A Short Proof of a Result on Polynomials

by Răzvan Gelca

In this note we want to present a short proof of a result that appeared in [1]. For a polynomial $f(x) = \prod_1^n (x - x_i)$, with distinct real roots $x_1 < x_2 < \cdots < x_n$, we let $d = \delta(f) = min_i(x_{i+1} - x_i)$ and $g(x) = f'(x)/f(x) = \sum_1^n 1/(x - x_i)$. If k is a real number then the roots of the polynomial $f' - kf$ are also real and distinct.

PROPOSITION. *If for some j, y_0 and y_1 satisfy $y_0 < x_j < y_1 \leq y_0 + d$ then y_0 and y_1 are not zeros of f and $g(y_0) < g(y_1)$.*

PROOF: The hypothesis implies that for all i, $y_1 - y_0 \leq d \leq x_{i+1} - x_i$. Hence for $1 \leq i \leq j-1$ we have $y_0 - x_i \geq y_1 - x_{i+1} > 0$ and so $1/(y_0 - x_i) \leq 1/(y_1 - x_{i+1})$; similarly for $j \leq i \leq n-1$ we have $y_1 - x_{i+1} \leq y_0 - x_i < 0$ and again $1/(y_0 - x_i) \leq 1/(y_1 - x_{i+1})$.

Finally $y_0 - x_n < 0 < y_1 - x_1$, so $1/(y_0 - x_n) < 0 < 1/(y_1 - x_1)$, and the result follows by addition of these inequalities.

COROLLARY. $\delta(f' - kf) > \delta(f)$.

PROOF: If y_0 and y_1 are zeros of $f' - kf$ with $y_0 < y_1$ then they are separated by a zero of f and satisfy $g(y_0) = g(y_1) = k$. Hence from the proposition we can not have $y_1 \leq y_0 + d$, so $y_1 - y_0 > d$ as required.

[1]. Walker, P., *Separation of the zeros of polynomials*, Amer. Math. Monthly (100)(1993)272–273.

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