

SBS Chapter 2: Limits & continuity

(SBS 2.1) Limit of a function

Consider a free falling body with no air resistance. Falls approximately $s(t) = 16t^2$ feet in t seconds.

- We already know how to find the *average velocity* over an interval of time.
- Now we want to know *instantaneous velocity* at $t = 2$ seconds, for example. We can express this as a *limit*.

Compute the average velocity over a smaller and smaller time interval near $t = 2$ seconds.

Start with \bar{v} as the average velocity over the interval $1.9 \leq t \leq 2$.

$$\begin{aligned}\bar{v} &= \frac{\text{distance traveled}}{\text{elapsed time}} \\ &= \frac{s(2) - s(1.9)}{2 - 1.9} \\ &= \frac{16(2)^2 - 16(1.9)^2}{0.1} \\ &= 62.4 \text{ ft/s}\end{aligned}$$

We can make similar computations with smaller and smaller intervals from above or below $t = 2$.

From below:

Interval	[1.9,2]	[1.99,2]	[1.999,2]
Length	0.1	0.01	0.001
Ave. velocity	62.4	63.84	63.98

From above:

Interval	[2,2.0001]	[2,2.001]	[2,2.01]
Length	0.0001	0.001	0.01
Ave. velocity	64.0016	64.016	64.16

Average velocity seems to be approaching 64 as we make the time intervals smaller. So we expect the instantaneous velocity at $t = 2$ will be 64.

Average velocity of the falling body over time interval $2 \leq t \leq 2+h$ is

$$\frac{\Delta s}{\Delta t} = \frac{s(2+h) - s(2)}{(2+h) - 2} = \frac{16(2+h)^2 - 16(2)^2}{h}.$$

Think of h as a small number, so $(2, 2+h)$ is a small interval above 2.

In this example, the average velocity has a limiting value of 64 as the length h of the time interval tends to zero.

We write:

$$\lim_{h \rightarrow 0} \frac{16(2+h)^2 - 16(2)^2}{h} = 64.$$

Informal limit definition:

The notation

$$\lim_{x \rightarrow c} f(x) = L$$

means: the function values $f(x)$ can be made arbitrarily close to a unique number L by choosing x sufficiently close to c (but not equal to c).

Other notation: Also written as $f(x) \rightarrow L$ as $x \rightarrow c$

Limit example: Evaluate (by table)

$$L = \lim_{x \rightarrow 0} \frac{2\sqrt{x+1} - x - 2}{x^2}$$

x	-0.5	-0.1	-0.01	-0.001	0	0.001	0.005
f(x)	-0.3431	-0.2633	-0.2513	-0.2501	undef	-0.2499	-0.2494

Table pattern suggests that the limit L is -0.25 .

- Note that we can have a limit of -0.25 as $f(x) \rightarrow 0$ even though $f(0)$ itself is undefined.
- Be careful with calculators (or computers) and subtracting very small numbers! Can introduce errors.

One-Sided Limits

Right-hand limit:

$$\lim_{x \rightarrow c^+} = L$$

if we can make $f(x)$ arbitrarily close to L by choosing x sufficiently close to c on a small interval (c, b) immediately to the right of c .

Left-hand limit:

$$\lim_{x \rightarrow c^-} = L$$

if we can make $f(x)$ arbitrarily close to L by choosing x sufficiently close to c on a small interval (a, c) immediately to the left of c .

Theorem. (*One-sided limit theorem*)

The two-sided limit $\lim_{x \rightarrow c} f(x)$ exists iff the two one-sided limits exist and are equal. Furthermore, if

$$\lim_{x \rightarrow c^-} f(x) = L = \lim_{x \rightarrow c^+} f(x),$$

then

$$\lim_{x \rightarrow c} f(x) = L$$

- Note that the limit does not depend on how the function behaves exactly at c .
- The function does not even have to be defined at $x = c$!

Example: consider 3 examples with same limit

$$\begin{aligned}f(x) &= \frac{x^2 - 1}{x - 1} \\g(x) &= \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases} \\h(x) &= x + 1\end{aligned}$$

Limits do not always exist:

If the limit of the function f fails to exist, $f(x)$ is said to *diverge* as $x \rightarrow c$.

- The function may grow arbitrarily large (or small) as $x \rightarrow c$
E.g., $\lim_{x \rightarrow 0} \frac{1}{x^2}$

A function f that increases or decreases without bound as x approaches c is said to *tend to infinity* as $x \rightarrow c$.

$$\lim_{x \rightarrow c} f(x) = +\infty \quad \text{if } f \text{ increases without bound}$$

$$\lim_{x \rightarrow c} f(x) = -\infty \quad \text{if } f \text{ decreases without bound}$$

- The function may oscillate as $x \rightarrow c$
E.g., $\lim_{x \rightarrow 0} \sin \frac{1}{x}$
divergence by oscillation

Formal definition of a limit (epsilon-delta definition):

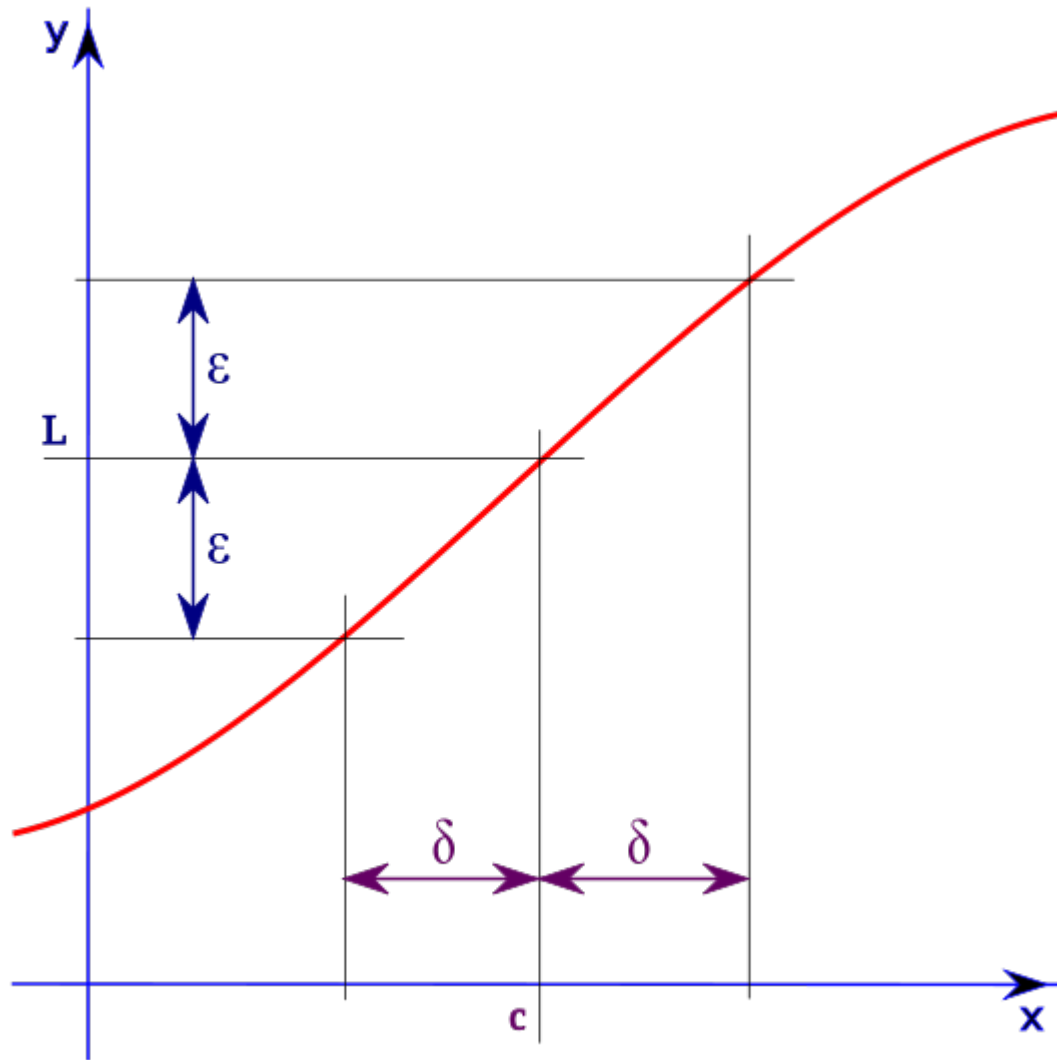
Definition. The *limit* statement:

$$\lim_{x \rightarrow c} f(x) = L$$

means that for each number $\varepsilon > 0$ there corresponds a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.

In other words:

- If the distance between x and c is small enough, then the distance between $f(x)$ and L is also small.
- If x is in $(c, c + \delta)$ or $(c - \delta, c)$, then $f(x)$ must be in the interval $(L - \varepsilon, L + \varepsilon)$.



If the limit exists, can always find a value of δ that makes the ϵ band as narrow as I ask it to be. If I can't always find such a δ , the limit does not exist.

Example. ($\varepsilon - \delta$ proof)

Show that $\lim_{x \rightarrow 1} (5x - 3) = 2$.

Need to show that for any ε , we can find a δ so that whenever

$0 < |x - 1| < \delta$, we also have $|f(x) - 2| < \varepsilon$.

Algebraic computation of limits (SBS 2.2)

Basic properties and rules for limits:

For any real number c , suppose f and g both have limits as $x \rightarrow c$, and let k be a constant:

Constant rule:

$$\lim_{x \rightarrow c} k = k$$

Limit of x rule:

$$\lim_{x \rightarrow c} x = c$$

Multiple rule:

$$\lim_{x \rightarrow c} [k f(x)] = k \lim_{x \rightarrow c} f(x)$$

Sum rule:

$$\lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$$

Difference rule:

$$\lim_{x \rightarrow c} [f(x) - g(x)] = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x)$$

Product rule:

$$\lim_{x \rightarrow c} f(x)g(x) = \left(\lim_{x \rightarrow c} f(x)\right)\left(\lim_{x \rightarrow c} g(x)\right)$$

Quotient rule:

$$\lim_{x \rightarrow c} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$$

if $\lim_{x \rightarrow c} g(x) \neq 0$

Power rule:

$$\lim_{x \rightarrow c} (f(x)^n) = \left(\lim_{x \rightarrow c} f(x)\right)^n$$

for n rational and limit on
right exists

Limit of a polynomial:

If P is a polynomial function, then

$$\lim_{x \rightarrow c} P(x) = P(c)$$

Proof: use first 5 rules.

Limit of a rational function:

If Q is a rational function

$Q(x) = \frac{P(x)}{D(x)}$ then

$$\lim_{x \rightarrow c} Q(x) = \frac{P(c)}{D(c)}$$

provided $\lim_{x \rightarrow c} D(x) \neq 0$

Proof: use Quotient Rule and Limit of a Polynomial rule.

Limits of trigonometric functions:

If c is any number in the domain of the given function, then

$$\lim_{x \rightarrow c} (\cos x) = \cos c$$

$$\lim_{x \rightarrow c} (\sec x) = \sec c$$

$$\lim_{x \rightarrow c} (\sin x) = \sin c$$

$$\lim_{x \rightarrow c} (\csc x) = \csc c$$

$$\lim_{x \rightarrow c} (\tan x) = \tan c$$

$$\lim_{x \rightarrow c} (\cot x) = \cot c$$

Examples of limits: sometimes need to manipulate function to find the limit

□ Fractional reduction

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}$$

Note that $f(1)$ is undefined. But the limit still exists as $x \rightarrow 1$.

□ Rationalization

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} = ?$$

$$\frac{\sqrt{x^2 + 100} - 10}{x^2} = \left(\frac{\sqrt{x^2 + 100} - 10}{x^2} \right) \cdot \left(\frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} \right)$$

One of the most common tricks in math is multiplying by “a clever choice of 1”

□ Piecewise functions

$$\lim_{x \rightarrow 2} f(x) = \begin{cases} x^2 + 1, & x \leq 2 \\ 2x + 1 & x > 2 \end{cases}$$

For a limit at the break, need to check left and right limits and see if they are equal.

Squeeze Rule

Theorem. (*Squeeze rule, a.k.a. sandwich theorem*)

If $g(x) \leq f(x) \leq h(x)$ for all x in an open interval about c (but not necessarily including c), and if

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L,$$

then

$$\lim_{x \rightarrow c} f(x) = L.$$

Example: use Squeeze rule to prove:

$$\lim_{x \rightarrow 0} x \sin \left(\frac{1}{x} \right) = 0$$

Can't use product rule since 2nd limit diverges (by oscillation).

For proof, recall $-|a| \leq a \leq |a|$ and $|\sin \frac{1}{x}| \leq 1$ for all $x \neq 0$:

Some special limits

(can prove with squeeze rule - see textbook)

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$$

(memorize these for now - we'll have a better way of doing them later)

Continuity (SBS 2.3)

Intuitively, continuity means “without jumps or breaks.”

Formal definition:

Definition. A function f is *continuous at a point* $x = c$ if the following three conditions are all satisfied:

1. $f(c)$ is defined
2. $\lim_{x \rightarrow c} f(x)$ exists
3. $\lim_{x \rightarrow c} f(x) = f(c)$

A function that is not continuous at c is said to have a *discontinuity* at that point.

The key idea of continuity is that if x is close to c , then $f(x)$ is close to $f(c)$.

Examples:

$$\begin{aligned}f(x) &= \frac{x^2 - 1}{x - 1} \\g(x) &= \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 4, & x = 1 \end{cases} \\h(x) &= x + 1\end{aligned}$$

All three have a limit of 2 as $x \rightarrow 1$, but only $h(x)$ is *continuous* at $x = 1$.

Some common discontinuity examples: holes, poles, jumps

□ hole:

- $\lim_{x \rightarrow c} f(x)$ exists, but $f(c)$ not defined
- $\lim_{x \rightarrow c} f(x)$ exists, $f(c)$ defined, but $f(c) \neq \lim_{x \rightarrow c} f(x)$

□ jump:

- left limit not equal to right limit

□ pole:

- $f(c)$ defined, but either left or right limit $\rightarrow \pm\infty$

Continuity Theorem

Theorem. *If f is a polynomial, rational function, power function, trigonometric function, or an inverse trigonometric function, then f is continuous at any number $x = c$ for which $f(c)$ is defined (i.e., f is continuous at each x in its domain).*

Theorem. If functions f and g are continuous at $x = c$, then the following functions are also continuous at $x = c$:

Scalar multiple: $kf(x)$ for any const k

Sum and difference: $f(x) + g(x)$ and $f(x) - g(x)$

Product: $f(x)g(x)$

Quotient: $\frac{f(x)}{g(x)}$ provided $g(c) \neq 0$

Composition: $f \circ g(x)$ provided g cont at c and f cont at $g(c)$

Theorem. (*Composition Limit Rule*)

*If $\lim_{x \rightarrow c} g(x) = L$ and f is continuous at L ,
then $\lim_{x \rightarrow c} f[g(x)] = f(L)$.*

I.e.,

$$\lim_{x \rightarrow c} f[g(x)] = f \left[\lim_{x \rightarrow c} g(x) \right] = f(L)$$

Applies similarly to left and right limits.

*Idea is that the limit of a continuous function is the
function of the limiting value.*

Continuity from the left and right:

Definition. The function f is *continuous from the right at a* iff

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

The function f is *continuous from the left at b* iff

$$\lim_{x \rightarrow b^-} f(x) = f(b)$$

I.e., continuous from the right at a iff

1. $f(a)$ is defined
2. $\lim_{x \rightarrow c^+} f(x)$ exists
3. $\lim_{x \rightarrow c^+} f(x) = f(a)$

(Similarly for left continuity.)

Continuity on an interval:

Definition. The function f is *continuous on the open interval* (a,b) if it is continuous at each number in the interval.

(Note that the end points are *not* in the interval.)

- If f is continuous on (a,b) and continuous from the right at a , then f is *continuous on the half-open interval* $[a,b)$.
- If f is continuous on (a,b) and continuous from the left at b , then f is *continuous on the half-open interval* $(a,b]$.
- If f is continuous on (a,b) , continuous from the right at a , and continuous from the left at b , then f is *continuous on the closed interval* $[a,b]$.

Checking continuity at “suspicious points”

Often, there are only a few points in the domain where a discontinuity may occur. We call such points **suspicious points**:

- where the defining rule for f changes (e.g., piecewise defined functions, absolute values)
- where substitution of $x = c$ causes division by 0

Example of checking suspicious points:

$$f(x) = \begin{cases} 2x + 5, & x > 2 \\ 15 - x^2, & x \leq 2 \end{cases}$$

Intermediate Value Theorem

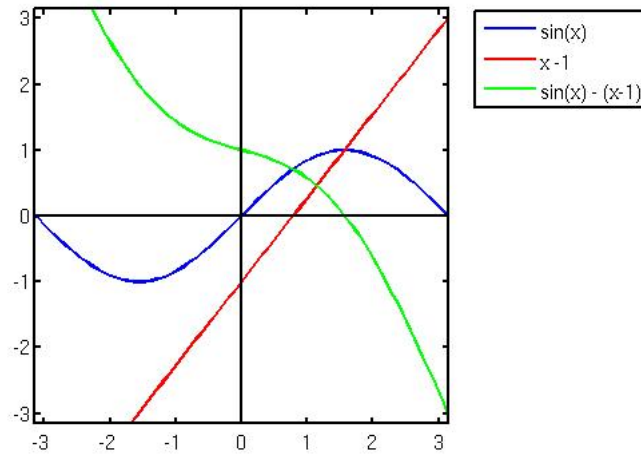
Theorem. (*Intermediate Value Theorem*) *If f is a continuous function on the closed interval $[a, b]$ and L is some number strictly between $f(a)$ and $f(b)$, then there exists at least one number c on the open interval (a, b) such that $f(c) = L$.*

In other words: if f is a continuous function on $[a, b]$ then $f(x)$ must take on all values between $f(a)$ and $f(b)$.

Important special case of the Intermediate Value Theorem:

Theorem. (*Root location theorem*) *If f is continuous on the closed interval $[a,b]$ and if $f(a)$ and $f(b)$ have opposite algebraic signs, then $f(c) = 0$ for at least one number c on the open interval (a,b) .*

Example: Show that $f(x) = \sin x - x + 1$ has a root on the interval $(\frac{\pi}{2}, \pi)$:



Proof:

- $f(x)$ is continuous on the interval $[\frac{\pi}{2}, \pi]$ (no suspicious points; it's continuous on whole real line).
- $f(\frac{\pi}{2}) = 1 - \frac{\pi}{2} + 1 = 2 - \frac{\pi}{2} > 0$
- $f(\pi) = 0 - \pi + 1 < 0$

- (i.e., $L = 0$ is strictly between $f(\frac{\pi}{2})$ and $f(\pi)$)
- Therefore $f(x)$ has at least one root in $(\frac{\pi}{2}, \pi)$ by Root location theorem (or by Intermediate Value Theorem)

Exponentials and Logarithms (SBS 2.4)

Recall that we can already define b^x for any rational number x :

- For natural (counting) numbers n , we have $b^n = b \cdot b \cdot b \cdots b$ (n factors).
- If $b \neq 0$, then $b^0 = 1$, and $b^{-n} = \frac{1}{b^n}$.
- If $b > 0$, then $b^{1/n} = \sqrt[n]{b}$.
- If m, n are integers and m/n is a reduced fraction, then $b^{m/n} = (b^{1/n})^m = \sqrt[n]{b^m}$.

These steps work us up from counting numbers to any rational number $\frac{m}{n}$.

We now want to extend the idea to all real numbers.

Completeness of the reals

For any real number x there is a sequence r_n of rational numbers such that

$$x = \lim_{n \rightarrow \infty} r_n.$$

That is, for any $\varepsilon > 0$ there is a number N such that $n > N \Rightarrow |x - r_n| < \varepsilon$.

As n gets bigger and bigger, r_n gets closer and closer to the real number x .

This means that we can approximate any real number to any degree of accuracy by a rational number.

Exponential functions

Definition. Let x be a real number and let r_n be a sequence of rational numbers such that $x = \lim_{n \rightarrow \infty} r_n$.

Then the *exponential function with base b* , $b > 0$ ($b \neq 1$), is given by

$$b^x = \lim_{n \rightarrow \infty} b^{r_n}$$

b is called the *base* and x is the *exponent*.

[Do not confuse these with power functions $f(x) = x^p$. Here the variable is in the exponent. In a power function, it is in the base.]

Properties of the exponential function

Let x and y be any real numbers, and let a and b be positive real numbers.

Equality rule: If $b \neq 1$, then $b^x = b^y$ iff $x = y$

Inequality rules: If $x > y$ and $b > 1$, then $b^x > b^y$
If $x > y$ and $0 < b < 1$, then $b^x < b^y$

Product rule: $b^x b^y = b^{x+y}$

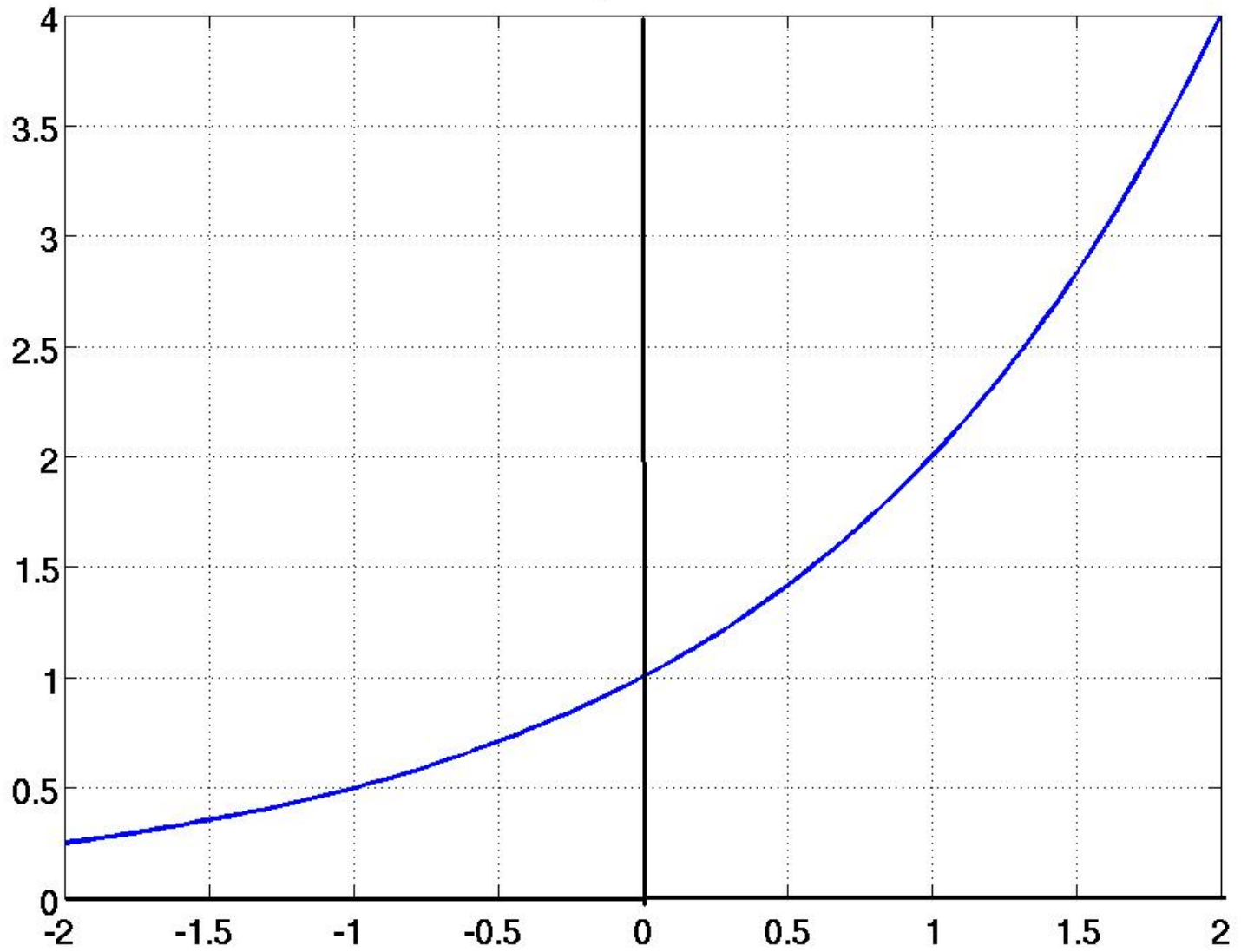
Quotient rule: $\frac{b^x}{b^y} = b^{x-y}$

Power rules: $(b^x)^y = b^{xy}$; $(ab)^x = a^x b^x$; $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

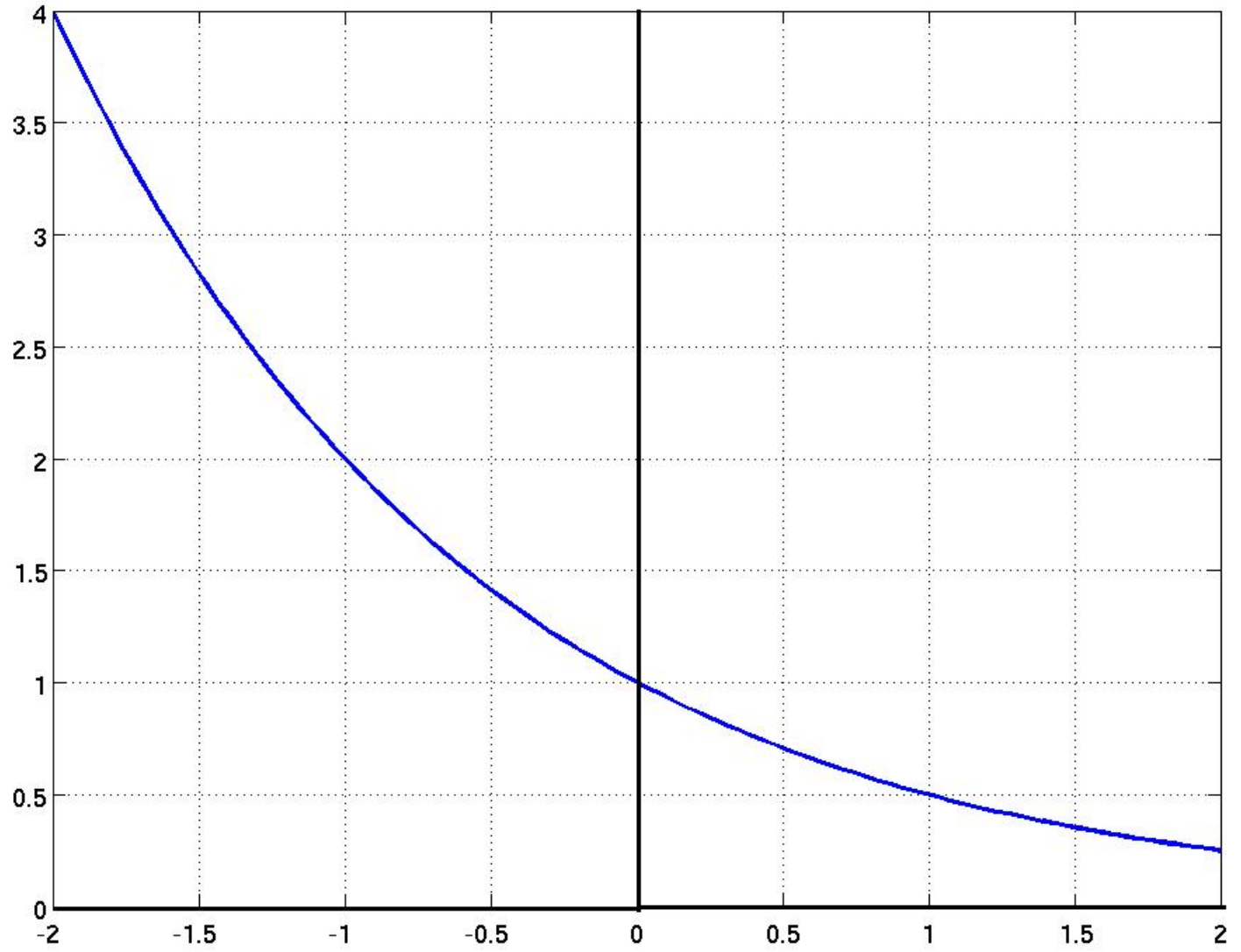
Graphical properties:

- The exponential function $f(x) = b^x$ is continuous for all real numbers x .
- $b^x > 0$
- The graph is always rising for $b > 1$ and always falling for $0 < b < 1$.

$$y = 2^x$$



$$y = (1/2)^x$$



Example of working with exponentials:

$$3^{x^2-x} = 9$$

$$3^{x^2-x} = 3^2$$

$$\Rightarrow x^2 - x = 2 \quad (\text{equality rule})$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2, -1$$

Logarithmic functions

$y = b^x$ ($b > 0$, $b \neq 1$) is monotonic, so the exponential has an inverse.

If $b > 0$ and $b \neq 1$, the **logarithm of x to the base b** is the function $y = \log_b x$ that satisfies $b^y = x$.

$$y = \log_b x \text{ means } b^y = x$$

Note that since $b^y = x$, this means that we can only take the logarithm of positive numbers.

I.e., the domain of the logarithm (which is the range of the exponential) is $(0, \infty)$.

Properties of the logarithmic function

Let x and y be real numbers, and assume $b > 0$ and $b \neq 1$.

Equality rule: $\log_b x = \log_b y$ if and only if $x = y$

Inequality rules: If $x > y$ and $b > 1$, then $\log_b x > \log_b y$
If $x > y$ and $0 < b < 1$, then $\log_b x < \log_b y$

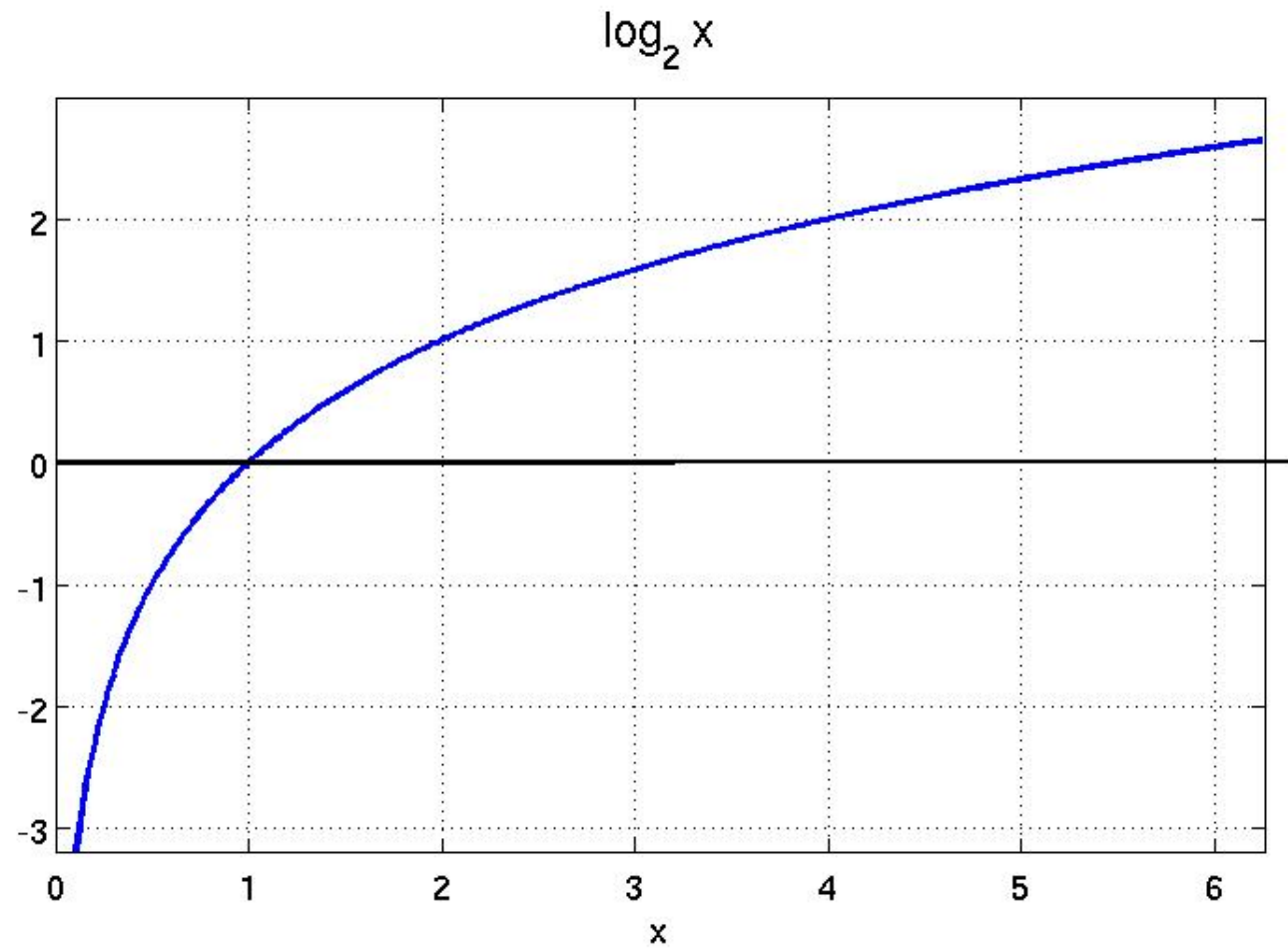
Product rule: $\log_b(xy) = \log_b x + \log_b y$

Quotient rule: $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

Power rule: $\log_b x^p = p \log_b x$ for any real number p

Inversion rules: $b^{\log_b x} = x$ and $\log_b b^x = x$

Special values: $\log_b b = 1$ and $\log_b 1 = 0$



$$y = \log_2(x)$$

Examples: solve for x

$$\log_x 16 = 2$$

$$\log_3 x + \log_3(2x + 1) = 1$$

Natural exponential and natural logarithm

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n$$

$$e \approx 2.7182818\dots$$

- Notation: $\exp(x) = e^x$ is the **natural exponential**
- **Natural Logarithm** is the log base e , $\log_e x$, and is often written $\ln x$
- **Common Logarithm** is the log base 10, $\log_{10} x$, and is written $\log x$
- **Beware**: the notation “ $\log x$ ” is often used to represent whichever base is the standard base in a particular field (or a particular course).

- But “ $\ln x$ ” always means the Natural Logarithm.
- **In this class**, $\log x$ or $\log_{10} x$ will always mean the common logarithm and $\ln x$ or $\log_e x$ will mean the natural logarithm. For any other base we write $\log_b x$.

Basic properties of the natural logarithm:
(logarithm base e)

- $\ln 1 = 0$
- $\ln e = \log_e e = 1$
- $e^{\ln x} = x$ for all $x > 0$
- $\ln e^y = y$ for all y
- $b^y = e^{y \ln b}$ for any $b > 0$ ($b \neq 1$)

Theorem. (*Change of base theorem*)

$$\log_b x = \frac{\ln x}{\ln b}$$

for any $b > 0$ ($b \neq 1$).

Important, for example, for taking logarithms on a calculator or in many software packages, which often only do common and natural logarithms.

Example: Find N such that $e^N = 10^{2x}$:

Example: an object moves along a straight line such that after t seconds, its velocity is given by

$$v(t) = 10\log_5 t + 3\log_2 t$$

in ft/sec. How long will it take for the velocity to reach 20 ft/sec?

What is so natural about the natural log?

Many important growth (and decay) processes are described in terms of natural exponentials and logarithms. For example,

- Can be used to describe some types of biological colony growth
e.g., exponential growth of E. Coli bacteria
- Describes continuous compound interest (see examples in 2.4)
- Newton's law of cooling (see problem 69 in 2.4)
- Many other applications such as disease propagation, radioactive decay...

Example: Exponential growth

The population of a particular bacterial colony at time t minutes is $P(t)$:

$$P(t) = P_0 e^{kt}$$

where P_0 is the initial population and k is a positive constant.

If the colony begins with 5000 individuals and has a population of 7000 after 20 minutes, what is the constant k , and what will the population be after 30 minutes?

Example: continuous compounding interest

If D dollars are compounded n times per year at an annual interest rate r , then the Future Value after t years is

$$A(t) = D \left(1 + \frac{r}{n}\right)^{nt}$$

If the compounding interest is continuous, the Future Value is

$$A(t) = De^{rt}$$