Solutions to Exam I Practice Questions

Math 1351-011

9/20/2007

1a. $|3x-5| = 4$

$3x - 5 = 4$ OR $3x - 5 = -4$
$3x = 9$ $3x = 1$
$x = 3$ $x = 1/3$

$\boxed{x = 1/3,\ 3}$

b. $9x + 3x^2 \leq 0$
$3x + x^2 \leq 0$
$x(3+x) \leq 0$

find critical values:
$x(3+x) = 0$
$x = 0, -3$

if $x < 0$ need $(x+3) > 0$ [neg · pos = neg < 0]
$x > -3$
if $x > 0$ need $(x+3) < 0$ [pos · neg = neg < 0]
$x < -3$
but can't have $x > 0$ and $x < -3$,
so must be first case.

$x(3+x) < 0$ for $-3 < x < 0$
$x(3+x) = 0$ for $x = 0, -3$

So $9x + 3x^2 \leq 0$ when $\boxed{-3 \leq x \leq 0}$

c. $|2-3x| \leq 6$

$-6 \leq 2 - 3x \leq 6$
$-8 \leq -3x \leq 4$
$8 \geq 3x \geq -4$ ← multiply by -1, so reverse the inequalities
$\boxed{8/3 \geq x \geq -4/3}$

2. Line passes through $P(1,2)$ and parallel to line passing through $Q(1,4)$ and $R(2,7)$.

Line $\overline{QR}$ has slope $m = \frac{\Delta y}{\Delta x} = \frac{7-4}{2-1} = 3$

Parallel line also has slope 3.
Use point-slope eqn. for line through $P(1,2)$ with slope $= 3$:

$$(y-2) = 3(x-1)$$
$$y-2 = 3x-3$$
$$\boxed{3x - y - 1 = 0}$$

Midpoint of line segment QR is

$$\left(\frac{x_Q + x_R}{2}, \frac{y_Q + y_R}{2}\right) = \left(\frac{1+2}{2}, \frac{4+7}{2}\right)$$
$$= \boxed{\text{mid point } \left(\frac{3}{2}, \frac{11}{2}\right)}$$

3. Line perpendicular to line passing through $Q(1,4)$ and $R(2,7)$, and which passes through midpoint of line segment $\overline{QR}$.

· midpt. of line segment $\overline{QR}$ is $\left(\frac{3}{2}, \frac{11}{2}\right)$ (see #2)
· slope of line through Q and R is 3 (see #2)

Slope of line perpendicular to line $\overline{QR}$ is $-1/3$.

$$(y - 11/2) = -1/3(x - 3/2)$$
$$6y - 33 = -2x + 3$$
$$2x + 6y - 36 = 0$$
$$\boxed{x + 3y - 18 = 0}$$

4. $x^2 - 6x + y^2 + 4y = -12$

$(x^2 - 6x \qquad) + (y^2 + 4y \qquad) = -12$

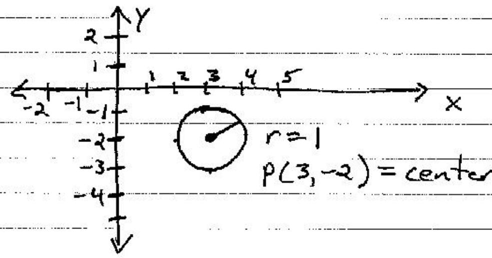
Complete the square

$(x^2 - 6x + 9) + (y^2 + 4y + 4) = -12 + 9 + 4$

$(x-3)^2 + (y+2)^2 = 1$

center $= (3, -2)$

radius $= \sqrt{1} = 1$



5. First find slope of each line

a. $y - 2x + 7 = 0$

$y = \underline{\underline{2}}x - 7 \qquad$ slope $= 2$

b. $2x + 2y - 4 = 0$

$x + y - 2 = 0$

$y = \underline{\underline{-}}x + 2 \qquad$ slope $= -1$

c. $y + 2 = \underline{\underline{2}}(x - 4) \qquad$ slope $= 2$

d. $x + 2y - 6 = 0$

$2y = -x + 6$

$y = \underline{\underline{-\frac{1}{2}}}x + 3 \qquad$ slope $= -\frac{1}{2}$

Lines a and c are parallel.
Line d is perpendicular to lines a and c

6. $f(s) = s^2 - 1, \quad g(t) = \dfrac{3t}{t-1}$

a. domain of f is $(-\infty, \infty)$

b. domain of g is $(-\infty, 1) \cup (1, \infty)$
(all real numbers except 1)

c.
$$
\begin{aligned}
f \circ g &= \left(\frac{3t}{t-1}\right)^2 - 1 \\
&= \frac{9t^2}{(t-1)^2} - 1 \\
&= \frac{9t^2 - (t-1)^2}{(t-1)^2} \\
&= \frac{9t^2 - t^2 + 2t - 1}{(t-1)^2} \\
&= \boxed{\frac{8t^2 + 2t - 1}{(t-1)^2}}
\end{aligned}
$$

d.
$$
\begin{aligned}
g \circ f &= \frac{3(s^2 - 1)}{s^2 - 1 - 1} \\
&= \boxed{\frac{3s^2 - 3}{s^2 - 2}}
\end{aligned}
$$

e.
$$
\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - 1 - (x^2 - 1)}{h} \\
&= \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h} \\
&= \frac{2hx + h^2}{h} \\
&= \boxed{2x + h}
\end{aligned}
$$

7. $f(t) = 2\cos t + 3$, $g(t) = \sin^{-1} t$

$$g(f(\pi)) = \sin^{-1}(2\cos\pi + 3)$$
$$= \sin^{-1}(2\cdot(-1)+3)$$
$$= \sin^{-1}(1)$$
$$= \frac{\pi}{2}$$

[Recall: domain of $\sin^{-1}$ is $-1 \le x \le 1$, so 1 is in domain. Range of $\sin^{-1}$ is $-\pi/2 \le x \le \pi/2$]

8. a. $f(x) = x^2 + 3x + 1$

$f(x)$ is not strictly monotonic on its domain, so f^{-1} does not exist. (Does not pass horiz. line test.)

To graph, find x and y intercepts:

$f(0) = 1$ $\quad (0,1) =$ y-intercept

$f(x) = 0$

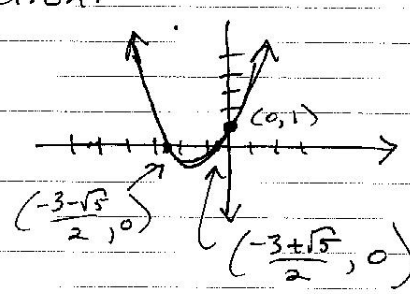
$$x^2 + 3x + 1 = 0$$
$$x = \frac{-3 \pm \sqrt{9 - 4\cdot1\cdot1}}{2}$$
$$x = \frac{-3 \pm \sqrt{5}}{2}$$

[Recall: quadratic formula $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$]

x-intercepts

$\left(\frac{-3-\sqrt{5}}{2}, 0\right)$, $\left(\frac{-3+\sqrt{5}}{2}, 0\right)$

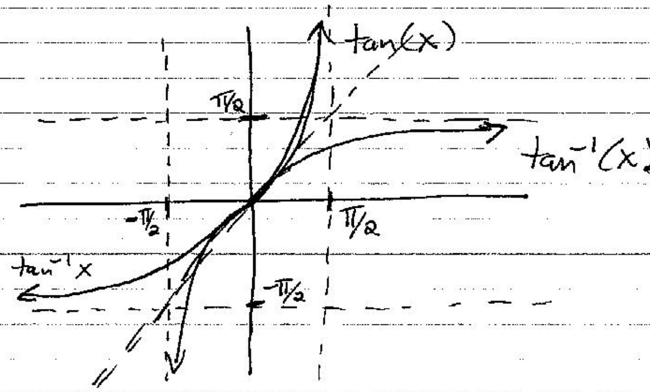
We also know the basic shape of a quadratic function.



8 b. $g(t) = \tan(t)$ on $(-\pi/2, \pi/2)$

tangent is strictly monotonically increasing on $(-\pi/2, \pi/2)$, so g^{-1} exists.

$\boxed{g^{-1}(t) = \tan^{-1}(t)}$ on $(-\infty, \infty)$



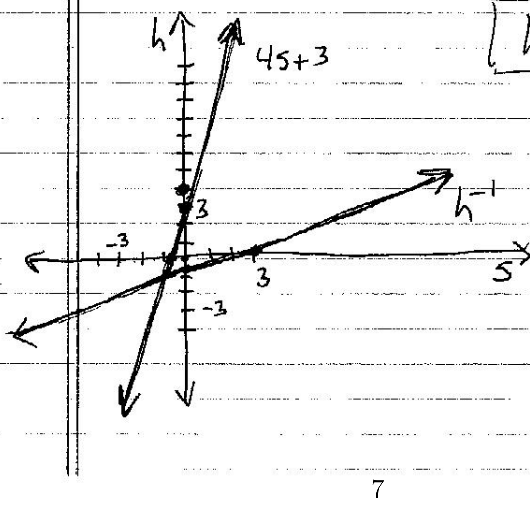
8c. $h(s) = 4s+3$ is strictly monotonic so h^{-1} exists

$y = 4s+3 \longrightarrow s = 4y+3$

$4y = s-3$

$y = \frac{1}{4}(s-3)$

$\boxed{h^{-1}(s) = \frac{1}{4}(s-3)}$



$h(0) = 3$

$4s+3 = 0$

$s = -3/4$

$h^{-1}(0) = -3/4$

$\frac{1}{4}(s-3) = 0$

$s-3 = 0$

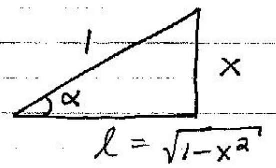
$s = 3$

9. Simplify $\tan(\sin^{-1} x)$

let $\alpha = \sin^{-1} x$ $\qquad -\frac{\pi}{2} \le \alpha \le \pi/2$

Then $\sin \alpha = x$

Consider $0 \le \alpha \le \pi/2$



$$x^2 + l^2 = 1^2$$
$$l^2 = 1 - x^2$$
$$l = \sqrt{1-x^2}$$

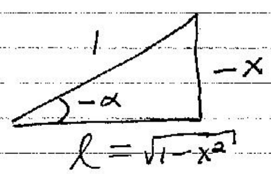
Therefore $\tan(\alpha) = \tan(\sin^{-1} x) = \dfrac{x}{\sqrt{1-x^2}}$

Now consider $-\pi/2 \le \alpha \le 0$

~~Let $\beta = \alpha$~~ $\qquad \cancel{0 \le \beta \le \pi/2}$

$\sin \alpha = x$

$\sin(-\alpha) = -x \qquad 0 \le -\alpha \le \pi/2$



$$(-x)^2 + l^2 = 1^2$$
$$l^2 = 1 - x^2$$
$$l = \sqrt{1-x^2}$$

Therefore $\tan(-\alpha) = \dfrac{-x}{\sqrt{1-x^2}}$

Recall $\tan(-\alpha) = -\tan(\alpha)$

Therefore $\tan(\alpha) = \tan(\sin^{-1} x) = \dfrac{x}{\sqrt{1-x^2}}$

10. a. $\lim_{x \to 2^+} f(x) = 3$

b. $\lim_{x \to 2^-} f(x) = 3$

c. $\lim_{x \to 2} f(x) = 3$ since left and right limits exist and are equal

d. $\lim_{x \to 4^+} f(x) = 4$

e. $\lim_{x \to 4^-} f(x) = 4$

f. $\lim_{x \to 4} f(x) = 4$ since left and right limits exist and are equal

g. $\lim_{x \to 4} f(x) = 4 = f(4)$

So f is continuous at $x = 4$

h. $\lim_{x \to -2^+} f(x) = 1$, $\lim_{x \to -2^-} f(x) = -2$

Therefore the limit does not exist at $x = -2$ and f is not continuous at $x = -2$

i. $\lim_{x \to 0} f(x) = 2 = f(0)$

So f is continuous at $x = 0$

j. f is continuous on $(-2, 8)$

Only suspicious point is $x = 4$, which was checked in (g).

k. f is continuous on $(-2, 4)$

$\lim_{x \to -2^+} f(x) = 1 = f(-2)$ so is right-cont. at $x = -2$

$\lim_{x \to 4^-} f(x) = 4 = f(4)$ so is cont. from left at $x = 4$

Therefore f is continuous on $[-2, 4]$

11. $\lim_{x \to 4} \frac{x^2 - 3x - 4}{x - 4}$

$= \lim_{x \to 4} \frac{(x+1)\cancel{(x-4)}}{\cancel{x-4}} = 5$

[Recall, can cancel $(x-4)$ since $x \neq 4$, just approaches 4]

$\lim_{x \to 0} \frac{\sin 3x}{2x} = \lim_{x \to 0} \left[\frac{3}{2} \cdot \frac{\sin 3x}{3x} \right]$

$= \frac{3}{2} \lim_{x \to 0} \frac{\sin 3x}{3x}$

$= \frac{3}{2}$ since $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\lim_{x \to 0} \frac{\sin^2 x}{x^2} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x}$

$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{1}$

$= 1 \cdot 1 = 1$

$\lim_{x \to 0} \frac{2 \sin 2x}{x \cos 3x} = \lim_{x \to 0} \frac{2}{\cos x} \cdot \frac{2}{1} \cdot \frac{\sin 2x}{2x}$

$= 2 \lim_{x \to 0} \frac{2}{\cos x} \cdot \lim_{x \to 0} \frac{\sin 2x}{2x}$

$= 2 \cdot 2 \cdot 1$

$= 4$

12. $\lim_{x \to 0} x \sin(\frac{1}{x})$

Recall $-|a| \le a \le |a|$

$$-|x \sin(\tfrac{1}{x})| \le x \sin(\tfrac{1}{x}) \le |x \sin(\tfrac{1}{x})|$$

for any $x \ne 0$ $|\sin(\frac{1}{x})| \le 1$

$$-|x| \le -|x||\sin(\tfrac{1}{x})| \le x \sin(\tfrac{1}{x}) \le |x||\sin(\tfrac{1}{x})| \le |x|$$

Therefore $-|x| \le x \sin(\frac{1}{x}) \le |x|$
on an open interval around 0 (but $x \ne 0$).

$$\lim_{x \to 0} -|x| = 0 \quad , \quad \lim_{x \to 0} |x| = 0$$

Therefore by Squeeze Theorem

$$\boxed{\lim_{x \to 0} x \sin(\tfrac{1}{x}) = 0}$$

13. a.) $f(x) = \frac{x-1}{2+\cos x}$

$x-1$ is continuous on $(-\infty,\infty)$
$2+\cos x$ is continuous on $(-\infty,\infty)$
Only suspicious points would be where we might divide by 0:

$2+\cos x = 0$ Can't happen since $-1 \le \cos x \le 1$

So $f(x)$ is continuous on $(-\infty,\infty)$.

b.) $g(x) = \frac{2x-x^2}{x^2-4}$

Rational function, so continuous except possibly at points where we might divide by 0.
Need to check $x = \pm 2$.

$g(2)$ not defined
$g(-2)$ not defined

$g(x)$ is continuous everywhere except $x = \pm 2$
i.e., continuous on $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

14. $f(x) = 2x^3 + x + 1$ has a root on $(-1, 1)$?

f is a polynomial, Therefore continuous on $(-\infty, \infty)$. In particular, it is continuous on The closed interval $[-1, 1]$.

$$f(-1) = -2 - 1 + 1 = -2 < 0$$

$$f(1) = 2 + 1 + 1 = 4 > 0$$

By root location theorem, $f(x)$ has at least one root on $(-1, 1)$.

15. $\cos x - \sin x = x$ has a solution on $(0, \pi/2)$?

Let $f(x) = (\cos x - \sin x) - x$

$f(x) = 0$ when given eqn. has a solution

$f(x)$ is continuous on $[0, \pi/2]$ (cont. on $(-\infty, \infty)$ in fact)

$$f(0) = \cos 0 - \sin 0 - 0 = 1 > 0$$
$$f(\pi/2) = \cos \pi/2 - \sin \pi/2 - \pi/2 = -1 - \pi/2 < 0$$

Therefore by root location Theorem $\cos x - \sin x = x$ has at least one solution on The interval $(0, \pi/2)$.