327), \\ &\quad (524.043793, -94.857639). \end{aligned} \tag{40}$$

Thus, no positive solutions exist for V^2. Hence, no Double Hopf bifurcation point exist for this example.

4.5. *Hopf-zero singularity*

Finally, we consider Hopf-zero singularity, which is determined from $a_7 = \Delta_6 = 0$. Using Eq. (29) and the second equation of Eq. (38) yields

$$\begin{aligned} V^2 &= \frac{3+\sqrt{109}}{70}T_L, \\ Q_1(T_L) &= \frac{1}{63233837890625}\left[\frac{13965396212411959\sqrt{109}}{3456000000\pi^{15}}T_L^5\right. \\ &\quad + \left(\frac{592509049275164777}{144000000\pi^{12}} + \frac{48306554339113\sqrt{109}}{1920000\pi^{12}}\right)T_L^4 \\ &\quad + \left(\frac{4653993307256909}{480000\pi^9} + \frac{188877393676254783 71\sqrt{109}}{1308000000\pi^9}\right)T_L^3 \\ &\quad + \left(\frac{185178722456914113 97}{5450000\pi^6} - \frac{35519483292986567\sqrt{109}}{1090000\pi^6}\right)T_L^2 \\ &\quad - \left(\frac{1463255478565608}{125\pi^3} - \frac{7528814115153510144\sqrt{109}}{1485125\pi^3}\right)T_L \\ &\quad \left. + 307911640221504 - \frac{33627879261312192\sqrt{109}}{11881}\right]. \end{aligned} \tag{41}$$

Since $Q_1(T_L) = 0$ does not have positive solutions for T_L, Hopf-zero singularity does not exist in this example.

Summarizing the above results shows that the model of induction machines can only have simple zero and Hopf singularities. When T_L is taken as 1, the critical point for the simple zero is $V_0 = \left((3+\sqrt{109})/70\right)^{1/2} \approx 0.4381830425$, while that for Hopf bifurcation is $V_h = 6.2395593195$ or $V_h = 7.3536924239$.

5. Conclusion

In this note, explicit necessary and sufficient conditions for determining the critical points of general differential equations have been derived in terms of system parameters. Zero singularity, Hopf and multiple Hopf bifurcations, and the combination of zero and Hopf singularity have been considered. An induction machine model is presented to show the advantage of using the new formulas. The results obtained in this note are quite useful in solving real physical and engineering problems.

Acknowledgment

The support received from the Natural Sciences and Engineering Research Council of Canada (NSERC) is greatly acknowledged.

References

Chen, G., Moiola, J. L. & Wang, H. O. [2000] "Bifurcation control: Theories, methods, and applications," *Int. J. Bifurcation and Chaos* **10**, 515–548.

Chua, L. O., Desoer, C. A. & Kuh, E. S. [1987] *Linear and Nonlinear Circuits* (McGraw-Hill, NY).

Guckenheimer, J. & Holmes, P. [1993] *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, 4th edition (Springer-Verlag, NY).

Krause, P. C., Wasynczuk, O. & Sudhoff, S. D. [2002] *Analysis of Electrical Machinery and Drive Systems* (IEEE Press and Wiley Interscience, NJ).

Marsden, J. E. & McCracken, M. [1976] *The Hopf Bifurcation and its Applications* (Springer-Verlag, NY).

Orlando, L. [1911] "Sul problema di Hurwitz relative alle parti realli delle radici di un'equazione algebrica," *Math. Ann.* **71**, p. 233.

Porter, B. [1967] *Stability Criteria for Linear Dynamical Systems* (Oliver & Boyd, London).

Takens, F. [1974] "Singularities of vector fields," *Publ. Math. IHES* **43**, 47–100.

Yu, P. [1998] "Computation of normal forms via a perturbation technique," *J. Sound Vibr.* **211**, 19–38.

Yu, P. [2000] "A method for computing center manifold and normal forms," *Proc. Int. Conf. Diff. Eqns.* (Berlin, 1999), Vol. 2, (World Scientific, Singapore), pp. 832–837.

Yu, P. [2001] "Symbolic computation of normal forms for resonant double Hopf bifurcations using multiple time scales," *J. Sound Vibr.* **247**, 615–632.

Yu, P. [2002] "Analysis on double Hopf bifurcation using computer algebra with the aid of multiple scales," *Nonlin. Dyn.* **27**, 19–53.

Yu, P. & Leung, A. Y. T. [2003] "The simplest normal form of Hopf bifurcation," *Nonlinearity* **16**, 277–300.