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On a low-dimensional model for ferromagnetism

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Abstract

In this paper, we present a low-dimensional, energy-based model for ferromagnetic hysteresis. It is based on the postulates of Jiles and Atherton for modeling hysteresis losses. As a state space model, the system is a set of two state equations, with the time-derivative of the average applied magnetic field $\dot{H}$ as the input, and the average magnetic field H and the average magnetization M as state variables. We show analytically that for a class of time-periodic inputs and initial condition at the origin, the solution trajectory converges to a periodic orbit. This models an observed experimental phenomenon.

Keywords: Hysteresis; Ferromagnetism; Low dimensional model; Periodic orbit

1. Introduction

In recent years, there has been a great deal of interest in the area of smart structures largely due to the availability of materials that show giant magnetostrictive, piezoelectric and thermo-elastic responses. This opens up the possibility of building aircraft wings, rotorcraft blades, air inlets and engine nozzles with embedded smart actuators and sensors, so that they can sense environmental or flow-regime changes and respond by changing their structure to optimize performance. The above applications are based on novel materials that show electro–magneto–thermo–visco–elasto–plastic constitutive relationships resulting in complex, rate-dependent hysteretic responses. Thus, modeling and control of their behavior

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is a challenge. We are interested in obtaining low dimensional models for magnetostrictive actuators that show a constitutive coupling in their elastic and magnetic behaviors.

There is a large literature on the modeling of rate-independent hysteresis using the Preisach formalism [17,5,25], with applications to the modeling of magnetic recording media [9]. There are also extensions of the basic Preisach formalism to include rate-dependent hysteresis in some of this literature. In contrast to such non-local memory based models, here we are interested in local-memory models of hysteretic behavior that would permit representation in the form of low dimensional dynamical systems.

In this paper, we study a low-dimensional model for thin ferromagnetic rods that is based in thermodynamics. This model is based on the work of Jiles and Atherton (J–A) [14]. Here we systematically derive the model equations starting from energy balance considerations and the postulates of Jiles and Atherton. We also prove analytically that for a class of periodic inputs that are continuous in time, the unique solution to this strongly non-linear model converges to a periodic orbit. Such orbits represent hysteresis loops. The period of the asymptotic oscillation is the same as that of the input.

After the original J–A model was introduced in 1983 [13], its features including the use of few physically related material parameters and computational efficiency attracted the attention of many researchers. Extensions to model magnetostrictive hysteresis were made by Sablik and Jiles [20], and to a vector ferromagnetic hysteresis model was made by Bergqvist [3]. Jiles himself generalized the model to include minor loop excursions [12]. Chiampi, Chiarabaglio and Repetto [6] used the model along with a fixed point technique to compute the magnetic field and magnetic flux density in a hollow cylinder and validated the results with the analytical solution. A comparison of the J–A and the Preisach models was made by Philips et al. [19]. A more recent comparison of the Preisach and J–A models was undertaken by Benabou et al. [2]. Deane [8] used the J–A model to study the dynamics of an inductor circuit with a ferromagnetic core and validated the results with experiment. The idea of the J–A model that hysteresis is caused by hindrances to domain-wall motion was used to obtain models for ferroelectric, piezoelectric and ferroelastic materials by Smith and Hom [22], Smith and Ounaies [23], and Massad and Smith [16].

2. Bulk ferromagnetic hysteresis theory

In this section, we develop the equations that constitute a model for bulk ferromagnetism i.e. we consider the magnetization to be volume averaged. We first start by discussing Langevin's model of paramagnetism. Next, we discuss the modification of this model by Weiss to explain lossless ferromagnetism. Finally we discuss Jiles and Atherton's postulates regarding hysteresis losses in a lossy ferromagnetic material and show how these postulates together with energy-balance principles yield equations for a model for bulk ferromagnetism.

Consider a collection of N atomic magnetic moments of magnitude m and suppose that they do not interact with each other, and are free to point in any direction. Further suppose that an external magnetic field of magnitude H is applied to this group of free moments. For such a sample, Langevin showed using Boltzmann's statistics that the average magnetic moment of the sample in the direction of the magnetic field is given by [18,7]

$$M_{\text{para}} = M_s \mathscr{L}(z_1), \tag{1}$$

where $\mathscr{L}(z_1) = \coth z_1 - 1/z_1$ is called the Langevin function, and $M_s = Nm$ is the maximum value of the average magnetization when all the moments are aligned together. z is given by

$$z_1 = \frac{mH}{kT}, \tag{2}$$

where T is the absolute temperature and k is Boltzmann's constant. The function $\mathscr{L}(\cdot)$ has the following properties:

1. $\mathscr{L}(\cdot)$ is a strictly increasing function with $-1 < \mathscr{L}(z) < 0$ for $z < 0$; $\mathscr{L}(0) = 0$; and $0 < \mathscr{L}(z) < 1$ for $z > 0$;
2. A very important property regarding the derivative of $\mathscr{L}(\cdot)$ is

$$\max_z \frac{\partial \mathscr{L}}{\partial z}(z) = \frac{\partial \mathscr{L}}{\partial z}(0) = \frac{1}{3}; \tag{3}$$

3. For $z \ll 1$, the Langevin function may be expanded as

$$\mathscr{L}(z) = \frac{z}{3} - \frac{z^3}{45} + \cdots \ .$$

Thus for small values of z we can neglect terms higher than the first one in the above equation and we have

$$M_{\text{para}} \approx \frac{Nm^2}{3kT} H.$$

The above relation is the well-known Curie Law explaining the $1/T$ dependence of the susceptibility of a paramagnetic substance on the temperature. Though Langevin's result fit the experimental observations for paramagnetic materials well, it grossly overestimated the magnetic field value required to saturate ferromagnetic materials. Weiss reasoned that the atomic magnetic moments in a ferromagnetic substance interact strongly with one another and tend to align themselves parallel to each other. The interaction is such as to correspond to an applied field of the order of magnitude of 10^9 A/m for iron [7]. The effect of an externally applied field is merely to change the direction of the spontaneous magnetization. The effect of the interaction of the neighbouring magnetic moments was modeled by Weiss as an additional magnetic field experienced by each moment. Weiss called this additional magnetic field *the molecular field*. By Weiss's postulate, the atomic moments experience an additional field of magnitude αM_{an} in the direction of the magnetic field, where M_{an} is the average magnetic moment of the sample in the direction of the field. The suffix 'an' stands for *anhysteretic* and the reason for this will be seen in a moment.

Repeating the calculations as the paramagnetic case, we get [7]

$$M_{\text{an}}(z) = M_s \mathscr{L}(z) = M_s \left(\coth z - \frac{1}{z} \right) \tag{4}$$

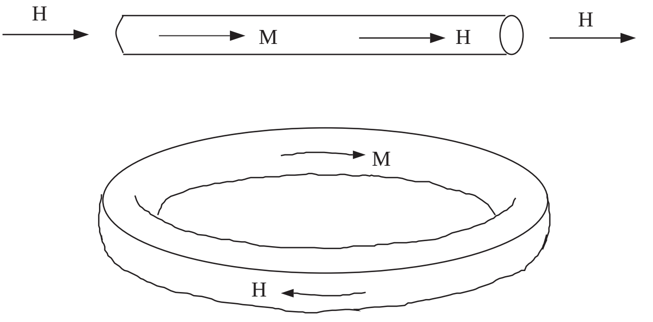

Fig. 1. Geometry of ferromagnetic rods considered.

z in the above equation is given by

$$z = \frac{m(H + \alpha M_{an})}{kT}, \tag{5}$$

where α is the molecular-field parameter. The magnetic field $H + \alpha M_{an}$ is the *effective magnetic field* in the body. Rewriting the above equation, we get

$$M_{an} = \frac{zkT}{m\alpha} - \frac{H}{\alpha}. \tag{6}$$

Then the magnetization M_{an} is given by the simultaneous solution of Eqs. (4) and (6) for a given value of H. The ferromagnetic solid considered was lossless, and hence the same curve in the (H, M)-plane is traced during both the increasing and decreasing branches for a periodic H (Fig. 3). This curve is called the "anhysteretic" curve.

In 1983, Jiles and Atherton [13] proposed a model for bulk ferromagnetic hysteresis. Their aim was to try and reproduce the bulk B–H curves observed in ferromagnetic rods or toroids. The theory was based on a modification of the Weiss molecular field model in which the changes in magnetization due to the motion of domain walls under an applied field were accounted for. In effect, they postulate an expression for the dissipation of energy during domain wall motion. This quantity is a troublesome quantity to calculate from first principles because of the diversity of phenomena that contribute to it and from practical considerations having to do with estimating the number of defects in a particular ferromagnet, etc. The contribution of Jiles and Atherton is to postulate a simple expression to account for the losses. This expression is very similar to the energy losses due to kinetic friction in that it says that the losses associated with magnetization changes for a magnetic body is proportional to the rate of change of magnetization.

Consider a ferromagnetic material that is in the shape of a thin toroid or rod (see Fig. 1). An external source is assumed to produce a uniform magnetic field H along the axis of the body as in Fig. 1. This field H is purely due to the external source (for example, a field generated by a current through a coil connected to a battery) and is not the effective magnetic field in the body. Suppose that the magnetization per unit volume, along the axis of the rod is denoted by M. H and M are scalar quantities denoting the magnitude of the magnetic field and magnetization per unit volume, along the axis of the specimen. A change in the field H brings about a corresponding change in the magnetization of the body in accordance with Maxwell's laws of electromagnetism. The work done by the external source δW_{bat}, is equal to the change in the internal energy of the material δW_{mag} and losses in the magnetization process δL_{mag}:

$$\delta W_{\mathrm{bat}} = \delta W_{\mathrm{mag}} + \delta L_{\mathrm{mag}}. \tag{7}$$

Brown [26] derives the work done by the battery in changing the magnetization per unit volume from M_1 to M_2. In our case, working with the average quantities H and M we get the work done per unit volume to be

$$\delta W_{\mathrm{bat}} = \int_{M_1}^{M_2} \mu_0 H \, \mathrm{d}M,$$

where μ_0 is the magnetic permeability of free space. This is the same as Chikazumi's expression [7]. We consider one "cycle" of the magnetization process as the change in the external magnetic field during a time interval $[0, T]$ so that $H(0)=H(T)$ and $M(0)=M(T)$. This is clearly possible for an *ideal* ferromagnetic material (as the one considered by Weiss) where the magnetization and magnetic field quantities are related through Eqs. (4)–(5). For other ferromagnetic materials, we will show later that it is indeed possible for this to happen. The work done by the battery during one *cycle* of the magnetization process is:

$$\delta W_{\mathrm{bat}} = \oint \mu_0 H \, \mathrm{d}M. \tag{8}$$

The relationship between the above energy expression and the usual expression of the work done can be derived easily.

$$\delta W_{\mathrm{bat}} = \oint \mu_0 H \, \mathrm{d}M = \oint \mu_0 H \, \mathrm{d}M + \oint \mu_0 H \, \mathrm{d}H = \oint H \, \mathrm{d}B,$$

where B is the *magnetic flux density* along the axis in the ferromagnetic body, and is related to H and M by $B=\mu_0(H+M)$. The above expression is not very useful for our purposes. Below, we obtain another equivalent expression for the work done by the battery. As $\oint \mu_0 H \, \mathrm{d}H$ and $\oint \mu_0 M \, \mathrm{d}M$ are loop integrals of exact differentials and hence equal to zero, we have

$$\oint \mu_0 H \, \mathrm{d}M = -\oint \mu_0 M \, \mathrm{d}H = -\oint \mu_0 M \, \mathrm{d}H - \alpha \oint \mu_0 M \, \mathrm{d}M = -\oint M \, \mathrm{d}B_e, \tag{9}$$

where the constant α can take any value and $B_e = \mu_0 H_e = \mu_0(H + \alpha M)$.

Eq. (9) is of interest because, in Weiss's molecular field theory for ideal ferromagnetic rods (no losses), $M \equiv M_{\mathrm{an}}$ is a function of B_e with $\alpha > 0$ the molecular field parameter. For an ideal ferromagnetic rod, M_{an} is given by Eq. (4), so that $M_{\mathrm{an}} = M_s \mathscr{L}(B_e/a)$. Using Eq. (9), we obtain the expression for δW_{mag} from the ideal case:

$$\delta W_{\mathrm{mag}} = -\oint M_{\mathrm{an}} \, \mathrm{d}B_e. \tag{10}$$

Thus for an ideal ferromagnet, δW_{bat} is equal to zero as we would expect it to be. Hence if H is a periodic function of time, then the same curve is traced for both the increasing

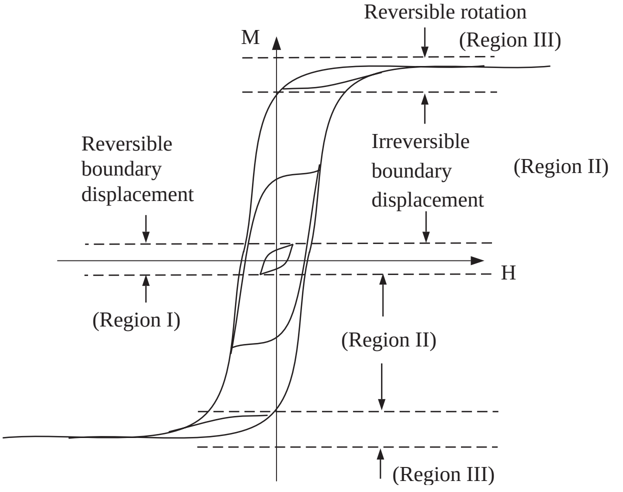

Fig. 2. Phenomenological modeling of hysteresis in ferromagnets.

and decreasing branches in the (H, M)-plane (Fig. 3). This curve is called the anhysteretic curve. One can think of this curve as the characteristic of an ideal ferromagnetic sample with no losses. In the following, we will call the value of M on the anhysteretic curve corresponding to a given value of H as the *anhysteretic magnetization* corresponding to H.

A typical ferromagnetic rod however, has losses. The *magnetization curve* or the M vs H characteristic for a typical ferromagnetic toroidal/thin rod sample is as shown in Fig. 2. Depending on whether the energy dissipated due to hysteresis is large or small, Bozorth [4] and Chikazumi [7] classify different parts of the magnetization curve as irreversible or reversible. For example in Fig. 2, the hysteresis loops in regions I and III tend to be smaller in area enclosed than the loops in region II. Bozorth classifies the three regions by identifying them with the following processes:

1. Reversible rotation of atomic magnetic moments(Region I);
2. Irreversible boundary displacement of domain walls in the rod (Region II);
3. Reversible boundary displacement of domain walls in the rod (Region III).

For a discussion of domain formation in the micromagnetic theory of magnetism please refer to Aharoni [1]. A quantitative model for hysteresis was proposed by Jiles and Atherton in 1983 along the lines of Chikazumi and Bozorth, with some significant differences however. For instance, they consider M to be comprised of an irreversible component M_{irr} and a reversible component M_{rev} so that:

$$M = M_{\mathrm{rev}} + M_{\mathrm{irr}}. \tag{11}$$

This is in contrast to Chikazumi who considers [7]:

$$\frac{\mathrm{d}M}{\mathrm{d}H} = \frac{\mathrm{d}M_{\mathrm{rev}}}{\mathrm{d}H} + \frac{\mathrm{d}M_{\mathrm{irr}}}{\mathrm{d}H}.$$

Next, Jiles and Atherton assume $M_{\rm rev}$ is related to the anhysteretic or ideal magnetization by

$$M_{\rm rev} = c(M_{\rm an} - M_{\rm irr}), \tag{12}$$

where $0 < c < 1$ is a parameter that depends on the material. If $c = 1$, we have $M = M_{\rm an}$. They hypothesize that the energy loss due to the magnetization is only due to $M_{\rm irr}$. We now consider one cycle of the magnetization process as the change in the external magnetic field during a time interval $[0, T]$ so that $H(0)=H(T)$, $M_{\rm rev}(0)=M_{\rm rev}(T)$ and $M_{\rm irr}(0)=M_{\rm irr}(T)$. At this point we will assume that this is possible and we will show later in Section 3 that this is indeed possible. Then Jiles and Atherton postulate that in one cycle the loss due to hysteresis is

$$\delta L_{\rm mag} = \oint k\delta(1-c)\,{\rm d}M_{\rm irr}. \tag{13}$$

In the above equation, k is a non-negative parameter, and δ is defined as

$$\delta \triangleq {\rm sign}(\dot{H}). \tag{14}$$

One can notice that for $k=0$, or $c=1$ we have $\delta L_{\rm mag}=0$. Jiles and Atherton further postulate that: *If the actual magnetization is less than the anhysteretic value and the magnetic field strength H is lowered, then until the value of M becomes equal to the anhysteretic value* $M_{\rm an}$, *the change in magnetization is reversible.*

That is,

$$\frac{{\rm d}M_{\rm irr}}{{\rm d}H} = 0 \quad \text{if} \begin{cases} \dot{H} < 0 & \text{and} \quad M_{\rm an}(H_e) - M(H) > 0, \\ \dot{H} > 0 & \text{and} \quad M_{\rm an}(H_e) - M(H) < 0. \end{cases} \tag{15}$$

As will be seen later, Eqs. (11)–(15) result in a model for magnetization that is numerically well-conditioned for periodic inputs. Without Eq. (15), the incremental susceptibility at the reversal points ${\rm d}M/{\rm d}H$ can become negative. This can be checked by numerical simulations. Ferromagnetic materials are characterized by a positive incremental susceptibility [4]. In fact, it is this feature that distinguishes paramagnetic and ferromagnetic materials from diamagnetic materials (that have negative incremental susceptibility).

By Eqs. (11) and (12) we get

$$M = (1-c)M_{\rm irr} + cM_{\rm an}. \tag{16}$$

Using the notation of Jiles and Atherton, let

$$\delta_M = \begin{cases} 0: \dot{H} < 0 \text{ and } M_{\rm an}(H_e) - M(H) > 0, \\ 0: \dot{H} > 0 \text{ and } M_{\rm an}(H_e) - M(H) < 0, \\ 1: \text{otherwise}. \end{cases} \tag{17}$$

Then by (15) and (16),

$$\frac{{\rm d}M}{{\rm d}H} = \delta_M(1-c)\frac{{\rm d}M_{\rm irr}}{{\rm d}H} + c\frac{{\rm d}M_{\rm an}}{{\rm d}H}. \tag{18}$$

From Eqs. (7)–(10) and (13), we obtain the following energy balance equation for one cycle of the magnetization process:

$$\oint \left(M_{\mathrm{an}} - M - k\delta(1-c)\,\frac{\mathrm{d}M_{\mathrm{irr}}}{\mathrm{d}B_e} \right) \mathrm{d}B_e = 0. \tag{19}$$

The above equation is valid for a cycle of the magnetization process as described earlier. We now make the *hypothesis* that the following equation is valid over any part of the magnetization cycle:

$$\int_{t_1}^{t_2} \left(M_{\mathrm{an}} - M - k\delta(1-c)\,\frac{\mathrm{d}M_{\mathrm{irr}}}{\mathrm{d}B_e} \right) \frac{\mathrm{d}B_e}{\mathrm{d}t}\mathrm{d}t = 0, \tag{20}$$

where $t_1, t_2 \in [0, T]$ with $t_2 > t_1$. We can see that Eq. (20) implies Eq. (19), but not vice versa. If we keep in mind that we are working with a full magnetization cycle, we can continue to work with Eq. (20). As Eq. (20) is valid for any $t_1, t_2 \in [0, T]$, the integrand must be zero:

$$M_{\mathrm{an}} - M - k\delta(1-c)\frac{\mathrm{d}M_{\mathrm{irr}}}{\mathrm{d}B_e} = 0. \tag{21}$$

Using Eqs. (18) and (21) we get after some formal manipulations that

$$\frac{\mathrm{d}M}{\mathrm{d}H} = \frac{\dfrac{k\delta}{\mu_0} c\,\dfrac{\mathrm{d}M_{\mathrm{an}}}{\mathrm{d}H} + \delta_M(M_{\mathrm{an}} - M)}{\dfrac{k\delta}{\mu_0} - \delta_M(M_{\mathrm{an}} - M)\alpha}. \tag{22}$$

Setting $k=0$ yields $\delta_M(M_{\mathrm{an}} - M)\frac{\mathrm{d}M}{\mathrm{d}H} = -\frac{\delta_M(M_{\mathrm{an}}-M)}{\alpha}$. As mentioned before, ferromagnetic materials show positive incremental susceptibility, that is $\mathrm{d}M/\mathrm{d}H > 0$. As $\alpha > 0$, for the above equation to make sense for all values of δ_M we must have

$$M_{\mathrm{an}} - M = 0 \quad \text{or} \quad M = M_{\mathrm{an}}. \tag{23}$$

Setting $c=1$ in Eq. (22) and using Eq. (18) we get (23) (one can also directly use Eq. (21) to see this). Thus $k=0$ or $c=1$ represent the lossless case. On the other hand, if $M_{\mathrm{an}} - M = 0$, then for (21) we must have $k=0$ or $c=1$. Hence for the ferromagnetic hysteresis model,

$$c = 1 \quad \text{or} \quad k = 0 \iff M = M_{\mathrm{an}}. \tag{24}$$

Rewriting Eq. (22) so that we have $\mathrm{d}M_{\mathrm{an}}/\mathrm{d}H_e$ in the numerator on the right-hand side we get

$$\frac{\mathrm{d}M}{\mathrm{d}H} = \frac{\dfrac{k\delta}{\mu_0} c\,\dfrac{\mathrm{d}M_{\mathrm{an}}}{\mathrm{d}H_e} + \delta_M(M_{\mathrm{an}} - M)}{\dfrac{k\delta}{\mu_0} - \delta_M(M_{\mathrm{an}} - M)\alpha - \dfrac{k\delta}{\mu_0}\alpha c\,\dfrac{\mathrm{d}M_{\mathrm{an}}}{\mathrm{d}H_e}}. \tag{25}$$

This equation is different from the one obtained by Jiles and Atherton [14] due to some apparent discrepancies in their derivations. We henceforth refer to it as the *bulk ferromagnetic hysteresis model* so as not to confuse it with the model in [14] that is popularly known

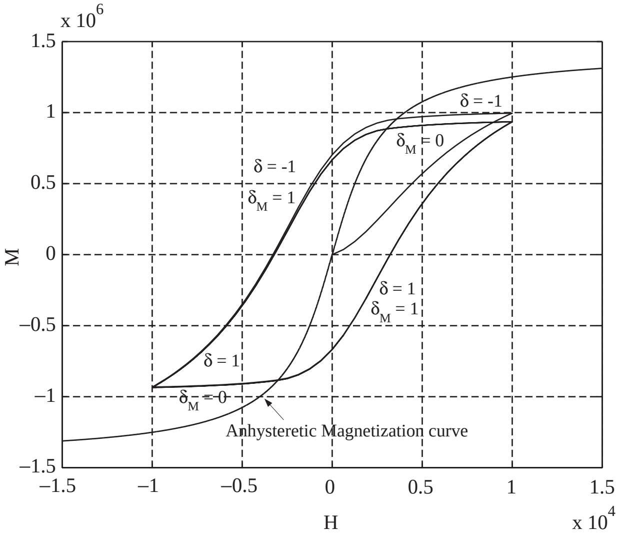

Fig. 3. M vs H relationship for an ideal and a lossy ferromagnet.

as J–A model. A main difference between the two is that for the J–A model, setting $k = 0$ *does not* yield $M = M_{\text{an}}$. For the sake of completeness we write down the other equations satisfied by the system:

$$M_{\text{an}}(H_e) = M_s \left(\coth \left(\frac{H_e}{a} \right) - \frac{a}{H_e} \right), \tag{26}$$

$$H_e = H + \alpha M, \tag{27}$$

$$\delta = \text{sign}(\dot{H}), \tag{28}$$

$$\delta_M = \begin{cases} 0: \dot{H} < 0 \quad \text{and} \quad M_{\text{an}}(H_e) - M(H) > 0, \\ 0: \dot{H} > 0 \quad \text{and} \quad M_{\text{an}}(H_e) - M(H) < 0, \\ 1: \text{otherwise}. \end{cases} \tag{29}$$

Eqs. (25)–(29) constitute the bulk ferromagnetic hysteresis model of this paper. There are 5 non-negative parameters in this model namely a, α, M_s, c, k. Also $0 < c < 1$. Fig. 3 shows the values taken by the discrete variables δ, δ_M at different sections of a representative hysteresis curve in the (H, M)-plane.

Remarks. 1. Note that in Eq. (26), the effective field is given by Eq. (27) and not by $H_e = H + \alpha M_{\text{an}}$ as for the ideal case.

2. The bulk ferromagnetic hysteresis model is *rate-independent* in the following sense. Suppose that $\phi : [0, T] \to [0, T]$ is an monotone-increasing function with $\phi(0) = 0$ and $\phi(T) = T$. Then ϕ can be considered to be a time reparametrization. Here $u \triangleq \dot{H}$ can be considered to be the parameter that is changed by external means (an input) so that the

system takes the form:

$$\dot{H} = u, \tag{30a}$$

$$\dot{M} = \frac{\mathrm{d}M}{\mathrm{d}H} u, \tag{30b}$$

with $\mathrm{d}M/\mathrm{d}H$ given by Eq. (25). Suppose that $u(\cdot)$ is a continuous function during a time interval $[0, T]$, $T > 0$, and $H(\cdot)$ and $M(\cdot)$ are the solutions of Eqs. (30a)–(30b). If the time axis is transformed according to ϕ then it is easy to see that the new solutions are simply $H \circ \phi(\cdot)$ and $M \circ \phi(\cdot)$. Thus the graph on the (H, M)-plane remains the same even if there is a time reparametrization. This property of the bulk ferromagnetic hysteresis model we call rate-independence.

3. Qualitative analysis of the model

The model (25)–(29) was derived by extracting a local law from the balance equation associated to loops in the (H, M) plane. For the model to be of value to an engineer interested in capturing the behavior of a rod of ferromagnetic material in computer simulations (as for instance practised in [6,19] with power applications in mind), it is necessary to demonstrate that it admits well-defined solutions. This is addressed in the existence and uniqueness theory below. Additionally we show that for a range of parameter values and a large class of periodic input signals, the model predicts convergence from the zero state in the (H, M) plane to a periodic solution of the type observed in experiments. These are among the main contributions of this paper.

First we prove an important property. Define state variables, $x_1 = H$, $x_2 = M$. Define

$$z \triangleq \frac{x_1 + \alpha x_2}{a}. \tag{31}$$

Denote $\mathscr{L}(z) = \coth(z) - 1/z$ and $\partial\mathscr{L}/\partial z(z) = -\mathrm{cosech}^2(z) + 1/z^2$. Then the state equations are:

$$\dot{x}_1 = u, \tag{32a}$$

$$\dot{x}_2 = g(x_1, x_2, x_3, x_4)u, \tag{32b}$$

where

$$x_3 = \mathrm{sign}(u), \tag{33a}$$

$$x_4 = \begin{cases} 0: x_3 < 0 \quad \text{and} \quad \coth(z) - \dfrac{1}{z} - \dfrac{x_2}{M_s} > 0, \\ 0: x_3 > 0 \quad \text{and} \quad \coth(z) - \dfrac{1}{z} - \dfrac{x_2}{M_s} < 0, \\ 1: \text{otherwise}, \end{cases} \tag{33b}$$

and

$$g(x_1, x_2, x_3, x_4) = \frac{\dfrac{kx_3}{\mu_0} \dfrac{cM_s}{a} \dfrac{\partial \mathscr{L}}{\partial z}(z) + x_4 M_s \left(\mathscr{L}(z) - \dfrac{x_2}{M_s} \right)}{\dfrac{kx_3}{\mu_0} - x_4 M_s \left(\mathscr{L}(z) - \dfrac{x_2}{M_s} \right) \alpha - \dfrac{kx_3}{\mu_0} \alpha \dfrac{cM_s}{a} \dfrac{\partial \mathscr{L}}{\partial z}(z)}. \tag{34}$$

The system (32a)–(34) has two continuous states: x_1 and x_2. $u(\cdot)$ is the input. x_3 and x_4 are discrete variables that are functions of x_1, x_2, u and time t. Therefore x_3 and x_4 are *not* discrete states. As the function g on the right-hand side of Eq. (32b) depends on x_3 and x_4, it is not continuous as a function of time. Therefore, the notion of solution to the system (32a)–(34) is in the sense of Carathéodory (see Appendix). A Carathéodory solution $(x_1, x_2)(t)$ to (32a)–(34) for t defined on a real interval I, satisfies (32a)–(34) for all $t \in I$ except on a set of Lebesgue measure zero, consisting of points where the right-hand side of (32b) is discontinuous. Note that if $u(t) = 0$ at those times t where $g(\cdot)$ is discontinuous, then one might consider applying the standard existence and uniqueness theorem for ODE's [15]. However we encounter a serious difficulty in the application of this theorem as we have to show that a Lipschitz inequality holds for the vector-field in a compact region that includes the origin in time and the (H, M) plane. Hence we use the notion of Carathéodory solution to Eqs. (32a)–(34), as it allows to show existence and extension of solutions first before considering uniqueness.

Theorem 3.1. *Consider the system of equations* (32a)–(34). *Let the initial condition* $(x_1, x_2)(t = 0) = (x_{1_0}, x_{2_0})$ *be on the anhysteretic curve*:

$$z_0 = \frac{x_{1_0} + \alpha x_{2_0}}{a},$$
$$x_{2_0} = M_s \left(\coth(z_0) - \frac{1}{z_0} \right). \tag{35}$$

Let the parameters satisfy

$$\frac{\alpha M_s}{3a} < 1, \tag{36a}$$

$$0 < c < 1, \tag{36b}$$

$$k > 0. \tag{36c}$$

Let $u(\cdot)$ *be a continuous function of* t, *with* $u(0) = 0$ *and* $u(t) > 0$ *for* $t \in (0, b)$, *where* $b > 0$ *and let* $(x_1(t), x_2(t))$ *denote the solution of* (32a)–(34). *Then* $(M_s \mathscr{L}(z(t)) - x_2(t)) > 0$ $\forall t \in (0, b)$. *If* $u(t) < 0$ *for* $t \in [0, b)$ *where* $b > 0$, *then* $(M_s \mathscr{L}(z(t)) - x_2(t)) < 0$ $\forall t \in (0, b)$.

Proof. We make a change of co-ordinates ψ from (x_1, x_2) to (z, y), where

$$z = \frac{x_1 + \alpha x_2}{a},$$
$$y = M_s \mathscr{L}(z) - x_2.$$

Denote $w = (z, y)$ and $x = (x_1, x_2)$. The domain of definition of the transformation $\psi : x \mapsto w$ is $\mathbb{R}^2$. The Jacobian of the transform is given by

$$\frac{\partial \psi}{\partial x} = \begin{bmatrix} \dfrac{1}{a} & \dfrac{\alpha}{a} \\ \dfrac{M_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z) & \dfrac{M_s \alpha}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z) - 1 \end{bmatrix}.$$

The determinant of $\partial\psi/\partial x$ is

$$\det\left(\frac{\partial \psi}{\partial x}\right) = -\frac{1}{a} \quad \forall x \in \mathbb{R}^2.$$

The results on existence, extension and uniqueness of solutions to the state equations in the transformed space carry over to the equations in the original state space. Denote $\dot{w} = f(t, w)$. The initial conditions in the transformed co-ordinates are

$$w_0 = (z_0, y_0) = \left(\frac{x_{1_0} + \alpha x_{2_0}}{a}, 0\right).$$

The state equations in terms of w are:

$$\dot{z} = f_1(t, w) \triangleq \left(\frac{1 + \alpha \bar{g}(z, y, x_3, x_4)}{a}\right) u, \tag{37a}$$

$$= \frac{\dfrac{1}{a}\dfrac{kx_3}{\mu_0}}{\dfrac{kx_3}{\mu_0} - \alpha\left(x_4 y + \dfrac{kx_3}{\mu_0}\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)\right)} u, \tag{37b}$$

$$\dot{y} = f_2(t, w) \triangleq \left(\frac{M_s}{a}\frac{\partial \mathscr{L}}{\partial z}(z) + \left(\frac{\alpha M_s}{a}\frac{\partial \mathscr{L}}{\partial z}(z) - 1\right)\bar{g}(z, y, x_3, x_4)\right) u, \tag{38a}$$

$$= \frac{\dfrac{M_s}{a}\dfrac{kx_3(1-c)}{\mu_0}\dfrac{\partial \mathscr{L}}{\partial z}(z) - x_4 y}{\dfrac{kx_3}{\mu_0} - \alpha\left(x_4 y + \dfrac{kx_3}{\mu_0}\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)\right)} u. \tag{38b}$$

where

$$x_3 = \mathrm{sign}(u), \tag{39a}$$

$$x_4 = \begin{cases} 0 : x_3 < 0 & \text{and} \quad y > 0, \\ 0 : x_3 > 0 & \text{and} \quad y < 0, \\ 1 : \text{otherwise}, \end{cases} \tag{39b}$$

where

$$\bar{g}(z, y, x_3, x_4) = \frac{\dfrac{kx_3}{\mu_0}\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z) + x_4 y}{\dfrac{kx_3}{\mu_0} - x_4 y \alpha - \dfrac{kx_3}{\mu_0}\alpha\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)}. \tag{40}$$

Let $(t, z, y) \in D = (-\delta_1, b) \times (-\infty, \infty) \times (-\varepsilon_1, \frac{k}{\mu_0}\frac{M_s(1-c)}{3a} + \varepsilon_1)$, where δ_1, ε_1 are sufficiently small positive numbers.

As $u(t)$ is only defined for $t \geqslant 0$, we need to extend the domain of $u(\cdot)$ to $(-\delta_1, 0)$. This can be easily accomplished by defining $u(t) = 0$ for $t \in (-\delta_1, 0)$. Then $f_1(t, w)$, $f_2(t, w)$ exist on D which can be seen as follows.

1. In the time interval $(-\delta_1, 0]$, $u(t) = 0$ by definition. Therefore $x_3 = 0$ by (39a) and $x_4 = 1$ by (39b). This implies that $\bar{g}(z, y, 0, 1) = -1/\alpha$ is well-defined on D. Therefore $f_1(t, w)$ and $f_2(t, w)$ are also well defined.
2. In the time interval $(0, b)$, $u(t) > 0$. Therefore $x_3 = 1$. Hence

$$\bar{g}(z, y, 1, x_4) = \frac{\dfrac{k}{\mu_0}\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z) + x_4 y}{\dfrac{k}{\mu_0} - x_4 y\alpha - \dfrac{k}{\mu_0}\alpha\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)}.$$

We have to ensure that f is well defined $\forall (z, y) \in (-\infty, \infty) \times (-\varepsilon_1, \frac{k}{\mu_0}\frac{M_s(1-c)}{3a} + \varepsilon_1)$.

(a) $x_4 = 0$ implies

$$\bar{g}(z, y, 1, 0) = \frac{\dfrac{k}{\mu_0}\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)}{\dfrac{k}{\mu_0} - \dfrac{k}{\mu_0}\alpha\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)}.$$

By (36a) and (36b), the denominator of $\bar{g}$ is always positive $\forall (z, y) \in (-\infty, \infty) \times (-\varepsilon_1, \frac{k}{\mu_0}\frac{M_s(1-c)}{3a} + \varepsilon_1)$. Hence $f_1(t, w)$ and $f_2(t, w)$ are well-defined.

(b) $x_4 = 1$ implies

$$\bar{g}(z, y, 1, 1) = \frac{\dfrac{k}{\mu_0}\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z) + y}{\dfrac{k}{\mu_0} - y\alpha - \dfrac{k}{\mu_0}\alpha\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)}.$$

By (36a), the denominator of $\bar{g}$ is always positive $\forall (z, y) \in (-\infty, \infty) \times (-\varepsilon_1, \frac{k}{\mu_0}\frac{M_s(1-c)}{3a} + \varepsilon_1)$ if we choose ε_1 small enough. Hence $f_1(t, w)$ and $f_2(t, w)$ are well-defined. □

Existence of a solution. We first show existence of a solution at $t = 0$. To prove existence, we show that $f(\cdot, \cdot)$ satisfies Carathéodory's conditions.

1. We have already seen that $f(\cdot, \cdot)$ is well defined on D. We now check whether $f_1(t, w)$ and $f_2(t, w)$ are continuous functions of w for all $t \in (-\delta_1, b)$.

(a) For $t \in (-\delta_1, 0]$, $f_1(t, w)$, $f_2(t, w)$ are both zero and hence trivially continuous in w.

(b) At $t > 0$, $x_3 = 1$. To check whether $f_1(t, w)$, $f_2(t, w)$ are continuous with respect to w, we only need to check whether $\bar{g}_t(\cdot)$ is continuous as a function of w.

$$\bar{g}_t(w) = \frac{\frac{k}{\mu_0}\frac{cM_s}{a}\frac{\partial \mathscr{L}}{\partial z}(z) + x_4 y}{\frac{k}{\mu_0} - x_4 y\alpha - \frac{k}{\mu_0}\alpha\frac{cM_s}{a}\frac{\partial \mathscr{L}}{\partial z}(z)}.$$

In the above expression, the only term that could possibly be discontinuous as a function of w is

$$h(w) \triangleq x_4 y.$$

By (39b), if $y \geqslant 0$, $x_4 = 1$ and if $y < 0$, $x_4 = 0$ (because $x_3 = 1$). Therefore

$$\lim_{y \to 0+} h(w) = \lim_{y \to 0-} h(w) = 0.$$

Hence, $f(\cdot, \cdot)$ satisfies Carathéodory's first condition for $t \in (-\delta_1, b)$.

2. Next, we need to check whether the function $f(t, w)$ is measurable in t for each w.

(a) For $t \in (-\delta_1, 0]$, $u(t) = 0$. Therefore for each w, $f(\cdot, w)$ is a continuous function of time t trivially.

(b) For $t > 0$, $u(t) > 0$. This implies by (39a) that $x_3 = 1$. Hence for each w, x_4 is also fixed. Therefore for each w

$$f_1(t, w) = K_1(w)u(t),$$
$$f_2(t, w) = K_2(w)u(t),$$

where $K_1(\cdot)$, $K_2(\cdot)$ are functions of w, implying that $f(t, w)$ is a continuous function of t as $u(\cdot)$ is a continuous function of t.

Hence, $f(\cdot, \cdot)$ satisfies Carathéodory's second condition for $t \in (-\delta_1, b)$.

3. For each $t \in (-\delta_1, b)$, $\bar{g}(\cdot)$ is continuous as a function of w. The denominator of $\bar{g}(\cdot)$ is bounded both above and below. The lower bound on the denominator of $\bar{g}(\cdot)$ in D is

$$A = \frac{k}{\mu_0}\left(1 - \frac{\alpha M_s}{3a}\right) - \alpha\varepsilon_1 \tag{41}$$

as $\partial\mathscr{L}/\partial z(z) \leqslant \frac{1}{3}$ (see (3)). Thus for all $(z, y) \in (-\infty, \infty) \times (-\varepsilon_1, \frac{k}{\mu_0}\frac{M_s(1-c)}{3a} + \varepsilon_1)$ we have,

$$|\bar{g}(t, w)| \leqslant \frac{1}{A}\left(\frac{k}{\mu_0}\frac{M_s}{3a} + \varepsilon_1\right).$$

Thus $g(\cdot, \cdot)$ is uniformly bounded in D. By (37a) and (38a), $f(\cdot, \cdot)$ is also uniformly bounded in D. Hence $f(\cdot, \cdot)$ satisfies Carathéodory's third condition for $(t, w) \in D$.

Hence by the Existence Theorem 6.1, for $(t_0, w_0) = (0, (0, 0))$, there exists a solution through (t_0, w_0).

Extension of the solution (We now extend the solution through (t_0, w_0), so that it is defined for all $t \in [0, b)$.). According to the Extension Theorem 6.2, the solution can be extended until it reaches the boundary of D. As $f(t, (z, y))$ is defined $\forall z$, we only need to ensure that $y(t)$ does not reach the boundary of the set $(-\varepsilon_1, \frac{kM_s(1-c)}{3\mu_0 a} + \varepsilon_1]$. We show this by proving that regardless of b, $y(\cdot)$ satisfies $0 \leqslant y(t) \leqslant \frac{kM_s(1-c)}{3\mu_0 a}$ $\forall t \in [0, b)$. This implies that the solution can be extended to the boundary of the time t interval.

1. We know that $y(0) = 0$. We will show that $y(t) > 0$ $\forall t \in (0, b)$. As $\dot{y}(0_+) > 0$, $\exists\, b_1 > 0 \ni y(t) > 0$ $\forall t \in (0, b_1)$. If this were not true then we could form a sequence of time instants $t_k \to 0$, with $t_k > 0 \ni y(t_k) \leqslant 0$ for k sufficiently large. Then

$$\lim_{t_k \to 0} \frac{y(t_k) - y(0)}{t_k - 0} = \lim_{t_k \to 0} \frac{y(t_k) - 0}{t_k} \leqslant 0$$

   which contradicts $\dot{y}(0_+) > 0$.
   Let b_1 denote the maximal time instant such that $y(t) > 0$ $\forall t \in (0, b_1)$. Suppose $b_1 < b$. Then $y(b_1) = 0$ by continuity of $y(\cdot)$. At $t = b_1$, $x_3 = 1$ by (39a) and $x_4 = 1$ by (39b). Therefore,

$$\dot{y}(b_1) = \left(\frac{M_s}{a} \frac{\partial \mathscr{L}}{\partial z}(z) - \frac{1 - \dfrac{\alpha M_s}{a} \dfrac{\partial \mathscr{L}}{\partial z}(z)}{\left(1 - \alpha \dfrac{cM_s}{a} \dfrac{\partial \mathscr{L}}{\partial z}(z)\right)} \frac{cM_s}{a} \frac{\partial \mathscr{L}}{\partial z}(z) \right) u(b_1).$$

   By (36a)–(36b) and (3),

$$\frac{1 - \dfrac{\alpha M_s}{a} \dfrac{\partial \mathscr{L}}{\partial z}(z)}{1 - c\dfrac{\alpha M_s}{a} \dfrac{\partial \mathscr{L}}{\partial z}(z)} < 1. \tag{42}$$

   By (42)

$$\begin{aligned} \dot{y}(b_1) &> \left(\frac{M_s}{a} \frac{\partial \mathscr{L}}{\partial z}(z) - \frac{cM_s}{a} \frac{\partial \mathscr{L}}{\partial z}(z) \right) u(b_1), \\ &= \frac{M_s}{a} \frac{\partial \mathscr{L}}{\partial z}(z)(1-c)u(b_1) \quad > 0 \quad \text{by (36b)}. \end{aligned}$$

   Therefore for some $\varepsilon > 0$ sufficiently small (with $\varepsilon < b_1$),

$$\begin{aligned} y(b_1 - \varepsilon) &= y(b_1) - \varepsilon \dot{y}(b_1) + o(\varepsilon^2) \\ &= 0 - \varepsilon \dot{y}(b_1) + o(\varepsilon^2) < 0, \end{aligned}$$

   which is a contradiction of the fact that $y(t) > 0$ $\forall t \in (0, b_1)$.
   Hence $y(t) > 0$ $\forall t \in (0, b)$.
2. We now verify that $y(t) \leqslant k/\mu_0 M_s(1-c)/3a$.
   As $u(t) > 0$ for $t \in (0, b)$, $x_3(t) = 1$ by (39a). We proved that $y(t) > 0$ for $t \in (0, b)$ implying that $x_4(t) = 1$. By expanding the right-hand sides of (37a) and (38a) with $x_3 = 1$

and $x_4 = 1$, we get

$$\dot{z}(t) = \frac{\frac{1}{a}\frac{k}{\mu_0}}{\frac{k}{\mu_0} - \alpha y - \frac{k}{\mu_0}\alpha \frac{cM_s}{a}\frac{\partial \mathscr{L}}{\partial z}(z)} u(t), \tag{43}$$

$$\dot{y}(t) = \frac{\frac{k}{\mu_0}\frac{(1-c)M_s}{a}\frac{\partial \mathscr{L}}{\partial z}(z) - y}{\frac{k}{\mu_0} - \alpha y - \frac{k}{\mu_0}\alpha \frac{cM_s}{a}\frac{\partial \mathscr{L}}{\partial z}(z)} u(t). \tag{44}$$

All possible relative maximum values of $y(\cdot)$ ($\triangleq y^k_{\max}$) occur for $t = t_k \in (0, b)$ such that $\dot{y}(t_k) = 0$. Denote the corresponding values of z by $z^k_{y_{\max}}$. By (44) and (3), we have for these values of t_k,

$$y^k_{\max} = \frac{k(1-c)}{\mu_0}\frac{M_s}{a}\frac{\partial \mathscr{L}}{\partial z}(z^k_{y_{\max}}) \leqslant \frac{k(1-c)}{\mu_0}\frac{M_s}{3a}. \tag{45}$$

Therefore the solution can be extended in time to the boundary of $[0, b)$. In the course of continuing the solutions, we also proved that $(M_s\mathscr{L}(z(t)) - x_2(t)) > 0\ \forall t \in (0, b)$.

Uniqueness. As $u(t) > 0$ for $t \geqslant 0$, $x_3 = 1$. As $y > 0$ for $t > 0$, $x_4 = 1$ for $t > 0$. We concentrate on this case below. At $t = 0$, $x_4 = 0$ and the Lipschitz constants obtained in the following analysis can again be used to show uniqueness.

Using A defined by (41), we can obtain a lower bound for the denominator of $f_1(t, w)$. With $w_1 = (z_1, y_1)$ and $w_2 = (z_2, y_2)$, we have

$$|f_1(t, w_1) - f_1(t, w_2)| \leqslant \frac{\frac{1}{a}\frac{k}{\mu_0}}{A^2}\left(\frac{k}{\mu_0}\frac{\alpha c M_s}{a}\left|\frac{\partial \mathscr{L}}{\partial z}(z_1) - \frac{\partial \mathscr{L}}{\partial z}(z_2)\right| + \alpha|y_1 - y_2|\right) u(t). \tag{46}$$

As $\partial\mathscr{L}/\partial z(z)$ is a smooth function of z, $\exists$ a non-negative constant $K \ni$ [15]

$$\left|\frac{\partial \mathscr{L}}{\partial z}(z_1) - \frac{\partial \mathscr{L}}{\partial z}(z_2)\right| \leqslant K|z_1 - z_2| \quad \forall z_1, z_2 \in (-\infty, \infty).$$

Hence

$$\begin{aligned} |f_1(t, w_1) - f_1(t, w_2)| &\leqslant \frac{\frac{1}{a}\frac{k}{\mu_0}}{A^2}\left(\frac{k}{\mu_0}\frac{\alpha c M_s}{a} K|z_1 - z_2| + \alpha|y_1 - y_2|\right) u(t) \\ &\leqslant \frac{\frac{1}{a}\frac{k}{\mu_0}}{A^2}\left(\frac{k}{\mu_0}\frac{c\alpha M_s}{a} K + \alpha\right) \|w_1 - w_2\| u(t). \end{aligned} \tag{47}$$

Now

$$|f_2(t,w_1)-f_2(t,w_2)| \leqslant \frac{u(t)}{A^2}\left(\left(\frac{k}{\mu_0}\right)^2 \frac{(1-c)M_s}{a}\left|\frac{\partial \mathscr{L}}{\partial z}(z_1)-\frac{\partial \mathscr{L}}{\partial z}(z_2)\right| + \frac{k}{\mu_0}|y_1-y_2| + \frac{k}{\mu_0}\frac{\alpha M_s}{a}\left|y_1\frac{\partial \mathscr{L}}{\partial z}(z_2)-y_2\frac{\partial \mathscr{L}}{\partial z}(z_1)\right|\right). \tag{48}$$

We can rewrite the last term with

$$y_1\frac{\partial \mathscr{L}}{\partial z}(z_2)-y_2\frac{\partial \mathscr{L}}{\partial z}(z_1)=y_1\left(\frac{\partial \mathscr{L}}{\partial z}(z_2)-\frac{\partial \mathscr{L}}{\partial z}(z_1)\right)+(y_1-y_2)\frac{\partial \mathscr{L}}{\partial z}(z_1).$$

Then Inequality (48) becomes

$$|f_2(t,w_1)-f_2(t,w_2)| \leqslant \frac{u(t)}{A^2}\left(\left(\frac{k}{\mu_0}\right)^2 \frac{(1-c)M_s}{a}K|z_1-z_2| + \frac{k}{\mu_0}|y_1-y_2| + \frac{k}{\mu_0}\frac{\alpha M_s}{a}\left(|y_1|K|z_1-z_2|+\left|\frac{\partial \mathscr{L}}{\partial z}(z_1)\right||y_1-y_2|\right)\right).$$

As $|y_1| \leqslant k(1-c)/\mu_0\, M_s/3a+\varepsilon_1$ and $\partial\mathscr{L}/\partial z(z_1) \leqslant \frac{1}{3}$ for all $(t,z_1,y_1)\in D$,

$$\begin{aligned}
&|f_2(t,w_1)-f_2(t,w_2)| \\
&\quad \leqslant \frac{u(t)}{A^2}\left[\left(\left(\frac{k}{\mu_0}\right)^2\frac{(1-c)M_s}{a}K+\frac{k}{\mu_0}\frac{\alpha M_s}{a}K\left(\frac{k(1-c)}{\mu_0}\frac{M_s}{3a}+\varepsilon_1\right)\right)|z_1-z_2| \right.\\
&\quad \left. +\frac{k}{\mu_0}\left(1+\frac{\alpha M_s}{3a}\right)|y_1-y_2|\right]\\
&\quad \leqslant \frac{u(t)}{A^2}\frac{k}{\mu_0}\left[\frac{k}{\mu_0}\frac{(1-c)M_s}{a}K+\frac{\alpha M_s}{a}\frac{k(1-c)}{\mu_0}\frac{M_s}{3a}K+\frac{\alpha M_s}{a}K\varepsilon_1\right.\\
&\quad \left. +1+\frac{\alpha M_s}{3a}\right]\|w_1-w_2\|.
\end{aligned} \tag{49}$$

By (47) and (49)

$$\|f(t,w_1)-f(t,w_2)\| \leqslant B\|w_1-w_2\|u(t), \tag{50}$$

where B is some positive constant. Hence there exists atmost one solution in D by Theorem 6.3.

For inputs $u(\cdot)$ with $u(t)<0$ for $t\in(0,b)$, the same proof can be repeated to arrive at the conclusion that $(M_s\mathscr{L}(z(t))-x_2(t))<0\ \forall t\in(0,b)$.

This completes the proof of Theorem 3.1. □

The following corollary continues the ideas contained in Theorem 3.1.

Corollary 3.1. *Consider the system of equations* (32a)–(34). *Let the initial condition* $(x_1,x_2)(t=0)=(x_{1_0},x_{2_0})$ *be on the anhysteretic curve* (*see* (35)). *Suppose the parameters satisfy* (36a)–(36c). *If* $u(t)>\varepsilon>0$ *for* $t\in(0,b)$ *then as* $b\to\infty$, $x_2(t)\to M_s$.

Proof. We again perform a change of co-ordinates $(x_1, x_2) \mapsto (z, y)$. By (37a)

$$\dot{z}(t) = \frac{1 + \alpha\bar{g}(z, y, x_3, x_4)}{a} u.$$

As in the proof of Theorem 3.1, $x_3 = 1$ and $x_4 = 1$ for all $t \in (0, b)$.With D defined as in the proof of Theorem 3.1, one can again repeat the arguments made earlier, to show that $\bar{g}(z, y, 1, 1) > 0$ for all $(z, y) \in D$. One can then make the conclusion that:

$$\dot{z}(t) > \frac{1}{a} u(t) > \frac{\varepsilon}{a}, \tag{51}$$

where $\bar{g}(z, y, x_3, x_4)$ is given by (40) and x_3, x_4 are defined by (39a) and (39b), respectively. Inequality (51) shows that $z(\cdot) \to \infty$ as $b \to \infty$. Hence it is sufficient to study the behavior of y as a function of z. Using Eqs. (43) and (44), we can obtain a differential equation for the evolution of y as a function of z:

$$y + \frac{1}{a} \frac{k}{\mu_0} \frac{\mathrm{d}y}{\mathrm{d}z} = \frac{k(1-c)M_s}{\mu_0 a} \frac{\partial \mathscr{L}}{\partial z}(z). \tag{52}$$

The initial condition for the above differential equation is $y(z = z_0) = 0$. Define

$$v(z) \triangleq \frac{k(1-c)M_s}{\mu_0 a} \frac{\partial \mathscr{L}}{\partial z}(z).$$

Clearly $v(z) > 0\ \forall z$. Employing Laplace transforms, we have

$$Y(s) = \frac{V(s)}{\frac{k}{a\mu_0} s + 1},$$

where the Laplace transform of $v(z)$, $y(z)$ are denoted as $V(s)$ and $Y(s)$, respectively. $V(s)$ exists for all $s \in \mathbb{C}$ because by definition of the Laplace transform

$$V(s) = \int_0^\infty v(z) \exp(-zs)\, \mathrm{d}z,$$

and $v(z)$ is an integrable function of z. By the Final-value theorem for Laplace transforms [21], $\lim_{z\to\infty} y(z) = \lim_{s\to 0} sY(s)$.

Therefore

$$\lim_{z\to\infty} y(z) = \lim_{s\to 0} \frac{sV(s)}{\frac{k}{a\mu_0} s + 1}.$$

Now (by another application of the Final-value theorem for Laplace transforms)

$$\begin{aligned} \lim_{s\to 0} sV(s) &= \lim_{z\to\infty} v(z) \\ &= \lim_{z\to\infty} \frac{k(1-c)}{\mu_0} \frac{\mathrm{d}\mathscr{L}}{\mathrm{d}z} = 0. \end{aligned} \tag{53}$$

Hence, $\lim_{z\to\infty} y(z) = 0$.

We conclude that $x_2(t) \to M_s$ as $t \to \infty$. □

There are some additional remarks that one can make from the proof of the corollary. Firstly, convergence of $y(\cdot)$ to zero is faster for smaller values of the ratio $k/a\mu_0$. Secondly, critical points of $y(\cdot)$ are obtained by setting $\dot{y}(t_{\text{cr}}) = 0$. Thus critical values of $y(\cdot)$ must satisfy:

$$y(t_{\text{cr}}) = \frac{k(1-c)M_s}{\mu_0 a}\frac{\partial \mathscr{L}}{\partial z}(z).$$

At $t = 0$, $\dot{y} > 0$ and so $y(\cdot)$ is increasing function initially. As $t \to \infty$, by Corollary 3.1 we have, $y \to 0$. In order to understand the behavior of $y(\cdot)$ at its critical points, we need to compute $\ddot{y}$ at the critical points. However, as $\ddot{y}$ involves the input signal u and its derivatives, it is more instructive to study $\mathrm{d}^2y/\mathrm{d}z^2$ at the critical points $z_{\text{cr}} = z(t_{\text{cr}})$. Note that at $t = t_{\text{cr}}$, $\mathrm{d}y/\mathrm{d}z = 0$ by Eq. (52). Using Eq. (52) we get

$$\frac{1}{a}\frac{k}{\mu_0}\frac{\mathrm{d}^2 y}{\mathrm{d}z^2}(z_{\text{cr}}) = \frac{k(1-c)M_s}{\mu_0 a}\frac{\mathrm{d}^2\mathscr{L}}{\mathrm{d}z^2}(z_{\text{cr}}) < 0 \quad \text{if } z_{\text{cr}} > 0, \text{ and } > 0 \quad \text{if } z_{\text{cr}} < 0.$$

Therefore the critical points *cannot* be maxima if $z_{\text{cr}} < 0$. We also have the condition that $\dot{z} > 0$. So if the initial condition satisfies $y(0) = 0$ and $z_0 > 0$ (that is, the initial condition is on the anhysteretic curve in the first quadrant of the (x_1, x_2) plane), then there can be atmost one maximum for $y(\cdot)$. If the initial condition satisfies $y(0) = 0$ and $z_0 < 0$ (that is, the initial condition is on the anhysteretic curve in the third quadrant of the (x_1, x_2) plane), then there cannot be any maxima for $y(\cdot)$ until the solution trajectory stays in the third quadrant. The above statements have to be appropriately changed if the input satisfies $u < 0$ instead of $u > 0$.

Next, we show a simple consequence of Theorem 3.1, that if the initial condition is on the positive x_1 axis with $u(t) > 0$ then we still have existence and uniqueness of solutions, and the conclusions of Corollary 3.1 also hold. This result is used in part two of this paper while analyzing the well-posedness of the magnetostriction model.

Corollary 3.2. *Consider the system of equations* (32a)–(34). *Suppose that the initial condition is* $(x_1, x_2)(t=0) = (x_{1_0}, 0)$ *where* $x_{1_0} > 0$, *and that the parameters satisfy* (36a)–(36c). *Then the following hold*:

- *let the input* $u(\cdot)$ *be a continuous function of* t *with* $u(t) > 0$ *for* $t \in (-\varepsilon, b)$, *where* $b > 0$ *and* $\varepsilon > 0$ *be a sufficiently small positive number. Let* $(x_1(t), x_2(t))$ *denote the solution of* (32a)–(34). *Then* $y(t) = (M_s\mathscr{L}(z(t)) - x_2(t)) > 0\ \forall t \in [0, b)$. *Else if* $u(t) < 0$ *for* $t \in (-\varepsilon, b)$ *where* $b > 0$, *then* $y(t) = (M_s\mathscr{L}(z(t)) - x_2(t)) < 0\ \forall t \in [0, b)$.
- *if* $u(t) > \varepsilon > 0$ *for* $t \in (0, b)$ *then as* $b \to \infty$, $x_2(t) \to M_s$.

Proof. We can choose the domain D as in Theorem 3.1 in order to show the existence and uniqueness of the solution. Proceeding exactly as in Theorem 3.1, we obtain the first assertion. Similarly, proceeding exactly as in Corollary 3.1 we obtain the second assertion. □

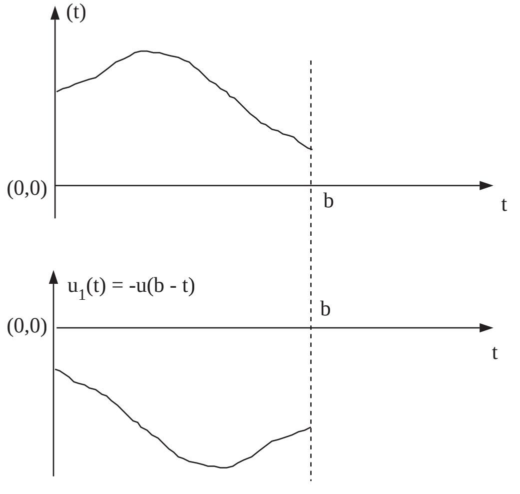

Fig. 4. Sample signals $u(\cdot)$ and $u_1(\cdot)$.

Next, suppose that an input $u(t) > 0$ for $t \in [0, b)$ has been applied to system (32a)–(34) with initial condition as in Theorem 3.1. Let

$$x_0 = (x_{1_0}, x_{2_0}) = \lim_{t \to b} (x_1, x_2)(t). \tag{54}$$

x_0 is well-defined because of the Extension Theorem 6.2. Define the set $\mathcal{O}_1$ as

$$\mathcal{O}_1 \triangleq \bigcup_{t \in (0,b)} x(t), \tag{55}$$

where $x(\cdot)$ is the solution of (32a)–(34). We now ask the following question: if the input to the system is reversed, then do we reach the origin (that was our starting point). The answer is no, as we shall show below. For this purpose, define (see Fig. 4)

$$u(b) = \lim_{t \to b} u(t), \tag{56}$$

$$u_1(t) = -u(b - t) \quad \text{for } t \in [0, b]. \tag{57}$$

Let the initial condition be x_0 as defined in (54). In the next theorem, we show that there exists a time $0 < b_1 < b$ such that $x_2(b_1) = M_s \mathscr{L}(\frac{x_1(b_1) + \alpha x_2(b_1)}{a})$. In other words, the solution trajectory intersects with the anhysteretic curve in the (x_1, x_2)-plane at time $b_1 < b$. The proof also shows that the solution obtained after reversing the input does not belong to the original solution set $\mathcal{O}_1$.

Theorem 3.2. *Consider the system of equations* (32a)–(34). *Let the initial condition* $(x_1, x_2)(t=0) = (x_{1_0}, x_{2_0})$ *where* (x_{1_0}, x_{2_0}) *is defined by* (54). *Let the parameters satisfy* (36a)–(36c). *Let* $u(t)$ *and* $u_1(t)$ *be defined by* (56)–(57). *If* $u_1(t)$ *is the input to the system* (32a)–(34) *for* $t \in [0, b]$, *then* $\exists\, b_1 > 0$ *such that* $b_1 < b$ *and* $x_2(b_1) = M_s \mathscr{L}(\frac{x_1(b_1) + \alpha x_2(b_1)}{a})$.

Proof. This proof of existence of a solution mimics that of Theorem 3.1, but we include it for completeness. However, extension of solutions and uniqueness have to be re-done.

As before, we make a change of co-ordinates from (x_1, x_2) to (z, y) where

$$z = \frac{x_1 + \alpha x_2}{a},$$
$$y = M_s \mathscr{L}(z) - x_2.$$

The Jacobian of this transform is non-singular $\forall (x_1, x_2) \in \mathbb{R}^2$ and hence the results on existence, extension and uniqueness of solutions to the state equations in the transformed space are applicable to the equations in the original state space. The state equations $\dot{w} = f(t, w)$ in terms of $w = (z, y)$ are given by (37a)–(39b), with $u_1(\cdot)$ as the input function instead of $u(\cdot)$ in Eqs. (43) and (44). The initial conditions in the transformed co-ordinates are

$$w_0 = (z_0, y_0) = \left(\frac{x_{1_0} + \alpha x_{2_0}}{a}, M_s \mathscr{L}(z_0) - x_{2_0} \right).$$

Let

$$D = \underbrace{(-\delta_1, b + \delta_1)}_{t} \times \underbrace{(-\infty, \infty)}_{z} \times \underbrace{\left(0, \frac{k}{\mu_0} \frac{M_s(1-c)}{3a} + \varepsilon_1\right)}_{y},$$

where δ_1, ε_1 are sufficiently small positive numbers.

We have to re-define $u_1(\cdot)$ so that it is well-defined over its domain $(-\delta_1, b + \delta_1)$. This can be easily accomplished by defining $u_1(t) = 0$ for $t \in (-\delta_1, 0) \cup (b, b + \delta_1)$. Then $f_1(t, w)$, $f_2(t, w)$ exist on D which can be seen as follows.

1. In the time interval $(-\delta_1, 0) \cup (b, b+\delta_1)$, $u_1(t) = 0$ by definition. Therefore $x_3 = 0$ by (39a) and $x_4 = 1$ by (39b). This implies that $\bar{g}(z, y, 0, 1) = -y/y$. Defining $\bar{g}(z, 0, 0, 1) = -1$ makes $\bar{g}(z, y, 0, 1)$ continuous as a function of y. This also makes $f_1(t, w)$ and $f_2(t, w)$ well defined.
2. In the time interval $[0, b]$, $u_1(t) < 0$. Therefore $x_3 = -1$. Hence

$$\bar{g}(z, y, -1, x_4) = \frac{\frac{k}{\mu_0} \frac{cM_s}{a} \frac{\partial \mathscr{L}}{\partial z}(z) - x_4 y}{\frac{k}{\mu_0} + x_4 y \alpha - \frac{k}{\mu_0} \alpha \frac{cM_s}{a} \frac{\partial \mathscr{L}}{\partial z}(z)}.$$

We have to ensure that f is well defined $\forall (z, y) \in (-\infty, \infty) \times (0, \frac{k}{\mu_0} \frac{M_s(1-c)}{3a} + \varepsilon_1)$.

(a) $x_4 = 0$ implies

$$\bar{g}(z, y, -1, 0) = \frac{\frac{k}{\mu_0} \frac{cM_s}{a} \frac{\partial \mathscr{L}}{\partial z}(z)}{\frac{k}{\mu_0} - \frac{k}{\mu_0} \alpha \frac{cM_s}{a} \frac{\partial \mathscr{L}}{\partial z}(z)}.$$

By (3), (36a) and (36b), the denominator of $\bar{g}$ is always positive $\forall (z, y) \in (-\infty, \infty) \times (0, \frac{k}{\mu_0}\frac{M_s(1-c)}{3a})$. Hence $f_1(t, w)$ and $f_2(t, w)$ are well-defined.

(b) $x_4 = 1$ implies

$$\bar{g}(z, y, -1, 1) = \frac{\dfrac{k}{\mu_0}\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z) - y}{\dfrac{k}{\mu_0} + y\alpha - \dfrac{k}{\mu_0}\alpha\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z(Z)}}.$$

Again, by (3), (36a) and (36b), the denominator of $\bar{g}$ is always positive $\forall (z, y) \in (-\infty, \infty) \times (0, \frac{k}{\mu_0}\frac{M_s(1-c)}{3a} + \varepsilon_1)$. Hence $f_1(t, w)$ and $f_2(t, w)$ are well-defined.

Existence of a solution. We first show existence of a solution at $t = 0$. As in Theorem 3.1, to prove existence, we show that $f(\cdot, \cdot)$ satisfies Carathéodory's conditions.

1. We have already seen that $f(\cdot, \cdot)$ is well defined on D. We now check whether $f_1(t, w)$ and $f_2(t, w)$ are continuous functions of w for all $t \in (-\delta_1, b + \delta_1)$.

   (a) For $t \in (-\delta_1, 0) \cup (b, b + \delta_1)$, $f_1(t, w)$, $f_2(t, w)$ are both zero and hence trivially continuous in w.

   (b) At $t \in [0, b]$, $x_3 = -1$. To check whether $f_1(t, w)$, $f_2(t, w)$ are continuous with respect to w, we only need to check whether $\bar{g}_t(\cdot)$ is continuous as a function of w, where the subscript t denotes the fact that the t variable is being held fixed.

$$\bar{g}_t(w, -1, x_4) = \frac{\dfrac{k}{\mu_0}\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z) - x_4 y}{\dfrac{k}{\mu_0} + x_4 y\alpha - \dfrac{k}{\mu_0}\alpha\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)}.$$

   In the above expression, the only term that could possibly be discontinuous as a function of w is

$$h(w) \triangleq x_4 y.$$

   By (39b), if $y \leqslant 0$, $x_4 = 1$ and if $y > 0$, $x_4 = 0$ (because $x_3 = -1$). Therefore

$$\lim_{y \to 0+} h(w) = \lim_{y \to 0-} h(w) = 0.$$

   Hence, $f(\cdot, \cdot)$ satisfies Carathéodory's first condition for $t \in (-\delta_1, b + \delta_1)$.

2. Next, we need to check whether the function $f(t, w)$ is measurable in t for each w.

   (a) For $t \in (-\delta_1, 0) \cup (b, b + \delta_1)$, $u_1(t) = 0$. Therefore for each w, $f(\cdot, w)$ is a continuous function of time t trivially.

   (b) For $t \in [0, b]$, $u_1(t) < 0$. This implies by (39a) that $x_3 = -1$. Hence for each w, x_4 is also fixed. Therefore for each w

$$f_1(t, w) = L_1(w)u_1(t),$$
$$f_2(t, w) = L_2(w)u_1(t),$$

where $L_1(\cdot)$, $L_2(\cdot)$ are only functions of w. This implies that $f(t, w)$ is a continuous function of t.

Hence, $f(\cdot, \cdot)$ satisfies Carathéodory's second condition for $t \in (-\delta_1, b + \delta_1)$.

3. For each $t \in (-\delta_1, b + \delta_1)$, $\bar{g}(\cdot)$ is continuous as a function of w. The denominator of $\bar{g}(\cdot)$ is bounded both above and below. The lower bound on $\bar{g}(\cdot)$ in D is

$$A = \frac{k}{\mu_0}\left(1 - \frac{c\alpha M_s}{3a}\right).$$

For all $(z, y) \in (-\infty, \infty) \times (0, \frac{k}{\mu_0}\frac{M_s(1-c)}{3a} + \varepsilon_1)$; $\partial\mathscr{L}/\partial(z) \leqslant \frac{1}{3}$ implying

$$|\bar{g}(t, w)| \leqslant \frac{1}{A}\left(\frac{k}{\mu_0}\ \frac{M_s}{3a}\right) \sup_{t\in(-\delta_1,b)} |u_1(t)|.$$

Thus $g(\cdot, \cdot)$ is uniformly bounded in D. By (37a) and (38a), $f(\cdot, \cdot)$ is also uniformly bounded in D. Hence $f(\cdot, \cdot)$ satisfies Carathéodory's third condition for $(t, w) \in D$.

Hence by the Existence Theorem 6.1, for $(t_0, w_0) = (0, (z_0, y_0)$, there exists a solution through $(t_0, w_0))$.

Extension of the solution (We now extend the solution through (t_0, w_0), so that it is defined for all $t \in [0, b + \delta_1))$. According to the Extension Theorem 6.2, the solution can be extended until it reaches the boundary of D. It obviously cannot reach the boundary of D in the z variable. We show that the solution reaches the boundary of D in the y variable.

As $y(0) > 0$ (owing to the choice of $y(0)$ as explained before the statement of this theorem and the conclusion of Theorem 3.1), there exists a time $\tau > 0$ such that $y(t) > 0\ \forall\ t \in [0, \tau)$. Suppose such a τ does not exist. Then we can choose a sequence $t_k \to 0_+$ with $y(t_k) \leqslant 0$ for k large enough, implying that $y(0) \leqslant 0$ (by continuity of $(z, y)(\cdot)$ at $t = 0$) which is a contradiction. Define

$$b_1 = \sup\{\tau | y(\tau) > 0 \text{ and } \tau \leqslant b\}. \tag{58}$$

Now one of two cases is possible:

- $b_1 < b$. This implies that at $t = b_1$, $y(b_1) = 0$. If this is not true and $y(b_1) > 0$, then we can choose $\varepsilon > 0$ sufficiently small such that $y(b_1 + \varepsilon) > 0$ contradicting (58).
- $b_1 = b$. We show that this is not possible.
  If $b_1 = b$ then clearly the solution can be extended to $[0, b)$. As the map $\psi : (x_1, x_2) \mapsto (z, y)$ is a diffeomorphism, we consider the behavior of the solution in terms of the variables $x = (x_1, x_2)$ for simplicity of analysis. Define the set $\mathscr{O}_2$ as

$$\mathscr{O}_2 = \bigcup_{t\in(0,b)} x(t).$$

Then we can make the following observations.

1. At time $t = b$

$$x_1(t = b) = 0. \tag{59}$$

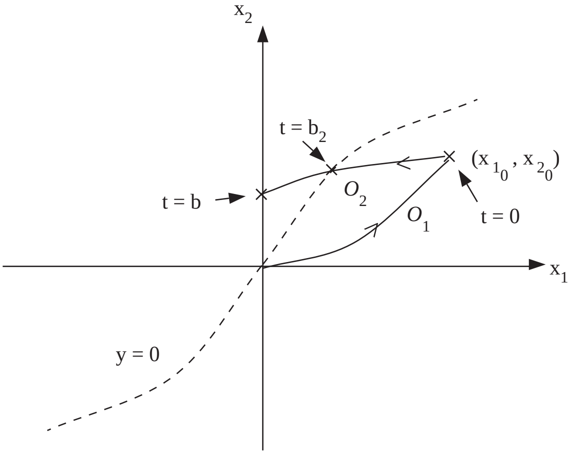

Fig. 5. Figure for the proof of Theorem 3.2.

This is because, the differential equation for x_1 is $\dot{x_1}(t) = u_1(t)$. As $u_1(t) = -u(t-b)$ we reach the starting condition for Theorem 3.1 at time $t = b$ (which was $x_1 = 0$).

2. The slopes of the curves $\mathcal{O}_1$ and $\mathcal{O}_2$ in the (x_1, x_2)-plane are always positive (refer to Fig. 5). The proof is as follows. By (32a)–(34)

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1}(x) = \frac{\dfrac{kx_3}{\mu_0}\dfrac{cM_s}{a}\dfrac{\partial\mathscr{L}}{\partial z}(z) + x_4 M_s\left(\mathscr{L}(z) - \dfrac{x_2}{M_s}\right)}{\dfrac{kx_3}{\mu_0} - x_4 M_s\left(\mathscr{L}(z) - \dfrac{x_2}{M_s}\right)\alpha - \dfrac{kx_3}{\mu_0}\alpha\dfrac{cM_s}{a}\dfrac{\partial\mathscr{L}}{\partial z}(z)}. \tag{60}$$

where $\mathscr{L}(z) = \coth(z) - 1/z$ and $\partial\mathscr{L}/\partial z(z) = -\mathrm{cosech}^2(z) + 1/z^2$. We have the following cases to consider:

(a) For $x \in \mathcal{O}_1$, except the point $(0, 0)$, we have $x_3 = 1$ and $x_4 = 1$.
By (36a) the denominator is positive (proved in Theorem 3.1 and by (58)). The first part of the numerator of the right-hand side of (60), is non-negative $\forall z$. The second part of the numerator is also positive as shown in Theorem 3.1. Thus $\mathrm{d}x_2/\mathrm{d}x_1(x) > 0$ for $x \in \mathcal{O}_1$,

(b) For $x \in O_2$, we have $x_3 = -1$ and $x_4 = 0$. In this case, we first cancel a factor of -1 between the numerator and the denominator. We showed the resulting denominator to be positive while considering the existence of the solution. The resultant numerator is always positive. With this, we conclude that $\mathrm{d}x_2/\mathrm{d}x_1(x) > 0$ for $x \in \mathcal{O}_2$.

Hence

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1}(x) > 0$$

for x belonging to the solution sets $\mathcal{O}_1$ and $\mathcal{O}_2$.

3. For all $x \in \mathcal{O}_1$,

$$0 < \frac{\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)}{1 - \alpha\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)} < \frac{\dfrac{k}{\mu_0}\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z) + M_s\left(\mathscr{L}(z) - \dfrac{x_2}{M_s}\right)}{\dfrac{k}{\mu_0} - M_s\left(\mathscr{L}(z) - \dfrac{x_2}{M_s}\right)\alpha - \dfrac{k}{\mu_0}\alpha\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)}.$$

   The first inequality is due to the assertion of the previous item. The second inequality is because the denominator is smaller and the numerator is larger in magnitude for the ratio on the right-hand side. Now the ratio in the middle is $\mathrm{d}x_2/\mathrm{d}x_1(x)$ for $x \in \mathcal{O}_2$, while the ratio on the right-hand side is $\mathrm{d}x_2/\mathrm{d}x_1(x)$ for $x \in \mathcal{O}_1$.
4. The point (x_{1_0}, x_{2_0}) belongs to both $\mathcal{O}_1$ and $\mathcal{O}_2$.
5. The projection of both the sets $\mathcal{O}_1$ and $\mathcal{O}_2$ on the x_1 axis is the set $[0, x_{1_0}]$. This is a consequence of (32a) and the definition of the input u_1.
   Items 2–5 imply that the curve $\mathcal{O}_2$ lies above the curve $\mathcal{O}_1$ in the (x_1, x_2)-plane except at the point (x_{1_0}, x_{2_0}) (see Fig. 5). Item 1 then implies that the curve $\mathcal{O}_2$ intersects with the anhysteretic curve $y = 0$ in the first quadrant of the (x_1, x_2)-plane. This means that there exists a time $b_2 < b$ such that $y(t = b_2) = 0$ and $y(t) < 0$ for $t \in (b_2, b]$. Hence the hypothesis that $b_1 = b$ is not possible.

Thus we have shown that $\exists\, 0 < b_1 < b$ such that $y(b_1) = 0$.

Uniqueness. The state equations for the time interval $[0, b_1]$ are:

$$\dot{z}(t) = \frac{\dfrac{1}{a}\dfrac{k}{\mu_0}}{\dfrac{k}{\mu_0} - \alpha\dfrac{k}{\mu_0}\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)}\, u_1(t), \tag{61a}$$

$$\dot{y}(t) = \frac{\dfrac{M_s}{a}\dfrac{k(1-c)}{\mu_0}\dfrac{\partial \mathscr{L}}{\partial z}(z)}{\dfrac{k}{\mu_0} - \alpha\dfrac{k}{\mu_0}\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)}\, u_1(t). \tag{61b}$$

We now show that the solution of (61a) and (61b) for $t \in [0, b_1]$) is unique. Denote $\dot{z} = f_1(t, w)$ and $\dot{y} = f_2(t, w)$ where $f_1(t, w)$ and $f_2(t, w)$ are defined by the right-hand sides of (61a) and (61b), respectively. As $u_1(t) < 0$ for $t \geqslant 0$, $x_3 = -1$. As $y > 0$ for $t \in [0, b_1]$, $x_4 = 0$. With $w_1 = (z_1, y_1)$ and $w_2 = (z_2, y_2)$, we have

$$|f_1(t, w_1) - f_1(t, w_2)| \leqslant \frac{1}{a}\frac{\dfrac{k}{\mu_0}}{A^2}\left(\frac{k}{\mu_0}\frac{\alpha c M_s}{a}\left|\frac{\partial \mathscr{L}}{\partial z}(z_1) - \frac{\partial \mathscr{L}}{\partial z}(z_2)\right|\right)|u_1(t)|. \tag{62}$$

As $\partial\mathscr{L}/\partial z(z)$ is a smooth function of z, $\exists$ a non-negative constant K such that [15]

$$\left|\frac{\partial \mathscr{L}}{\partial z}(z_1) - \frac{\partial \mathscr{L}}{\partial z}(z_2)\right| \leqslant K|z_1 - z_2|$$

so that

$$|f_1(t,w_1)-f_1(t,w_2)| \leqslant \frac{\frac{k}{\mu_0}}{A^2}\,\frac{k}{\mu_0}\,\frac{c\alpha M_s}{a} K\|w_1-w_2\||u_1(t)|. \tag{63}$$

Now

$$|f_2(t,w_1)-f_2(t,w_2)| \leqslant \frac{u(t)}{A^2}\left(\frac{k}{\mu_0}\right)^2 \frac{(1-c)M_s}{a}\left|\frac{\partial \mathscr{L}}{\partial z}(z_1)-\frac{\partial \mathscr{L}}{\partial z}(z_2)\right||u_1(t)|.$$

Therefore

$$|f_2(t,w_1)-f_2(t,w_2)| \leqslant \frac{u(t)}{A^2}\left(\frac{k}{\mu_0}\right)^2 \frac{(1-c)M_s}{a} K\|w_1-w_2\||u_1(t)|.$$

By the above inequality and (63)

$$\|f(t,w_1)-f(t,w_2)\| \leqslant B\|w_1-w_2\||u_1(t)|, \tag{64}$$

where B is some positive constant. Hence there exists atmost one solution in D by Theorem 6.3. This concludes the proof of Theorem 3.2. □

We now study the system described by Equations (32a)–(34), together with the input given by

$$u(t)=U\cos(\omega t). \tag{65}$$

Next, we prove the existence of a periodic orbit to which the solution to the system of Eqs. (32a)–(34) with u as in (65) converges. Using Theorems 3.1 and 3.2 we show that:

1. Starting from $(x_1,x_2)=(0,0)$, $x_2(t)$ increases for $t\in[0,\pi/2\omega]$ and satisfies $x_2(t)<M_s\mathscr{L}(z(t))$. This implies that when x_2 is considered as a function of x_1 during this time interval, x_2 lies below the anhysteretic curve in the first quadrant of the (x_1,x_2) plane.
2. For $t\in[\pi/2\omega,3\pi/2\omega]$, the solution first intersects the anhysteretic curve in the first quadrant of the (x_1,x_2) plane at a time t_1^* such that $\pi/2\omega<t_1^*<\pi/\omega$. After this time, $x_2(t)>M_s\mathscr{L}(z(t))$. An important fact to be shown is that $x_2(3\pi/2\omega)>-x_2(\pi/2\omega)$.
3. For $t\in[3\pi/2\omega,5\pi/2\omega]$, the solution intersects the anhysteretic curve in the third quadrant of the (x_1,x_2) plane provided the ratio $k/a\mu_0$ is small enough. Furthermore, if the time is t_2^* when this intersection takes place, then we show that $0>x_2(t_2^*)>-x_2(t_1^*)$ using existence and uniqueness of solutions and the fact that $x_2(3\pi/2\omega)>-x_2(\pi/2\omega)$.
4. For $t\in[5\pi/2\omega,7\pi/2\omega]$, we show that the solution intersects the anhysteretic curve in the first quadrant of the (x_1,x_2) plane at a time t_3^*. An important fact that we prove is that $0<-x_2(t_2^*)<x_2(t_3^*)<x_2(t_1^*)$.
5. By repeating the analysis in the previous steps, we show that the solution trajectory of the system intersects with the anhysteretic curve in the first quadrant of the (x_1,x_2) plane during the intervals

$$\left[\frac{(2n+1)\pi}{2\omega},\frac{(2n+3)\pi}{2\omega}\right]$$

where $n \in \mathbb{N}$. Furthermore these intersection points satisfy:

$$0 < -x_2(t_2^*) < x_2(t_{2n+3}^*) < x_2(t_{2n+1}^*) < x_2(t_1^*); \quad n \in \mathbb{N}.$$

Thus we have a monotonically decreasing sequence of positive numbers that lies in the compact set $[-x_2(t_2^*), x_2(t_1^*)]$. Thus there exists a limit x_{2_∞} to this sequence that lies in the same compact set.

This shows that the Ω limit set is a periodic orbit in the (x_1, x_2)-plane. Since x_3 and x_4 depend on x_1, x_2, we conclude that the system of Eqs. (32a)–(34) with u as in (65) and the origin as initial condition, have asymptotically periodic solutions.

3.1. Analysis of the model for $t \in [0, 5\pi/2\omega]$

Lemma 3.1. *Consider the system described by Eqs.* (32a)–(34) *with the input given by* (65), *and* $(x_1(0), x_2(0)) = (0, 0)$. *Suppose the parameters satisfy conditions* (36a)–(36c). *In the time interval* $[0, \pi/2\omega]$, *there exists a unique solution and it satisfies the condition* $|x_2(t)| < M_s$.

Proof. Choosing $b = \pi/2\omega$, we apply Theorem 3.1 as the initial condition is on the anhysteretic curve and $u(\cdot) > 0$ in the time interval $(0, \pi/2\omega)$. The conclusion of Theorem 3.1 implies that $x_2(t) < M_s\ \forall t \in [0, \pi/2\omega]$. □

By the Extension Theorem 6.2, the solution trajectory reaches the boundary of the rectangle D (see Theorem 3.1 for the definition of D) in the time variable. Hence

$$x\left(\frac{\pi}{2\omega}\right) = (x_1, x_2)\left(\frac{\pi}{2\omega}\right) \tag{66}$$

$$\triangleq \lim_{t \to \pi/2\omega -} (x_1, x_2)(t) \quad \text{is well-defined.} \tag{67}$$

Lemma 3.2. *Consider the system described by Eqs.* (32a)–(34) *with the input given by* (65), *and* $(x_1(0), x_2(0)) = (0, 0)$. *Suppose the parameters satisfy conditions* (36a)–(36c). *In the time interval* $[\pi/2\omega, 3\pi/2\omega]$, *there exists a unique solution and it satisfies the condition* $|x_2(t)| < M_s$.

Furthermore, $x(3\pi/2\omega)$ *lies in the third quadrant in the* (x_1, x_2) *plane.*

Proof. Let $\tau \triangleq t - \pi/2\omega$ and $\varepsilon \triangleq t$. Define $u_1(\tau) = U\cos(\omega\tau + \pi/2)$ for $\tau \in [0, \pi/2\omega]$, and $u(\varepsilon) = U\cos(\omega\varepsilon)$ for $\varepsilon \in [0, \pi/2\omega]$. If the input $u_1(\tau)$ is applied to the system (32a)–(33b) with initial condition $x(\tau = 0) = x(t = \pi/2\omega)$ where $x(t = \pi/2\omega)$ is given by (67), then the conditions of Theorem 3.2 are satisfied (with $u(\varepsilon)$ taking the place of $u(t)$). This implies that there exists $0 < \tau^* < \pi/2\omega$ such that $x_2(\tau = \tau^*) = M_s \mathscr{L}(\frac{x_1(\tau=\tau^*)+\alpha x_2(\tau=\tau^*)}{a})$. If we define $t_1^* \triangleq \tau^* + \pi/2\omega$, then the intersection with the anhysteretic curve takes place at $t = t_1^*$ (see Fig. 6).

Let $\mu \triangleq t - \pi/2\omega - t_1^*$. Now define $u(\mu) = U\cos(\omega(\mu + t_1^*) + \pi/2)$, for $\mu \in [0, \pi/\omega - t_1^*]$. Then with initial condition at $x(\mu = 0) = x(\tau = t_1^*)$, the conditions of Theorem 3.1 is satisfied.

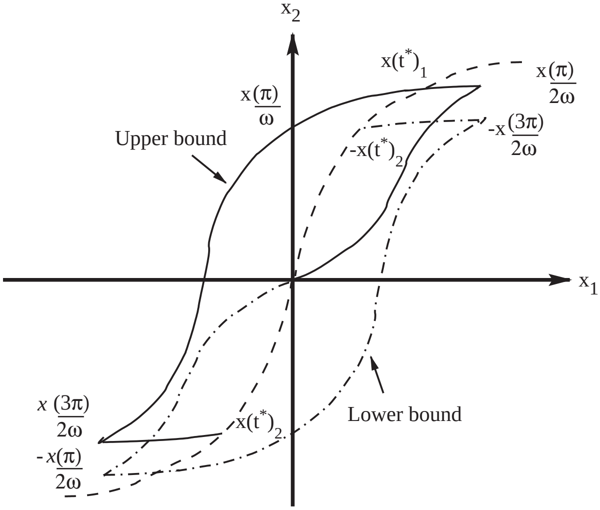

Fig. 6. Figure for the proof of Lemma 3.2.

The conclusions of Theorem 3.1 imply that $x_2(t) < M_s\ \forall t \in [\pi/2\omega, 3\pi/2\omega]$. Again by the extension theorem,

$$x\left(t = \frac{3\pi}{2\omega}\right) = (x_1, x_2)\left(\frac{3\pi}{2\omega}\right) \tag{68}$$

$$\triangleq \lim_{\mu \to (\pi/\omega - t_1^*)-} (x_1, x_2)(\mu) \quad \text{is well-defined.} \tag{69}$$

For the last part of the lemma, refer to Fig. 6. In the figure, the dashed line denotes the anhysteretic curve satisfying $x_2 = M_s \mathscr{L}(z)$. The solution trajectory for the time interval $[0, t_2^*]$ is shown by a solid curve. The solution curve for $[0, t_2^*]$ has been multiplied by -1 and shown with a dash–dot line. This curve can be obtained by applying the input $-u(t)$ to the system with the same initial conditions at $t = 0$.

Our analysis after Corollary 3.1 shows that $y(\cdot)$ cannot have any minima during the time interval $[t_1^*, \pi/\omega]$ (remember that here $\text{sign}(u(\cdot)) = -1$ and so our analysis after Corollary 3.1 has to be re-interpreted for this case). Next, note that $x_2(\pi/\omega) < x_2(\pi/2\omega)$ because $\dot{x_2}(t) < 0$ during the interval $(\pi/2\omega, \pi/\omega)$. Further, we must have $0 < x_2(\pi/\omega)$, because by the last statement of Theorem 3.1 we have $y(\pi/\omega) < 0$. As $x_1(\pi/\omega) = 0$, we can have $x_2(\pi/\omega) = 0$ only if $(x_1, x_2)(\pi/\omega)$ lies on the anhysteretic curve which would then imply $y(\pi/\omega) = 0$.

Let us now compare the solution trajectory $x(\cdot)$ during the interval $[\pi/\omega, 3\pi/2\omega]$ with the solution trajectory $\widehat{x}(\cdot)$ with input $-u(\cdot)$ during the interval $[0, \pi/2\omega]$. This comparison will lead us to the proof of the lemma. For the first case, lets re-define time to be $\sigma = t - \pi/\omega$.

Then in both cases, the system is described by:

$$\dot{x_1}(\sigma) = -U\cos(\omega\sigma),$$
$$\dot{x_2}(\sigma) = -\frac{\dfrac{k}{\mu_0}\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z) - y}{\dfrac{k}{\mu_0} + y\alpha - \dfrac{k}{\mu_0}\alpha\dfrac{cM_s}{a}\dfrac{\partial \mathscr{L}}{\partial z}(z)}\, U\cos(\omega\sigma),$$

where $\sigma = t$ for the second case. Thus the two solutions satisfy the same differential equations albeit with different initial conditions. The initial condition for the first case is $x_1(\sigma=0)=0$ and $x_2(\sigma=0)=x_2(t=\pi/\omega)>0$; while the initial condition for the second case is $\widehat{x_1}(\sigma=0)=0$ and $\widehat{x_2}(\sigma=0)=0$. Therefore, we must have $x_1(\sigma)=\widehat{x_1}(\sigma)$ and $x_2(\sigma)>\widehat{x_2}(\sigma)$ for all $\sigma \in [0, \pi/2\omega]$. Otherwise, there will be an intersection of the two trajectories which cannot happen by the existence and uniqueness of solutions to the above differential equations that we proved earlier in Theorem 3.1. This analysis shows that $x_2(\sigma)>\widehat{x_2}(\sigma)$ for all $\sigma \in [0, \pi/2\omega]$. This implies that

$$y(\sigma) = M_s\mathscr{L}(z) - x_2(\sigma) < \widehat{y}(\sigma) = M_s\mathscr{L}\left(\frac{\widehat{x_1}(\sigma)+\alpha\widehat{x_2}(\sigma)}{a}\right) - \widehat{x_2}(\sigma) < 0.$$

We need to show that $x_2(t=3\pi/2\omega)<0$ in order to conclude the proof of this lemma. We show this by comparing $\mathrm{d}x_2/\mathrm{d}x_1$ for the two cases discussed above. In both cases, we have

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1}(\sigma) = \frac{\dfrac{k}{\mu_0}\dfrac{cM_s}{a}\dfrac{\mathrm{d}\mathscr{L}}{\mathrm{d}z}(z(\sigma)) - y(\sigma)}{\dfrac{k}{\mu_0} + \alpha y(\sigma) - \dfrac{k}{\mu_0}\alpha\dfrac{cM_s}{a}\dfrac{\mathrm{d}\mathscr{L}}{\mathrm{d}z}(z(\sigma))}.$$

We have already shown that $y(\sigma)<\widehat{y}(\sigma)<0$. This implies that $\mathrm{d}x_2/\mathrm{d}x_1(\sigma) > \mathrm{d}\widehat{x_2}/\mathrm{d}\widehat{x_1}(\sigma)$. Thus

$$\left(x_2\left(t=\frac{\pi}{\omega}\right) - x_2\left(t=\frac{3\pi}{2\omega}\right)\right) > \left(0 - \widehat{x_2}\left(t=\frac{\pi}{2\omega}\right)\right).$$

Now, $\widehat{x_2}(t=\pi/2\omega)=-x_2(t=\pi/2\omega)$ and by our earlier analysis, $x_2(t=\pi/\omega)<x_2(t=\pi/2\omega)$. Therefore, combining these inequalities, we must have $x_2(t=3\pi/2\omega)<0$. □

The last conclusion of Lemma 3.2 is needed for proving the next lemma. If $x_2(3\pi/2\omega)$ is not less than 0, then the solution of the next time interval $[3\pi/2\omega, 5\pi/2\omega]$ could intersect the anhysteretic curve in the first quadrant instead of the third quadrant in the (x_1, x_2) plane (see Fig. 6). Note that in the next lemma, we have an additional condition on the parameters (namely, $k/a\mu_0$ being small enough) that we have not seen earlier.

Lemma 3.3. *Consider the system described by Eqs.* (32a)–(34) *with input given by* (65), *and* $(x_1(0), x_2(0)) = (0,0)$. *Suppose the parameters satisfy Eqs.* (36a)–(36c). *If the ratio* $k/a\mu_0$ *is small enough, then in the time interval* $[3\pi/2\omega, 5\pi/2\omega]$, *there exists a unique solution and it satisfies the condition* $|x_2(t)| < M_s$.

Furthermore, if

$$x\left(t=\frac{5\pi}{2\omega}\right)=(x_1,x_2)\left(\frac{5\pi}{2\omega}\right) \tag{70}$$

$$\triangleq \lim_{t\to 5\pi/2\omega-}(x_1,x_2)(t), \tag{71}$$

then, $x_1(5\pi/2\omega)=x_1(\pi/2\omega)$ *while* $x_2(5\pi/2\omega)<x_2(\pi/2\omega)$.

Proof. Let $\tau\triangleq t-3\pi/2\omega$and $\varepsilon\triangleq t-t_1^*$. Define $u_1(\tau)=U\cos(\omega\tau+3\pi/2)$ for $\tau\in[0,\pi/\omega]$, and $u(\varepsilon)=U\cos(\omega\varepsilon)$ for $\varepsilon\in[0,3\pi/2\omega-t_1^*]$. If the input $u_1(\tau)$ is applied to the system (32a)–(33b) with initial condition $x(\tau=0)=x(t=3\pi/2\omega)$ where $x(t=3\pi/2\omega)$ is given by (69), then the conditions of Theorem 3.2 are satisfied (with $u(\varepsilon)$ taking the place of $u(t)$). Theorem 3.2 then implies that there exists $0<\tau^*<3\pi/2\omega-t_1^*$ such that $x_2(\tau=\tau^*)=M_s\mathscr{L}(\frac{x_1(\tau=\tau^*)+\alpha x_2(\tau=\tau^*)}{a})$.Define $t_2^*\triangleq\tau^*+3\pi/2\omega$ (see Fig. 6). We would like this intersection of the solution trajectory with the anhysteretic curve to take place in the third quadrant of the (x_1,x_2) plane. For this, consider the quantity $\mathrm{d}x_2/\mathrm{d}x_1(\tau)$ for $0<\tau<\tau^*$ (during this interval, $x_3(\tau)=1$ and $x_4(\tau)=0$):

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1}(\tau)=\frac{\dfrac{k}{\mu_0}\dfrac{cM_s}{a}\dfrac{\mathrm{d}\mathscr{L}}{\mathrm{d}z}(z(\tau))}{\dfrac{k}{\mu_0}-\dfrac{k}{\mu_0}\alpha\dfrac{cM_s}{a}\dfrac{\mathrm{d}\mathscr{L}}{\mathrm{d}z}(z(\tau))}.$$

By making the ratio $k/a\mu_0$ small enough, we can make $\mathrm{d}x_2/\mathrm{d}x_1(\tau)$ as close to zero as we please. This combined with the fact that $x_2(3\pi/2\omega)<0$ from Lemma 3.2 implies that we can make $(x_1,x_2)(t_2^*)$ lie in the third quadrant of the (x_1,x_2) plane. Next, we claim that:

$$0>x_2(t_2^*)>-x_2(t_1^*).$$

We can prove our claim by comparing the solution for the system with input $U\cos(\omega t)$ during the interval $[3\pi/2\omega,t_2^*]$ with the solution for the system with input $-U\cos(\omega t)$ during the interval $[\pi/2\omega,t_1^*]$. In these two cases, the differential equation satisfied by the systems is the same whilst the initial conditions are different. In the first case, the initial condition is $x(3\pi/2\omega)=(-U/\omega,x_2(3\pi/2\omega))$ while in the second case, the initial condition is $\widehat{x}(\pi/2\omega)=(-U/\omega,-x_2(\pi/2\omega))$. By existence and uniqueness of solutions proved earlier in Theorem 3.2 the two solutions cannot intersect. This and the fact that $x(3\pi/2\omega)>-x(\pi/2\omega)$ then imply our claim (see Fig. 6).

Let $\mu\triangleq t-t_2^*$. Define $u(\mu)=U\cos(\omega\mu)$, for $\mu\in[0,5\pi/\omega-t_2^*]$. Then with initial condition at $x(\mu=0)=x(t=t_2^*)$, the conditions of Theorem 3.1 is satisfied. Then the conclusions of Theorem 3.1 imply that $|x_2(t)|<M_s\ \forall t\in[3\pi/2\omega,5\pi/2\omega]$. □

3.2. Proof of limiting periodic behavior of the model for sinusoidal inputs

Using the Lemmas 3.1–3.3 we can prove the main result of this paper.

Theorem 3.3. *Consider the system given by Eqs.* (32a)–(34), *with input given by Eq.* (65). *Suppose that* (36a)–(36c) *are satisfied and the ratio* $k/a\mu_0$ *is small enough.*

If $(x_1, x_2)(0) = (0, 0)$, *then there exists a unique solution to the system, and furthermore* $|x_2(t)| \leqslant M_s \forall t \geqslant 0$. *Thus the solution trajectory lies in the compact region* $[-U/\omega, U/\omega] \times [-M_s, M_s]$ *in the* (x_1, x_2)*-plane. Furthermore, the* Ω*-limit set of this trajectory is a periodic orbit of period* $2\pi/\omega$.

Proof. By Lemmas 3.1–3.3, we have shown that if $k/a\mu_0$ is small enough, then

$$|x_2(t)| \leqslant M_s \quad \forall t \in \left[0, \frac{5\pi}{2\omega}\right].$$

Let us consider the solution during the time interval $[t_2^*, 5\pi/2\omega]$. By using the same techniques used in the proofs of Lemmas 3.2 and 3.3, we can show the following:

- the variable $y(t)$ does not have any critical points during the interval $[t_2^*, 2\pi/\omega]$;
- during the interval $[2\pi/\omega, 5\pi/2\omega]$, the variable $x_2(t)$ is bounded above by $x_2(\tau)$ the solution of the same differential equation for the time interval $[0, \pi/2\omega]$ with initial condition at the origin. It is also bounded below the solution to the same differential equation with input $-U\cos(\omega t)$ for the interval $[\pi/\omega, 3\pi/2\omega]$, with initial condition at the origin (see Fig. 6 for an illustration);
- by the previous item, we have
$$-x_2\left(\frac{3\pi}{2\omega}\right) < x_2\left(\frac{5\pi}{2\omega}\right) < x_2\left(\frac{\pi}{2\omega}\right);$$
- if we now consider the solution to the differential equation during the time interval $[5\pi/2\omega, 7\pi/2\omega]$ we see that the solution must intersect with the anhysteretic curve at a time t_3^* such that $0 < t_3^* < 3\pi/\omega$. Furthermore, the point of intersection must satisfy
$$0 < -x_2(t_2^*) < x_2(t_3^*) < x_2(t_1^*);$$
- continuing the solution further from the time $t = t_3^*$ to $t = 7\pi/2\omega$ we see that by Theorem 3.1 we must have $|x_2(t)| < M_s$.

Proceeding in this manner and considering time intervals $[\frac{(2n+1)\pi}{2\omega}, \frac{(2n+3)\pi}{2\omega}]$, and $[\frac{(2n+3)\pi}{2\omega}, \frac{(2n+5)\pi}{2\omega}]$, respectively for $n = 0, 1, 2, \ldots,$ we can show existence and the uniqueness of solution and the fact that $|x_2(t)| < M_s$ (one can also use the principle of induction to prove this formally). If we focus on the solutions during the time intervals $[\frac{(2n+1)\pi}{2\omega}, \frac{(2n+3)\pi}{2\omega}]$, we obtain a sequence $\{x_2(t_{2n+1}^*); n \in \mathbb{N}\}$ that satisfies:

$$0 < -x_2(t_2^*) < x_2(t_{2n+3}^*) < x_2(t_{2n+1}^*) < x_2(t_1^*); \quad n \in \mathbb{N}.$$

Thus we have a monotonically decreasing sequence of positive numbers that lies in the compact set $[-x_2(t_2^*), x_2(t_1^*)]$. Thus there exists a limit x_{2_∞} to this sequence that lies in the same compact set.

Next consider the sequence in a slightly different manner. Let $\theta = \omega t$, with $\theta + 2\pi$ identified with θ. Then the non-autonomous system given by Eqs. (32a)–(34) with input given by (65), can be transformed into an autonomous one with the auxiliary equation,

$\dot{\theta} = \omega$. Define the set:

$$\mathscr{M} = \left\{ (x_1, x_2) | x_2 = \mathscr{L}\left(\frac{x_1 + \alpha x_2}{a}\right) \text{ and } x_1, x_2 \geqslant 0 \right\}.$$

Given $x \equiv (x_1, x_2) \in \mathscr{M}$, solve Eqs. (32a)–(34) with initial condition $x(0) = x$ and input $u(t) = U\cos(\omega t + \phi)$ with ϕ chosen so that $x_1(t)$ achieves its maximum values for $t = \frac{(2n+1)\pi}{2}$; $n \in \mathbb{N}$. Let $\widehat{t_1} > 0$ denotes the smallest time such that the solution trajectory intersects the anhysteretic curve in the first quadrant of the (x_1, x_2) plane (that this happens if $k/a\mu_0$ is small enough is shown just as in Lemmas 3.1–3.3).

Define the map $\Phi : \mathscr{M} \to \mathscr{M}$ by defining $\Phi(x) = x(\widehat{t_1})$. The map Φ is a Poincaré map. Then, the sequence $\{x(t^*_{2k-1}); k \in \mathbb{N}\}$ obtained above is just,

$$x(t^*_{2k+1}) = \Phi(x(t^*_{2k-1})) = \Phi^k(x(t^*_1)); \quad k \in \mathbb{N}.$$

The limit point,

$$x_\infty \equiv (x_{1_\infty}, x_{2_\infty}) = \lim_{k \to \infty} \Phi^k(x(t^*_1)).$$

We can show $\Phi(x_\infty) = x_\infty$ by a contradiction argument. Let $(\Phi_1(x), \Phi_2(x)) \equiv \Phi(x)$. By our earlier analysis, we have $\Phi_2(x_\infty) \leqslant x_{2_\infty}$. If $\Phi_2(x_\infty) < x_{2_\infty}$, then we must have $\Phi_2^k(x_\infty) \leqslant \Phi(x_{2_\infty}) < x_{2_\infty}$ where $k \in \mathbb{N}$. Therefore x_∞ cannot be a limit point and we have proved our claim.

Thus a solution trajectory for system (32a)–(34) with initial condition $x(0) = x_\infty$ and input $u(t) = U\cos(\omega t + \phi)$ with ϕ chosen so that $x_1(t)$ achieves its maximum values for $t = \frac{(2n+1)\pi}{2}$; $n \in \mathbb{N}$, satisfies $x(2n\pi) = x_\infty$ for $n \in \mathbb{N}$.

Next, trivially we have

$$|x_1(t)| \leqslant \frac{U}{\omega} \quad \forall t \geqslant 0$$

so that the solution lies in the compact set $[-U/\omega, U/\omega] \times [-M_s, M_s]$ in the (x_1, x_2)-plane.

This concludes the proof of Theorem 3.3. $\square$

Theorems 3.1 and 3.2 are the two main theorems used in proving the above theorem. As the bulk ferromagnetism model is *rate-independent* (please see the remarks at the end of Section 2), it is not necessary for the input $u(\cdot)$ to be co-sinusoidal for Theorems 3.1 and 3.2 to be valid. Therefore we can considerably strengthen the above theorem by enlarging the class of inputs for which it is valid, without significant change in the proof. We now define the class of inputs for which the theorem would be valid. Consider the set $\mathscr{F}$ of functions $u(t) = U(t)\cos((2\pi/\omega)t)$ where $U(t) > 0$ is a $T \equiv 2\pi/\omega$ periodic function satisfying:

$$\int_0^T u(\tau)\,\mathrm{d}\tau = 0,$$

and

$$\min_{t \in [0,T]} \int_0^t \int_0^s u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = -\max_{t \in [0,T]} \int_0^t \int_0^s u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.$$

The second condition above ensures that $\min_{t\in[0,T]} x_1(t) = -\max_{t\in[0,T]} x_1(t)$ which was used in Theorem 3.2. From the set $\mathscr{F}$ we can obtain other periodic functions by means of time re-parametrizations. For any continuous, piecewise monotone function *f* defined on $[0, T]$, we can partition $[0, T]$ into sub-intervals by choosing $0 = \tau_1 < \cdots < \tau_n = T$, so that *f* is strictly monotone on each sub-interval $[t_k, t_{k+1}]$; $k = 1 \cdots n-1$. Denote the set of such partitions by $\mathscr{P}_f$; an element in the set $\mathscr{P}_f$ by $\{\tau_1, \ldots, \tau_n\}$; and define a function $N : \mathscr{P}_f \to \mathbb{N}$ by setting $N(P) = n$ where $P = \{\tau_1, \ldots, \tau_n\}$. The number $N(P)$ is always finite as *f* is a continuous, piecewise monotone function. One can define a partial ordering relation $\leqslant$ on this set as follows. For $P_1, P_2 \in \mathscr{P}_f$:

$$P_1 \leqslant P_2 \quad \text{if and only if} \quad \tau_k \in P_1 \Rightarrow \tau_k \in P_2.$$

One can construct a minimal partition $P_f = \{0 = \tau_1, \ldots, \tau_q = T\}$ for any continuous, piecewise monotone function *f* such that $P_f \leqslant P$ for every $P \in \mathscr{P}_f$. It is that partition for which *f* fails to be monotone on the intervals $[\tau_k - \varepsilon, \tau_{k+1} + \varepsilon]$; $k = 2, \ldots, q-2$ for $\varepsilon > 0$. For example, the minimal partition corresponding to the function $U\cos(\omega t)$ where $U > 0$ is $\{0, \pi/\omega, 2\pi/\omega\}$. For a continuous, piecewise monotone function *f*, let the minimal partition be $\mathscr{P}_f = \{0 = \tau_1, \ldots, \tau_{N(P_f)} = T\}$. If $Q = \{0 = s_1, \ldots, s_{N(P_f)} = T\}$ is any other partition of $[0, T]$ then we can define monotone increasing functions $\psi : [0, T] \to [0, T]$ with $\psi(\tau_i) = \psi(s_i)$; $i = 1, \ldots, N(P_f)$. For example, one can define $\psi(\tau)$ for τ in the interval $[\tau_i, \tau_{i+1}]$ to be

$$\psi(\tau) = s_i + \frac{s_{i+1} - s_i}{\tau_{i+1} - \tau_i}\,(\tau - \tau_i). \tag{72}$$

Denote the set $\Psi_{f,Q}$ of functions $\Psi : [0, T] \to [0, T]$ that satisfy (72). For each $\psi \in \Psi$, one can define another function g_Q on $[0, T]$ by composing *f* with ψ:

$$g_Q = f \circ \psi.$$

It is clear that the function $g_Q(\cdot)$ is a continuous, piecewise monotone function defined on $[0, T]$ with minimal partition Q. Finally, denote by $\mathscr{U}$ the set of all possible functions that can be obtained from the set $\mathscr{F}$ by time re-parametrizations. We can strengthen Theorem 3.3 for input signals $u \in \mathscr{U}$ without any significant change in the proof.

Theorem 3.4. *Consider the system given by Eqs.* (32a)–(34). *Let the input* $u(\cdot) : \mathbb{R} \to \mathbb{R}$ *belong to the set* $\mathscr{U}$ *defined above. Suppose that the parameters satisfy* (36a)–(36c) *and the ratio* $k/a\mu_0$ *is sufficiently small.*

If $(x_1, x_2)(0) = (0, 0)$, *then the* Ω*-limit set of this trajectory is a periodic orbit of period* T.

Proof. The proof is essentially same as that of Theorem 3.3. □

Remarks.

1. If Theorem 3.1 is reproved for their set of equations, then by using the same method, we can show that the Ω limit set is a periodic orbit for the J–A model.

2. The important difference between the bulk ferromagnetic hysteresis model of this paper and the J–A model [14] is that $k = 0$ *does not* represent the lossless case for the latter.
3. It is very important to note that the model does not show the property of minor-loop closure. This implies that for inputs that do not vary between the same maximum and minimum values, the solution might not exist. The J–A model shows the same problem. Jiles's proposed fix to the J–A model [12] can be used for the bulk ferromagnetic hysteresis model also, but this approach is somewhat ad hoc and arbitrary.

4. Conclusion

In this paper, we derived a low-dimensional model for bulk ferromagnetic hysteresis from energy-balance principles and the J–A postulates for hysteretic losses. We also showed that for a large class of periodic inputs and initial condition at the origin, the Ω-limit set of the solution is a periodic orbit in the (H, M) plane provided the parameters satisfy (36a)–(36c) with the ratio $k/a\mu_0$ small enough. This shows that the model is numerically well-conditioned for a large class of periodic inputs.

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Appendix

Below we collect basic results concerning existence and uniqueness of solutions to ODEs with right-hand sides that are not continuous in time. The relevant theory can be found in [11,10].

Carathéodory conditions: Suppose D is an open set in $\mathbb{R}^{n+1}$. Let $f : D \to \mathbb{R}^n$, and let

1. the function $f(t, x)$ be defined and continuous in $x \in \mathbb{R}^n$ for almost all $t \in \mathbb{R}$;
2. the function $f(t, x)$ be measurable in t for each x;
3. on each compact set U of D, $|f(t, x)| \leqslant m_U(t)$, where the function $m_U(t)$ is integrable.

The equation $\dot{x}(t) = f(t, x(t))$; $x(t_0) = x_0$, where $x(t)$ is a scalar or a vector; $(t_0, x_0) \in D$; and the function f satisfies the above conditions is called a *Carathéodory equation* [10]. We say that $t \to x(t)$ is a solution in the sense of Carathéodory if $x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\,ds$ for $(t, x(t)) \in D$.

Theorem 6.1 (*Hale [11] Existence of solutions*). *If D is an open set in* $\mathbb{R}^{n+1}$ *and f satisfies the Carathéodory conditions on D, then, for any* (t_0, x_0) *in* D, *there is a solution of* $\dot{x} = f(t, x)$, *through* (t_0, x_0).

Theorem 6.2 (*Hale [11] Extension of solutions to a maximal set*). *If D is an open set in* $\mathbb{R}^{n+1}$, *f satisfies the Carathéodory conditions on* D, *and* ϕ *is a solution of* $\dot{x} = f(t, x)$ *on some interval, then there is a continuation of* ϕ *to a maximal interval of existence. Furthermore, if* (a, b) *is a maximal interval of existence of* $\dot{x} = f(t, x)$, *then* $x(t)$ *tends to the boundary of D as* $t \to a$ *and* $t \to b$.

Theorem 6.3 (*Hale [11] Uniqueness of solutions*). *If D is an open set in* $\mathbb{R}^{n+1}$,*f satisfies the Carathéodory conditions on* D, *and for each compact set U in* D, *there is an integrable function* $k_U(t)$ *such that*

$$\|f(t, x) - f(t, y)\| \leqslant k_U(t)\|x - y\|, \quad (t, x) \in U, \quad (t, y) \in U.$$

Then for any (t_0, x_0) *in* U, *there exists a unique solution* $x(t, t_0, x_0)$ of *the problem*

$$\dot{x} = f(t, x), \quad x(t_0) = x_0.$$

The domain E in $\mathbb{R}^{n+2}$ *of definition of the function* $x(t, t_0, x_0)$ *is open and* $x(t, t_0, x_0)$ *is continuous in E.*

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