Approximate Inversion of Hysteresis : Theory and Numerical Results

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Abstract

In previous work, we had proposed a low (6) dimensional model for a thin magnetostrictive actuator that was suitable for real-time control. One of the main results of this modeling effort was the separation of the rate-independent hysteretic effects from the rate-dependent linear effects. The hysteresis phenomenon may also be captured by a (modified) Preisach operator with the average magnetic field as the input.

If one can find an inverse for the Preisach operator, then the composite system can be approximately linearized. In this paper, we propose a new algorithm for computation of the inverse for the classical Preisach model. Prior approaches depended on the linearization of the operator at the operating point. As numerical differentiation is involved, this approach can cause divergence. Our algorithm does not linearize the Preisach operator, but makes use of its strictly incrementally increasing property. Convergence of the algorithm is proved using the contraction mapping principle.

1 Introduction

The Preisach operator is a mathematical construction that has been used successfully over the years to model the phenomenon of hysteresis occurring in magnetics, superconductivity, elasto-plastic deformations etc [1, 2, 3]. Though it does not provide any physical insight into the physical phenomenon, it provides a means of developing phenomenological models that are capable of producing behaviour similar to physical systems. It is of great interest to the smart structures and controls community because of its utility in developing low order models that can be used for designing real-time controllers.

The general structure of models for piezostriction, magnetostriction and electrostriction that capture hysteresis and dynamic behaviour is shown in Figure 1. In this figure, $G(s)$ is the Laplace-transform representation of a linear system while $\mathcal{W}$ denotes a rate-independent hysteretic non-linearity. The operator $\mathcal{W}$ could be a Preisach operator or some modification of the Preisach operator or it could a similar type of operator depending on the phenomenon being modeled – for instance Galinaitis and Rogers [4] use the Krasnoselskii-Pokrovskii (KP) operator in their model for piezostriction. Reimers [5] has used (a modification of) the Della Torre-Kadar-Oti (DOK) operator to model magnetostriction in Terzinol-D. In our previous work [6, 7], we have shown that a key component of a low-order model for magnetostriction in Terfenol-D has a structure resembling Figure 1.

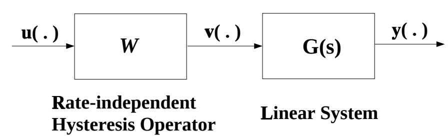

Figure 1: Structure of models for smart actuators

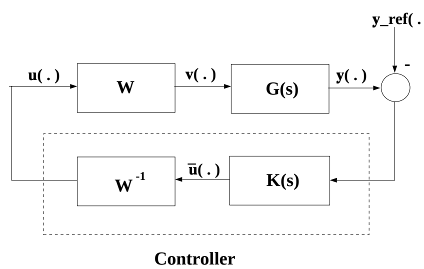

Figure 2: Controller design schematic

A basic idea for controller design for such systems is to design an right-inverse operator as shown in Figure 2. $\mathcal{W}^{-1}$ is the right-inverse of the rate-independent hysteresis operator $\mathcal{W}$ and hence the signal $v(\cdot) = \bar{u}(\cdot)$. Thus the controller design problem is reduced to designing a linear controller $K(s)$ for the linear system $G(s)$. This idea can be found in the works of Tao and Kokotovic [8] and Rogers [4]. In the work of Tao and Kokotovic, the hysteresis non-linearity is considered to be composed of linear segments and thus does not possess the important non-linear property known as *minor loop closure*. This very important property is found in physical phenomena exhibiting hysteresis like ferromag-

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netism, magnetostriction, piezostriction etc. to name a few. The Preisach model does possess this property and that is one of the reasons for the great popularity of the Preisach model among researchers in the smart-structures community.

We propose a constructive procedure to compute the inverse that is based on the Banach contraction mapping principle and utilises the incrementally strictly increasing property of the Preisach operator. Thus the inverse operator computation utilises the information of the full nonlinear Preisach operator and is not based on a linearisation or local information about the operator around an operating point. A very important property of our inverse algorithm is that we can show the tracking of a specified output signal with any specified (non-zero but arbitrarily small) tolerance. The algorithm can also be used for hysteresis operators other than the Preisach operator as long as they are incrementally strictly increasing.

In Section 2 we very briefly describe the Preisach operator. In section 3 we show how the contraction mapping principle can be used the construction of inverses of operators that are incrementally strictly increasing. In Section 5 we present numerical simulations where the output of a forced Van der Pol oscillator is used as the desired trajectory and we compute the corresponding input required for trajectory tracking.

2 The Preisach operator

Consider a simple hysteretic element shown in Figure 3. The relationship between the 'input' variable u and the 'output' variable v at each instant of time t can be described by:

$$v = +1 \quad \text{if } u > \alpha, \tag{1-a}$$
$$v = -1 \quad \text{if } u < \beta, \tag{1-b}$$
$$v \quad \text{remains unchanged} \quad \text{if } \beta \le u \le \alpha \tag{1-c}$$

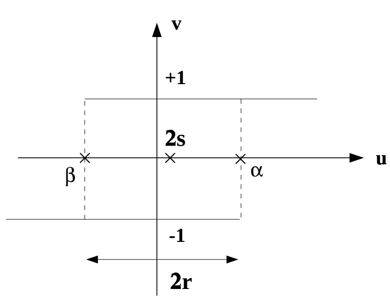

Figure 3: Illustration of the *hysteresis* phenomenon.

Call the operator relating $u(\cdot)$ to $v(\cdot)$ as $\mathcal{R}_{\beta,\alpha}[u](\cdot)$, where we now view the input and output variables as functions of time. This operator is sometimes referred to as an *elementary Preisach hysteron* as it is a basic block from which the Preisach operator will be constructed. We now outline this construction. Suppose $u(\cdot) \in C[0,T]$ is the input to the elementary hysteron. The Preisach operator is defined as :

$$w(t) = \iint_{\alpha \ge \beta} \mu(\beta,\alpha)\mathcal{R}_{\beta,\alpha}[u](t)\, d\beta\, d\alpha. \tag{2}$$

where $\mu(\cdot,\cdot)$ is a density function. The support of this measure is usually taken to be a compact subset of $\mathbb{R}^2$. This representation is the most natural for the Preisach operator and is closest to Preisach's original definition [2]. The Preisach operator has *non-local* memory and it "remembers" the dominant maximum and minimum values of the past input. For an exposition of this and other basic properties of the Preisach operator, please refer to Mayergoyz [2] and Brokate [1].

The memory effect of the Preisach can be captured by curves in the Preisach (α,β)-plane. These curves have a staircase structure when the input is piecewise monotone and continuous. As it is difficult to describe them using the α, β variables, one typically transforms them to (r,s) variables with: $r = \frac{\alpha-\beta}{2}$ and $s = \frac{\alpha+\beta}{2}$.

Then the elementary hysteron can be re-described in the (r,s) variables as follows:

$$v = +1 \quad \text{if } u > s+r, \tag{3-a}$$
$$v = -1 \quad \text{if } u < s-r, \tag{3-b}$$
$$v \quad \text{remains unchanged} \quad \text{if } s-r \le u \le s+r \tag{3-c}$$

Note that r satisfies $r \ge 0$ while $-\infty \le s \le \infty$. Then the Preisach operator is re-written as:

$$w(t) = \int_0^{\infty}\int_{-\infty}^{\infty} \omega(r,s)\mathcal{R}_{s-r,s+r}[u](t)\, ds\, dr. \tag{4}$$

With the (r,s) variables as co-ordinates of the Preisach plane, the memory curve can be expressed as a function of r. $\omega(\cdot,\cdot)$ is the measure $\mu(\cdot,\cdot)$ in (2) expressed in the new co-ordinates.

For further understanding of the Preisach operator, Brokate and Sprekels express it in terms of the *Play* operator [1]. *At the heart of the analysis is the observation that for a given value of* r*, the variation of the memory curve* $\psi(r)(\cdot)$ *as a function of time, can be described by a play operator.* This very important development in the study of the Preisach operator enables one to show properties such as Lipschitz continuity and the existence of an inverse. We only present the main results in this paper. In the version of this paper that was reviewed, we presented a completed proof of Brokate and Sprekel's existence theorem for the inverse. It will be published elsewhere for lack of space.

3 The contraction mapping principle and construction of inverse operators

If a non-linear operator $g : X \to X$ (where (X,d) is a a complete metric, vector space) has a Lipschitz constant $\rho < 1$, then the Banach fixed point theorem can be used to obtain a constructive inverse for the operator $f = I - g$ where $I : X \to X$ is the identity operator. To see this let $x, y \in X$ and we need to solve $f(x) = y$.

Rewrite this equation as: $x = g(x) + y$. Since g has a Lipschitz constant $\rho < 1$, there exists a fixed point for the successive approximations $x_{n+1} = g(x_n) + y$ by the Banach fixed point theorem [9]. This solution is unique and therefore the operator f has an inverse.

We are interested in the converse of the above fact - namely, under what conditions does an operator g exist (with a Lipschitz constant $\rho < 1$) such that $f = I - g$, if we know that f is invertible and has a Lipschitz constant $\rho + 1$. The answer is that in addition to the above conditions, f must be incrementally strictly increasing. For example for $x_1, x_2 \in \mathbb{R}$ with $x_1 \geq x_2$, suppose there exists $1 > k > 0$ such that

$$k(x_1 - x_2) \leq f(x_1) - f(x_2) \leq (x_1 - x_2)$$

Then we can write, $0 \leq (x_1 - f(x_1)) - (x_2 - f(x_2)) \leq (1 - k)(x_1 - x_2)$. For $x_1 \leq x_2$ we have $0 \geq (x_1 - f(x_1)) - (x_2 - f(x_2)) \geq (1 - k)(x_1 - x_2)$. Therefore, $|g(x_1) - g(x_2)| \leq (1 - k)\,|x_1 - x_2|$, where $g(x) = x - f(x)$. Hence g has a Lipschitz constant $1 - k < 1$. For general Banach spaces, we need to define an ordering on the space, before we can speak of strictly increasing operators.

Definition 3.1 *(Order Cones) [9] Let X be a Banach space and let K be a subset of X. Then K is called an order cone if and only if:*

- *K is closed, nonempty and $K \neq \{0\}$;*
- *$a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in K \Rightarrow ax + by \in K$;*
- *$x \in K$ and $-x \in K \Rightarrow x = 0$.*

Definition 3.2 *(Ordering in Banach Spaces) [9] Let $K \subset X$ be an order cone in a Banach space X. Then define:*

- *$x \leq y$ if and only if $y - x \in K$,*
- *$x < y$ if and only if $x \leq y$ and $x \neq y$,*
- *$x \nleq y$ if and only if $x \leq y$ is false.*

Definition 3.3 *(Normal Cones) The order cone K is called normal if and only if there is a number $c > 0$ such that for all $x, y \in X$, $0 \leq x \leq y$ implies $\|x\| \leq c\,\|y\|$.*

Example 3.1 *Let $X = C[0,T]$. Then $K = C_+[0,T] = \{f \in C[0,T] : f(x) \geq 0 \text{ on } [0,T]\}$ is a normal cone with $\|f\| \leq \|g\|$ if $0 \leq f \leq g$.*

Definition 3.4 *(Incrementally Strictly Increasing Operators) An operator $F : X \to X$ is called incrementally strictly increasing, if for $x_1\, x_2 \in X$ with $x_1 \geq x_2$, there exists $k_1, k_2 > 0$ such that $k_1\,(x_1 - x_2) \leq F(x_1) - F(x_2) \leq k_2\,(x_1 - x_2)$.*

Lemma 3.1 *Let $X = C[0,T]$. Let $F : X \to X$ be an incrementally strictly increasing operator with constants k_1, k_2 as defined in Definition 3.4. Then $\|(k_2x_1 - F(x_1)) - (k_2x_2 - F(x_2))\| \leq (k_2 - k_1)\|x_1 - x_2\|$ for all $x_1, x_2 \in X$.*

Proof It was seen in Example 3.1 that X is a normal cone. For $x_1, x_2 \in X$ with $x_1 \leq x_2$, the claim is obvious. Similarly for $x_1 \geq x_2$. For general x_1, x_2, let $D \subset [0,T]$ be defined as $D_1 = \{t \in [0,T] \mid x_1(t) \leq x_2(t)\}$. D_1 is a closed set. Let $\|\cdot\|_{D_1}$ be the norm on X restricted to $X_1 = \{x(t) |\, x(\cdot) \in X;\ t \in D_1\}$. Then

$$\|(k_2x_1 - F(x_1)) - (k_2x_2 - F(x_2))\|_{D_1} \leq (k_2 - k_1)\|x_1 - x_2\|_{D_1}. \tag{5}$$

Define $D_2 = \{t \in [0,T] \mid x_1(t) \geq x_2(t)\}$. D_2 is also a closed set. Now define $\|\cdot\|_{D_2}$ be the norm on X restricted to $X_2 = \{x(t) |\, x(\cdot) \in X;\ t \in D_2\}$. Then

$$\|(k_2x_1 - F(x_1)) - (k_2x_2 - F(x_2))\|_{D_2} \leq (k_2 - k_1)\|x_1 - x_2\|_{D_2}. \tag{6}$$

Now $[0,T] = D_1 \bigcup D_2$ and $\|\cdot\| = \max\{\|\cdot\|_{D_1}, \|\cdot\|_{D_2}\}$. By (5 - 6),

$$\|(k_2x_1 - F(x_1)) - (k_2x_2 - F(x_2))\| \leq (k_2 - k_1)\|x_1 - x_2\|. \tag{7}$$

□

With the above lemma, it is an easy matter to construct an inverse for operators that satisfy the conditions of the lemma. This is done in Theorem 3.1. A more abstract version of the same theorem can be found in Krasnoselskii and Zabreiko [10]. They consider the case when $F(\cdot)$ satisfies

$$B_1(x_1 - x_2) \leq F(x_1) - F(x_2) \leq B_2(x_1 - x_2) \tag{8}$$

where B_1 and B_2 are linear operators with positive inverses. Our proof of Theorem 3.1 is much simpler and is instructive in its own right.

Theorem 3.1 *Let $X = C[0,T]$. Let $F : X \to X$ be incrementally strictly increasing operator as defined in Lemma 3.1. Then there exists a unique solution $x \in X$ to the operator equation*

$$F(x) = y \tag{9}$$

where $y \in X$. This solution is the limit of the iteration:

$$x_{n+1} = \frac{1}{k_2}\,(y + (k_2x_n - F(x_n))) \tag{10}$$

Proof Define $G \triangleq k_2I - F$, where I is the identity operator on X. Then we can write (9) as

$$x = \frac{1}{k_2}(y + G(x)) \tag{11}$$

By Lemma 3.1, the Lipschitz constant of $\frac{1}{k_2}G$ is $\frac{k_2 - k_1}{k_2} < 1$. Therefore we can solve for x by the iteration:

$$
\begin{aligned}
x_{n+1} &= \frac{1}{k_2}\left(y + G(x_n))\right) \\
&= \frac{1}{k_2}\left(y + (k_2 x_n - F(x_n))\right)
\end{aligned}
$$

□

Theorem 3.1 allows us to decompose original system into a feed-back system as shown in Figure 4 for the purpose of computing the inverse. The strictly increasing property of the system is taken advantage of to achieve this. Control theorists familiar with Zames [11] work, can see that the feed-back system is stable because the product of the incremental gains of the two systems ($\frac{k_2 - k_1}{k_2}$) is less than 1 !

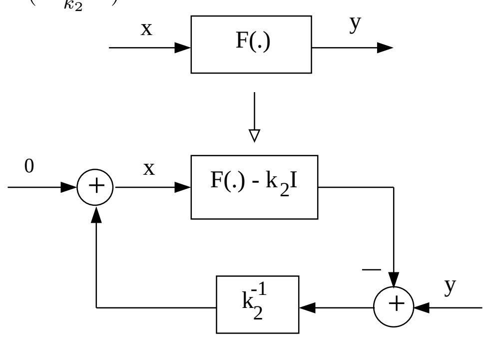

Figure 4: Decomposition of a incrementally strictly increasing system into a feedback system

4 Algorithm for the inverse of the Preisach Operator

In this section, we extend the results of Section 3 to make them applicable to the Preisach Operator. The need for the extension is outlined in Remark 4.1. But first, we reproduce two results from Brokate and Sprekels that show that the Preisach operator is Lipschitz continuous and incrementally strictly increasing.

4.1 Lipschitz continuity and the incrementally strictly increasing property of the Preisach operator

Brokate and Sprekels have showed the following regularity property of the Preisach operator.

Proposition 4.1 *(Lipschitz continuity property for the Preisach operator) [1] Let $\mathcal{W}$ be the Preisach operator having the initial memory curve ψ_{-1} [1]. If*

$$
C_1 \triangleq \int_0^\infty \sup_{s\in\mathbb{R}} |\omega(r,s)|\, d|\nu|(r) \;<\; \infty \qquad (12)
$$

then $\mathcal{P}$ is Lipschitz continuous on $C[0,T]$, with Lipschitz constant $2C_1$.

It was also shown by Brokate and Sprekels [1] that the Preisach operator is incrementally strictly increasing, but does not satisfy the condition of Lemma 3.1.

[1] See Brokate and Sprekels [1] for the definition of the set of initial memory curves

Let $I = [a,b]$. Define the quantity $\xi_I(x)$ as the infimum of the difference between the outputs of the Preisach operator, when the inputs are $v_1,\, v_2 \in C[0,T]$ with $v_1(t) \geq v_2(t)$ and $v_1(t),\, v_2(t) \in [a,b]$ for all $t \in [0,T]$. For a more precise definition using level functions of the Preisach operator, please refer to [1].

Lemma 4.1 *[1] Let $I = [a,b]$ and let $\mathcal{W}$ denote a piecewise strictly increasing Preisach operator having a density function ω which satisfies*

$$
\omega(r,s) \geq \beta(r) > 0, \qquad \forall\,(r,s) \in R_\epsilon = (0,\epsilon) \times (a-\epsilon, b+\epsilon),
$$

for some $\epsilon > 0$. Then

$$
\xi_I(x) \geq \int_0^{\frac{x}{2}} (x - 2r)\beta(r)\, dr \;>\; 0, \qquad \text{if } x \leq \epsilon, \quad (13)
$$

Remark 4.1

The above lemma shows that $\xi_I(x) > 0$ for $x \neq 0$. Thus the Preisach operator is incrementally strictly increasing. We can also see that

$$
\xi_I(x) \leq x^2 \sup_{(r,s)\,\in\, R_\epsilon} \omega(r,s), \qquad \text{if } x < \epsilon, \qquad (14)
$$

as noted by Brokate and Sprekels. This means that we cannot have $\xi_I(x) \geq \gamma\, x$ for some $\gamma > 0$ and $x < \epsilon$ and therefore cannot use the theory developed in the previous section directly. Figure 5 illustrates the above discussion. In figure, the solid curve represents the memory curve $\psi(t)$ at time t; $v(t)$ is the input value at time t and it is also the intercept of $\psi(t)$ with the s axis. x denotes the change (decrease) in the input leading to a value $v(t+\Delta t)$ at time $t+\Delta t$. The area of the shaded triangle is a measure of the change in the output value of the Preisach operator from time t to time $t+\Delta t$. In the language of functional analysis that we adopted before, we are studying continuous functions $v_1(\cdot)$, $v_2(\cdot)$ defined on the interval $[0, t+\Delta t]$, with $v_1(\tau) = v_2(\tau)$ for $\tau \in [0,t]$;$v_1(\tau) = v_1(t)$ for $\tau \in [t, t+\Delta t]$ while $v_2(\tau) = v_1(t) + \frac{\tau - t}{\Delta t} x$.

As one can observe, the area of the triangle is proportional to x^2. Now for the sake of illustration, choose the interval I so that the corresponding region of the Preisach plane to which the allowed inputs belong ($= \{(s-r) \in I\} \cup \{(s+r) \in I\}$) is a strict subset of the support of the density function ω. Then $\inf \omega(r,s)$ is strictly bounded away from zero over the compact set of allowed input values in the Preisach plane. Denote this value by μ. Therefore the change in the output value of the Preisach operator is $> \mu\, \frac{x^2}{2}$.

To extend Theorem 3.1 to the Preisach case when the operator $\mathcal{W}$ is incrementally strictly increasing, but there does not exist a $k_1 > 0$ such that $k_1(x_1 - x_2) \leq \mathcal{W}(x_1) - \mathcal{W}(x_2)$ for $(x_1 - x_2) > 0$, we make the key observation that given an $\epsilon > 0$ we can find $k_1 > 0$ such that $\xi_I(x) \geq k_1 x$ if $x \geq \epsilon$. This is because the facts:

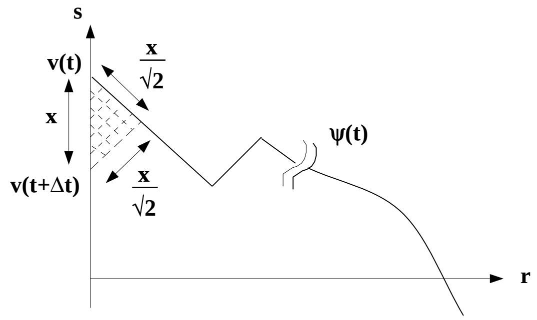

Figure 5: Graphical illustration of the incrementally strictly increasing property.

- the ratio $\frac{\xi_I(x)}{x}$ is a well-defined continuous function of x;
- $x \geq \epsilon$ and bounded from above because of the restriction on the input magnitudes

imply that $\inf \frac{\xi_I(x)}{x}$ is well-defined and greater than zero. □

Theorem 4.1 *Let $X = C_I[0,T]$, where $I = [a,b]$. Let $F : X \to Y$, where Y is the range of F. Let F be an incrementally strictly increasing operator with constants $k_1,\ k_2$ as defined in Definition 3.4. Let $\epsilon > 0$ and the operator equation*

$$F(x) = y \tag{15}$$

where $y \in Y$, be given. Consider the algorithm:

- $x_0 \in X$;
- *while* $\|x_n - x_{n-1}\| \geq \epsilon$:
$$x_{n+1} = \frac{1}{k_2}\left(y + (k_2 x_n - F(x_n))\right). \tag{16}$$

The sequence $\{x_n\}$ terminates at z which satisfies $\|z - x\| \leq \epsilon$ where x is the solution of (15). The rate of convergence is linear.

Proof The proof is essentially the same as that of Theorem 3.1. The one difference is that we need to show that x_{n+1} also belongs to X when x_n belongs to X. This is because the definition of X now only allows for continuous functions on $[0,T]$ with values in $[a,b]$.

That $x_{n+1} \in C[0,T]$ is obvious. Next, suppose $F(v) = y$ where $v \in X$. (We do not worry here about whether v is unique or not. Later it will become obvious that it must be.) Suppose that $x_n \geq v$ so that $F(x_n) \geq y$. Consider $(x_{n+1} - v)$ and $(x_n - v)$. We show that $0 \leq (x_{n+1} - v) \leq (x_n - v)$. The second inequality is true because:

$$\begin{aligned} k_2\,(x_{n+1} - v) - k_2\,(x_n - v) &= k_2\,(x_{n+1} - x_n), \\ &= y - F(x_n) \\ &\leq 0. \end{aligned}$$

We need to prove the first inequality. Now,

$$\begin{aligned} k_2\,(x_{n+1} - v) &= k_2\,(x_n - v) + y - F(x_n) \\ &\geq 0 \end{aligned}$$

since $x_n \geq v$. Similarly we can show that $(x_n - v) \leq (x_{n+1} - v) \leq 0$ if $x_n \leq v$. Thus by Definition 3.3 and Example 3.1,

$$\|x_{n+1} - v\| \leq \|x_n - v\| \tag{17}$$

when $x_n \geq v$ or $x_n \leq v$. For the case when $x_n \not\geq v$ or $x_n \not\leq v$, we can divide the interval $[0,T]$ into regions where $x_n \geq v$, $x_n = v$ and $x_n \leq v$ respectively. Then applying the same arguments as before, we show that Equation 17 holds whenever x_n and v belong in X. Equation (17) further implies that x_{n+1} belongs in X. Now observe that if the algorithm 16 is continued with $\epsilon = 0$, then by the proof of Theorem 3.1, $x_n \to v$ and v must be unique.

If $\epsilon > 0$, the sequence x_n terminates at some z that satisfies $\|z - v\| \leq \epsilon$. □

It is important to note that the condition $F(x_1) - F(x_2) \leq k_2\,(x_1 - x_2)$ does not follow from the Lipschitz continuity of F : $\|F(x_1) - F(x_2)\| \leq k_2\,\|x_1 - x_2\|$ and $F(x_1) - F(x_2) \geq k_1\,(x_1 - x_2)$.

Example 4.1 $X = C[0,1]$. *Let* $a(t) = t + 1$, *and* $b(t) = \frac{3}{2}t + \frac{1}{2}$. *Then* $b \geq \frac{1}{2}a \gg 0$ *and* $\|b\| \leq \|a\|$ *but* $b \not\leq a$.

However if the operator F satisfied the following stronger Lipschitz continuity condition

$$\sup_{s\in[0,t]} |F(x_1)(s) - F(x_2)(s)| \leq k_2 \sup_{s\in[0,t]} |x_1(s) - x_2(s)| \quad \forall\ t \in [0,T] \tag{18}$$

then we can conclude $F(x_1) - F(x_2) \leq k_2\,(x_1 - x_2)$. This fact can be proved as follows.

Lemma 4.2 *Let $X = C[0,T]$. If the incrementally strictly increasing operator $F : X \to X$ satisfies (18), where $x_1,\ x_2 \in X$ then for $x_1 \geq x_2$ we have*

$$F(x_1) - F(x_2) \leq k_2\,(x_1 - x_2) \tag{19}$$

Proof Obviously, for $T = 0$ we have the result. For $T > 0$, let $t \in (0,T]$ and $\{t_n\}$ a strictly increasing sequence converging to t, with $t_n < t$ for all $n \in \mathbb{N}$. From (18) we conclude that

$$\sup_{s\in[0,t_n]} (F(x_1)(s) - F(x_2)(s)) \leq k_2 \sup_{s\in[0,t_n]} (x_1(s) - x_2(s)) \tag{20}$$

and

$$\sup_{s\in[0,t]} (F(x_1)(s) - F(x_2)(s)) \leq k_2 \sup_{s\in[0,t]} (x_1(s) - x_2(s)) \tag{21}$$

Equation (21) implies

$$\begin{aligned} &\max\{\text{LHS of (20)}, \sup_{s\in[t_n,t]} (F(x_1)(s) - F(x_2)(s))\} \\ \leq \quad & k_2 \max\{\text{RHS of (20)}, \sup_{s\in[t_n,t]} (x_1(s) - x_2(s))\} \end{aligned} \tag{22}$$

Equations (20) and (22) imply

$$\sup_{s\in[t_n,t]} (F(x_1)(s) - F(x_2)(s)) \leq k_2 \sup_{s\in[t_n,t]} (x_1(s) - x_2(s))$$

Taking the limit as $t_n \to t$ we obtain

$$F(x_1)(t) - F(x_2)(t) \leq k_2 (x_1(t) - x_2(t)) \tag{23}$$

As (23) is true for all $t \in [0, T]$ we get the claim. □

Theorem 4.2 *Let $X = C_I[0, T]$, where $I = [a, b]$. Let $\mathcal{W} : X \to Y$, where $\mathcal{W}$ is a strictly increasing, strongly Lipschitz continuous Preisach operator (with Lipschitz constant k_2), some initial memory curve ψ_{-1}, and Y is the range of $\mathcal{W}$. Let $\epsilon > 0$ and the operator equation*

$$\mathcal{W}(x) = y \tag{24}$$

where $y \in Y$, be given. Consider the algorithm:

- $x_0 \in X$;
- *while* $\|x_n - x_{n-1}\| \geq \epsilon$:

$$x_{n+1} = \frac{1}{k_2} (y + (k_2 x_n - \mathcal{W}(x_n))). \tag{25}$$

The sequence $\{x_n\}$ terminates at z which satisfies $\|z - x\| \leq \epsilon$ where x is the solution of (24). The rate of convergence is linear.

Proof The only obstacle to the direct application of Theorem 4.1 is the fact that $\frac{\xi_I(x)}{x} \to 0$ as $x \to 0$ where $I = [a, b]$. But as pointed out in Remark 4.1, we can find $k_1 > 0$ such that $\xi_I(x) \geq k_1 x$ if $x > \epsilon$. Thus for $x > \epsilon$ we apply Theorem 4.1. □

4.2 The Moving Preisach model

An important modification of the Preisach model that is extensively used in the Magnetics community is the moving Preisach model. The output function w of the moving model is defined implicitly through the equation

$$w = \mathcal{W}[v + \alpha w] \tag{26}$$

where $\mathcal{W}$ is a Preisach operator and $\alpha > 0$. In magnetics, the input v is the average magnetic field while the output w is the average magnetization. A modification of the input as in Equation (26) is to account for exchange interactions in a ferro-magnet in a bulk model. In the context of bulk modeling in magnetism (as opposed to micromagnetics where one encounters the exchange interaction field), it is related to the "molecular-field" hypothesis originally due to Weiss [6]. In the context of Preisach operators, Della Torre seems to have proposed the moving model modification first [1].

It is easy to see that if $\mathcal{W}$ is incrementally strictly increasing we can apply Theorem 4.1 to approximately solve Equation (26). Let z be the solution obtained by applying the algorithm outlined in Theorem 4.1. Then we see that

$$v = z - \alpha w \tag{27}$$

which can be computed since w and z are known.

5 Numerical simulations

In this section, we show the results of simulations when the algorithm described in Section 4 is applied to the Preisach operator. In reference to Figure 2 we assume both the linear system and the controller to be identity so as to highlight our work.

The Preisach plane was discretized with grid points at $(-2.995 + 0.01m, -2.995 + 0.01n)$ with $1 \leq m \leq n \leq 599$. Note that the co-ordinates are given in terms of $\alpha = s + r$ and $\beta = s - r$ respectively. The measure is taken to be uniform with value 0.01 at the grid points for the sake of the discussion.

The desired signal was taken to be output of a forced Van der Pol oscillator. The equation of the system is

$$\ddot{x} + p(x^2 - 1)\dot{x} + \omega_0^2 x = f \sin(\omega t). \tag{28}$$

The values of the parameters are $p = f = 5$, $\omega_0^2 = 1$ and $\omega = 2.466$. The desired output for the Preisach operator was taken to be x. The above equation was integrated using the standard Runge-Kutta method with time step 0.02; initial time 0 and final time 100. The error tolerance on the output was taken to be 1.25% of the maximum value, which turns out to be about 2.25×10^3 as the maximum output is 1.8×10^5.

Figure 6 shows the output of the desired signal and the actual output of the Preisach operator.Figure 7 shows the difference between the desired and the actual signals. One can see that the error is within the required bounds. Figure 8 shows the output versus the actual output.

6 Discussion of Results and Conclusion

The most important difference between our algorithm and prior approaches [4, 5] is that we have proved that the error between actual output and the desired output, is **guaranteed** to be within a certain specified tolerance. We have also shown that we can improve the tolerance by simply discretizing the input signal in a finer way.

The next important difference is more subtle. If one uses a linearizing approach to inversion [4, 5] then we have to make sure that the variation in the desired output is small, so that the linearization method remains valid. This in turn means that the desired output signal must be discretized *by value*. This further implies that the sampling times will be irregular. On the other

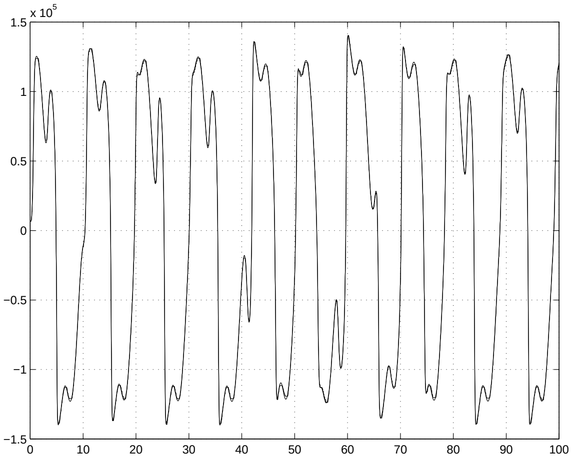

Figure 6: Desired versus actual output signals.

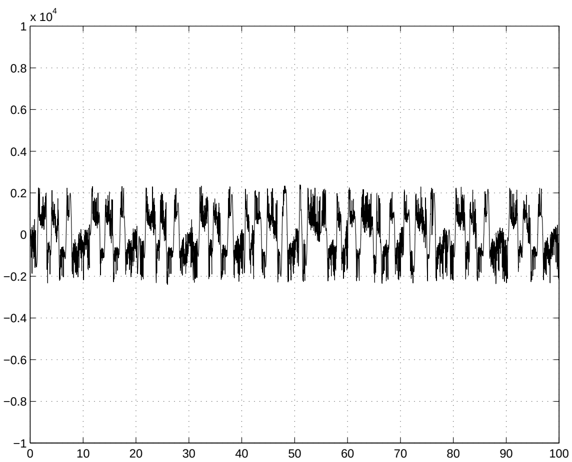

Figure 7: Difference between the desired and actual output signals.

hand, since our algorithm is non-linear and is not based on linearization, we are allowed to have regularly spaced sampling intervals in time. This is a highly desirable feature for real-time control purposes.

In conclusion, we have proposed a novel algorithm for the inverse of a Preisach operator. Such operators are very useful in describing the hysteretic relationship between the input and output variables in a smart structure. The contribution of this paper is that now the problem of designing controls for systems with hysteresis as shown in Figure 1 is reduced to a control design problem for a linear system.

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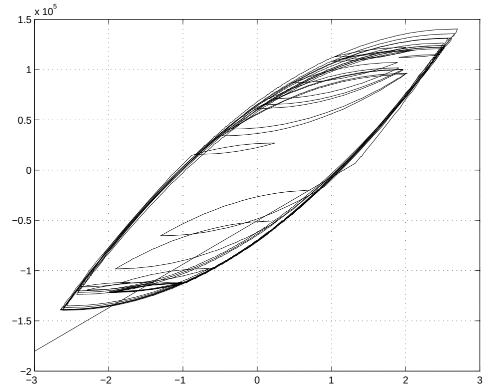

Figure 8: Output versus Input graph shows hysteresis.

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