REGULAR POLYGONS

Răzvan Gelca
Texas Tech University

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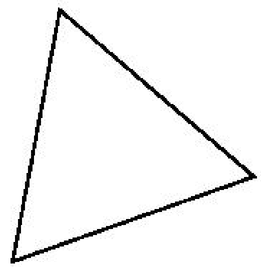

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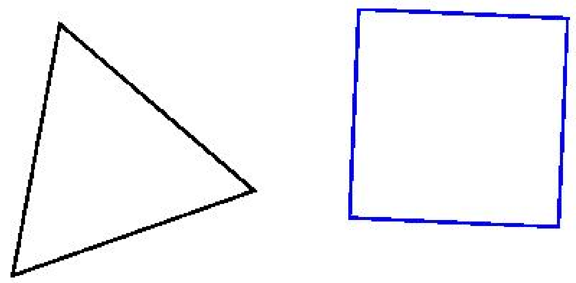

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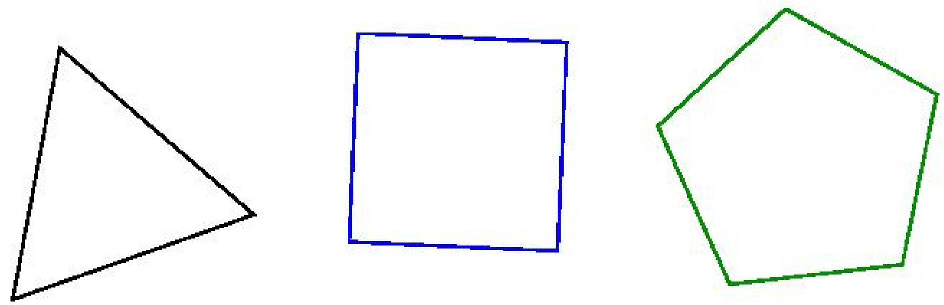

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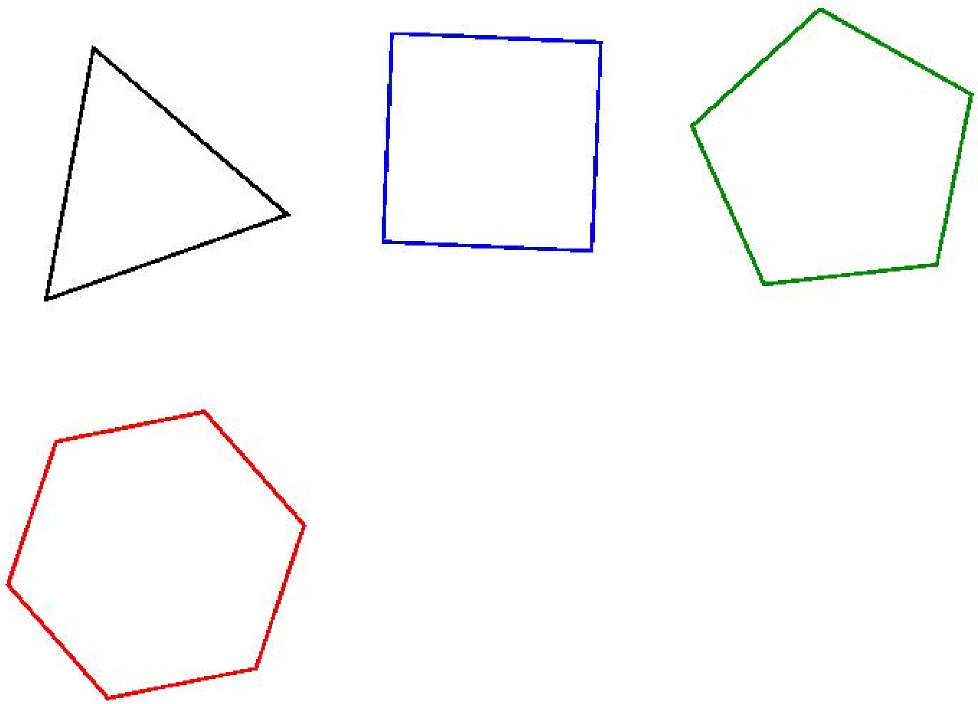

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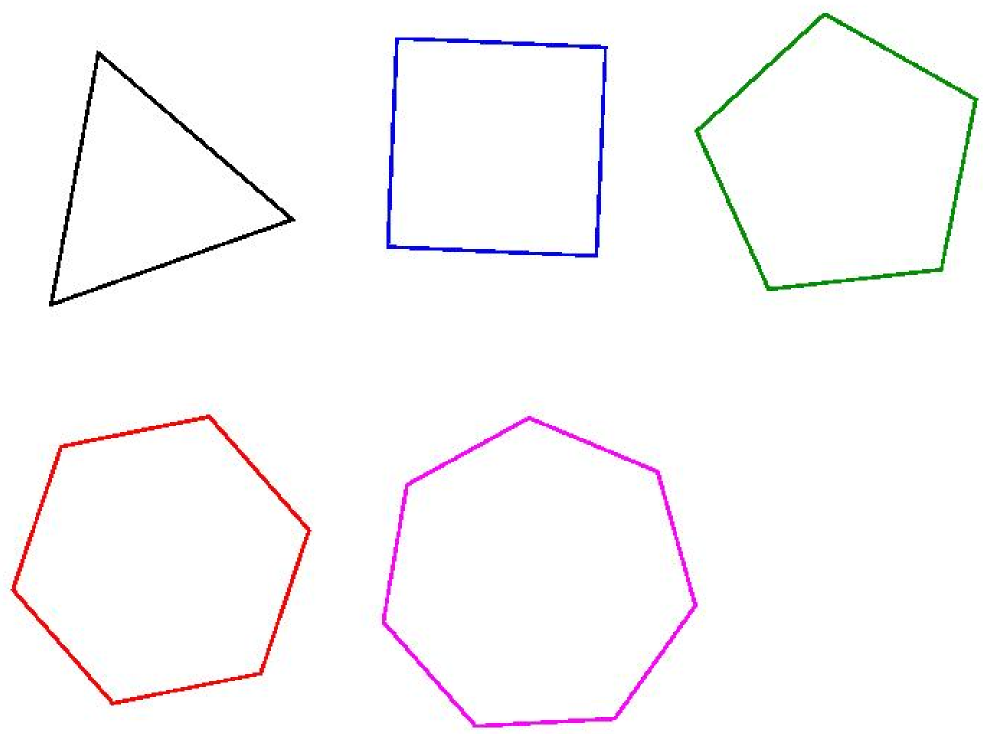

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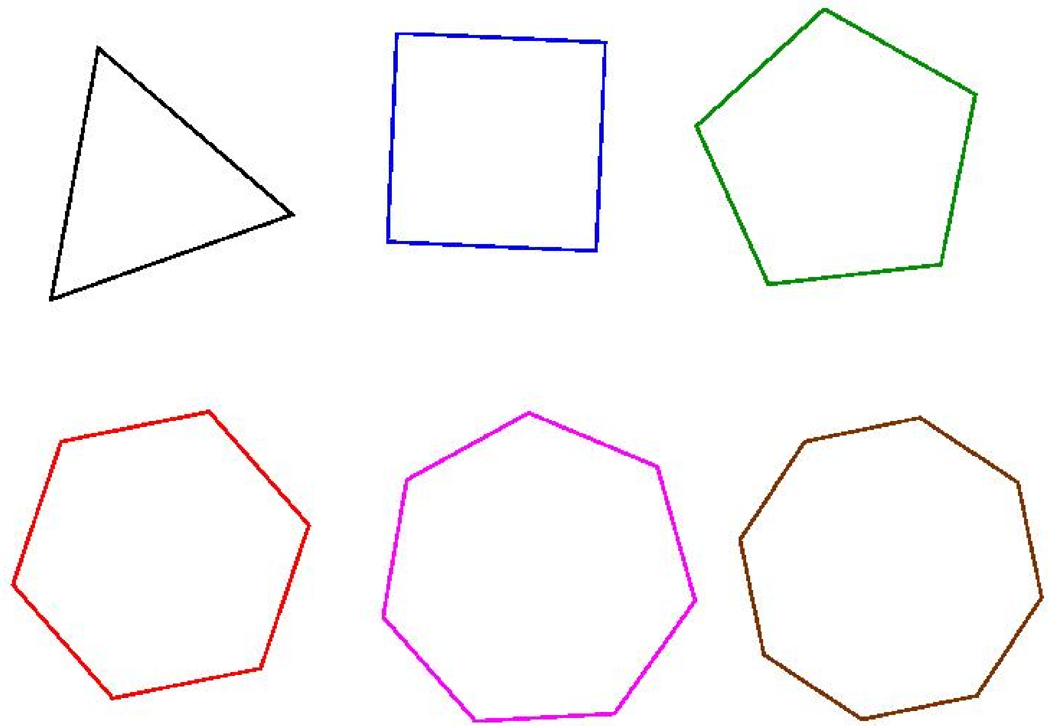

Construct an equilateral triangle.

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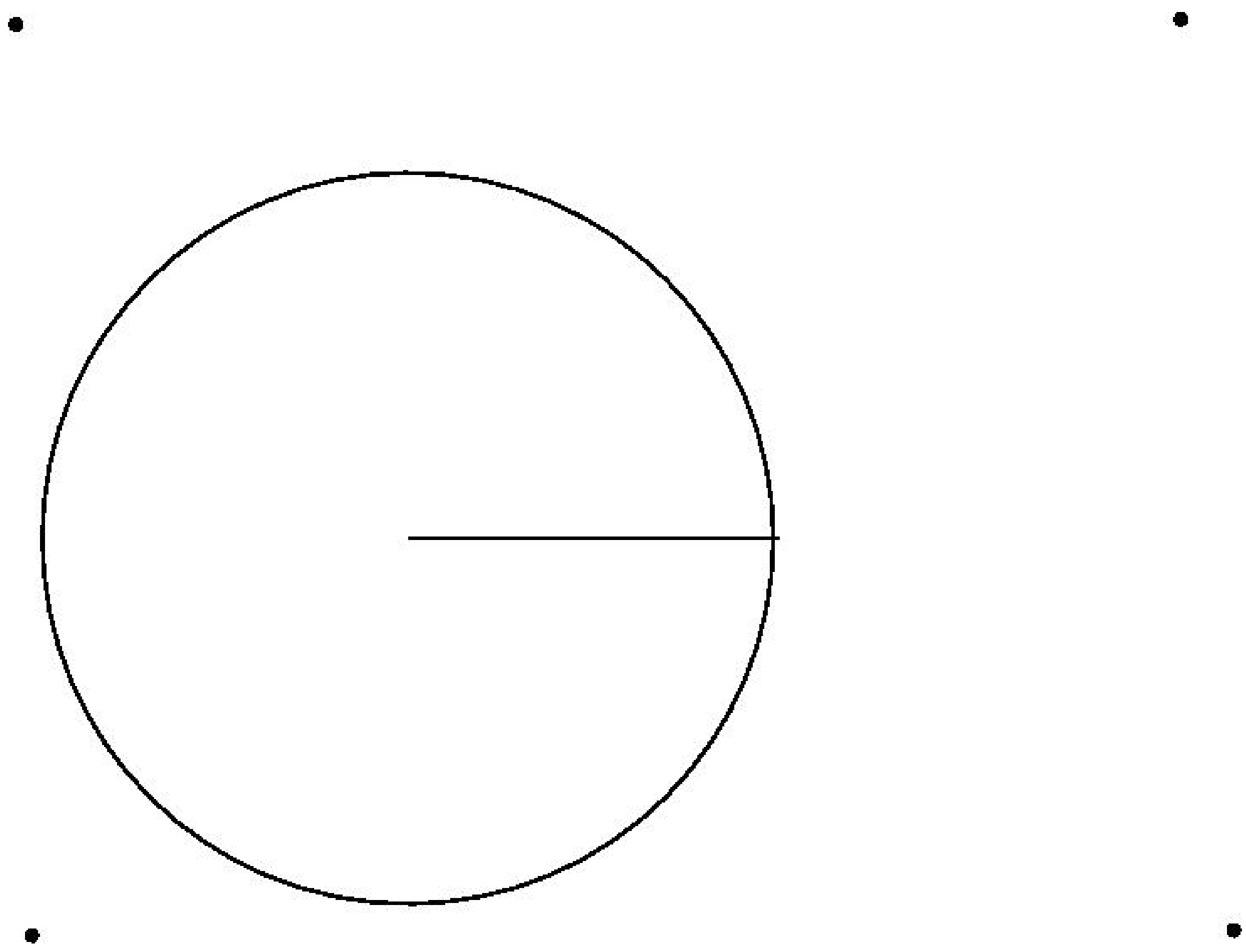

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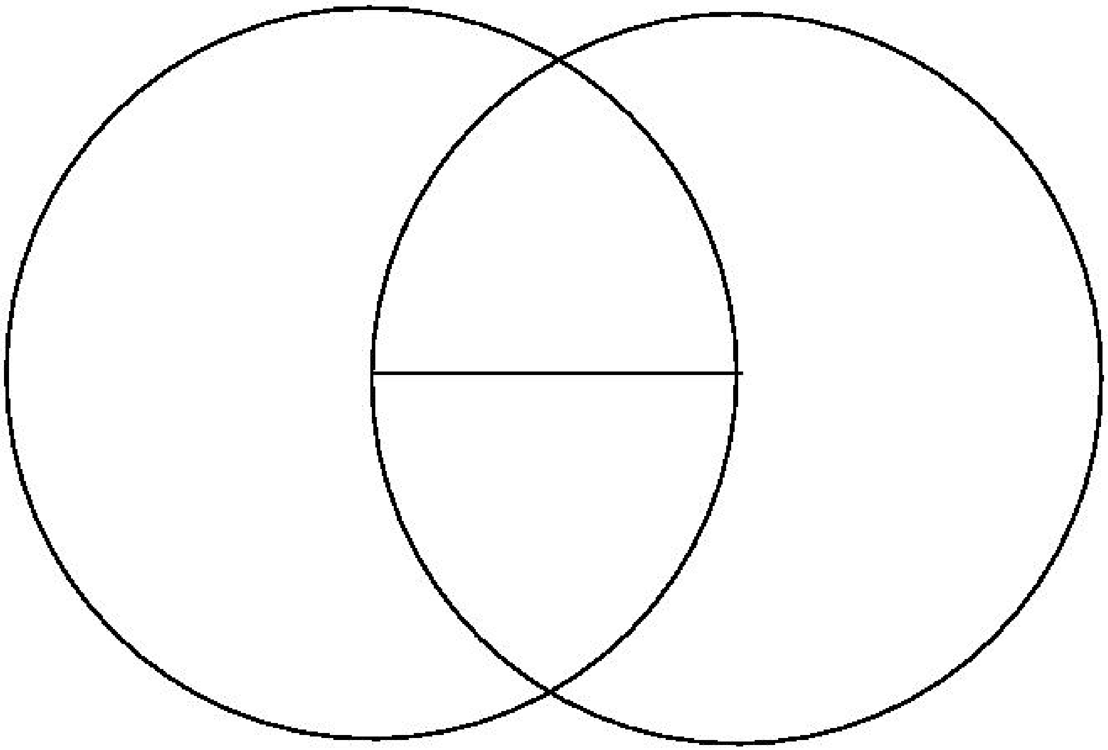

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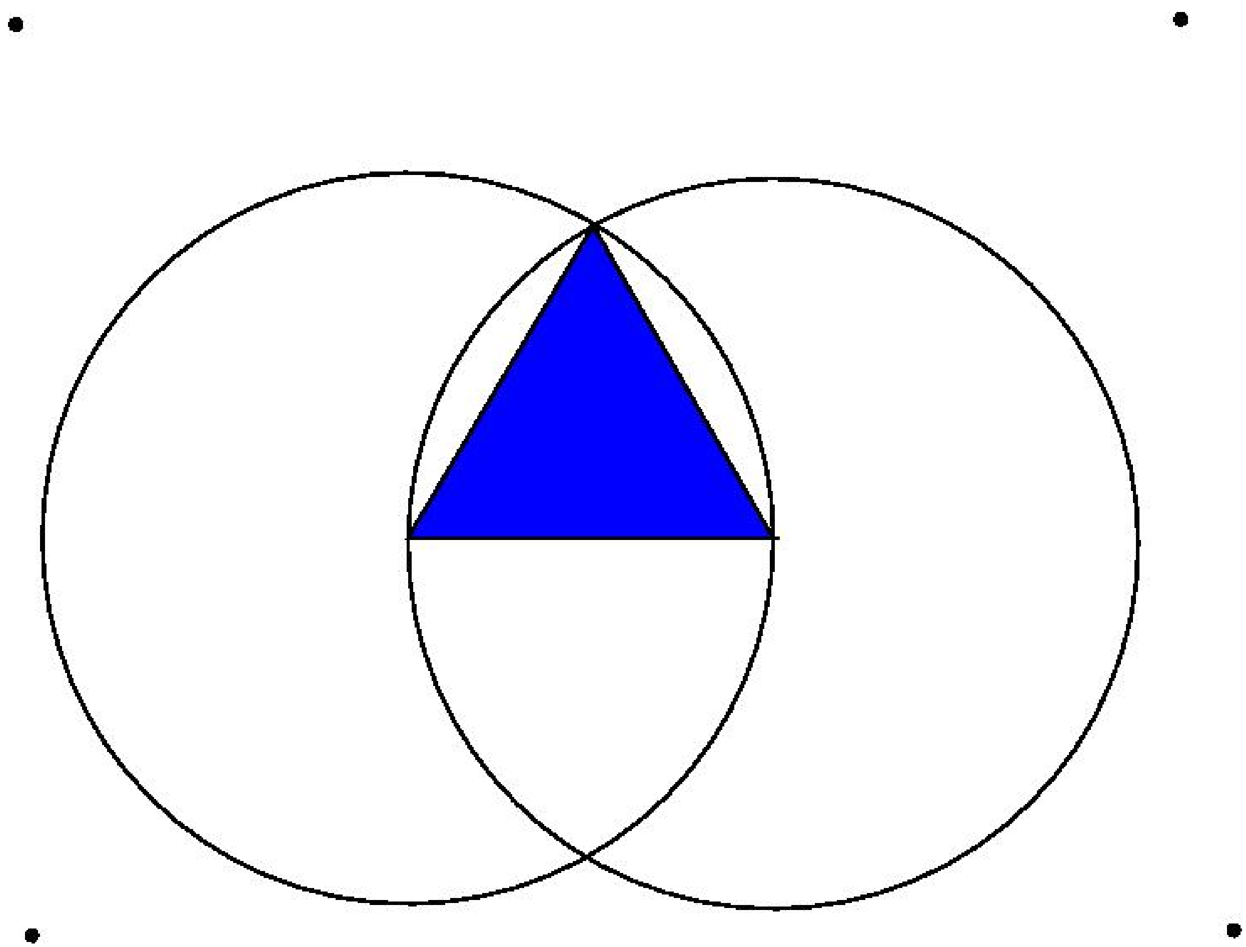

Construct a square.

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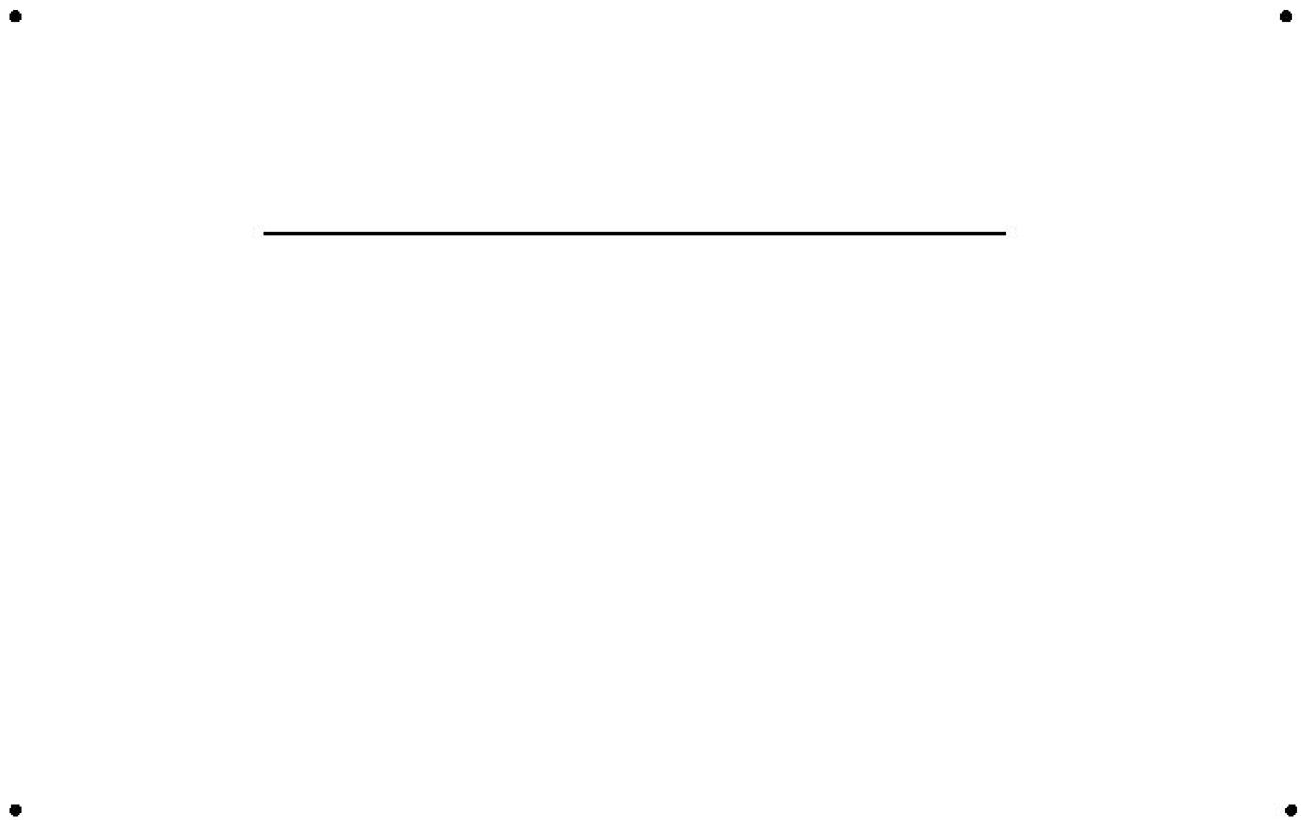

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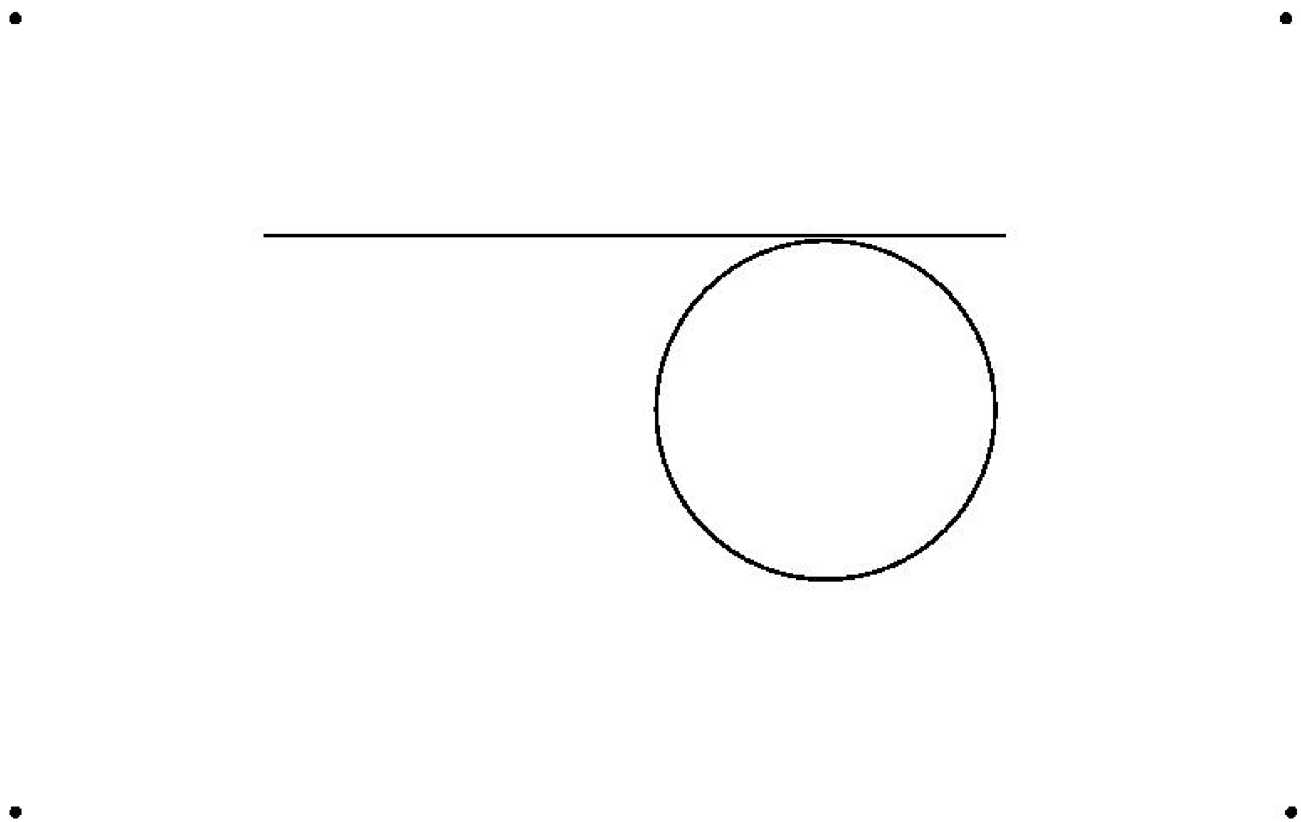

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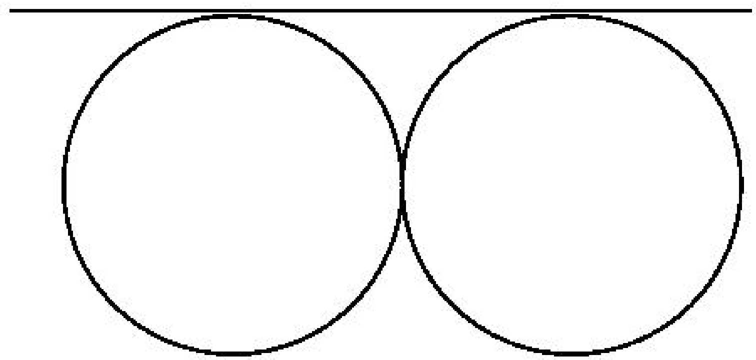

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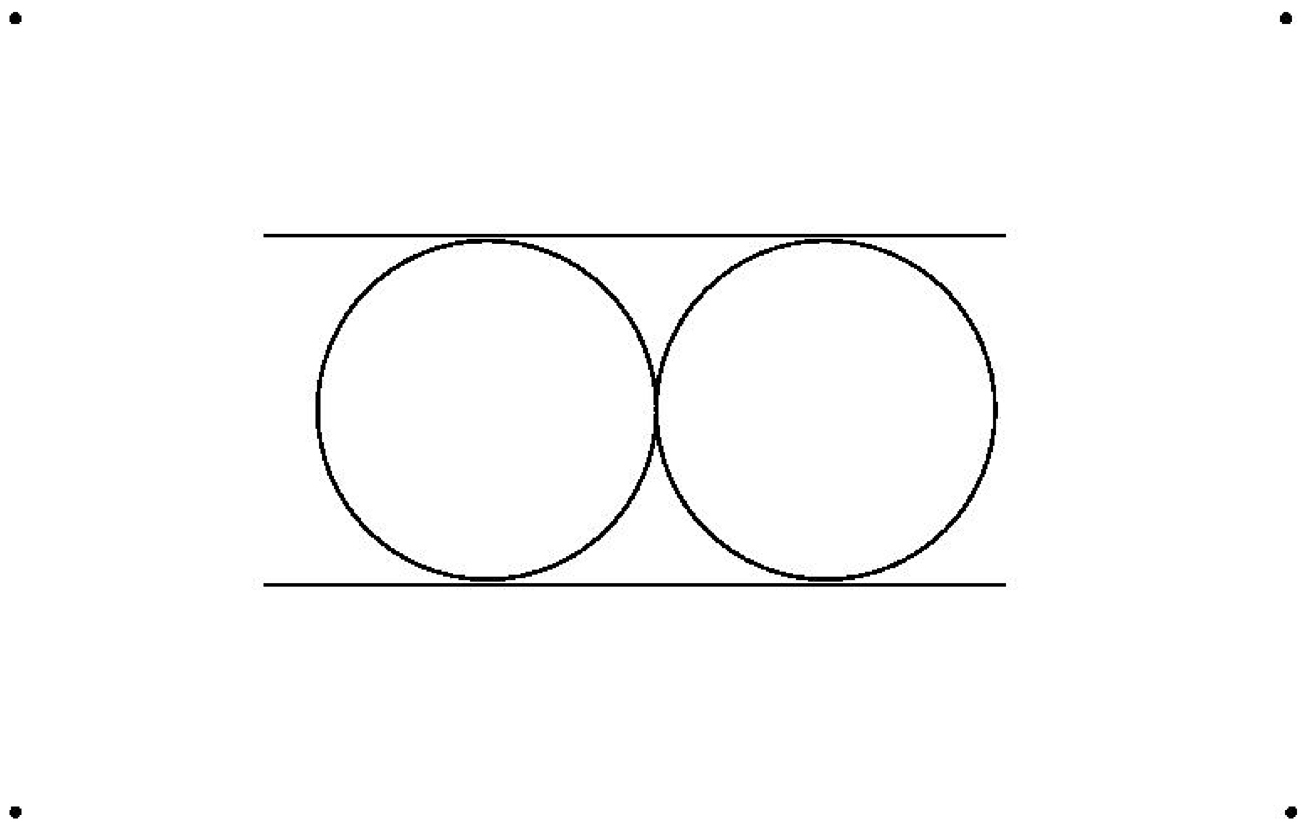

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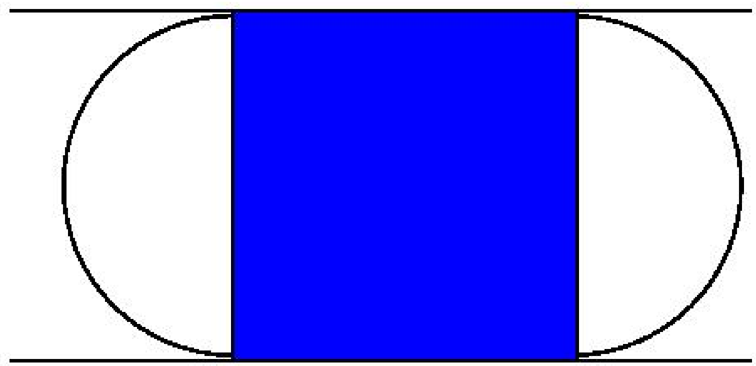
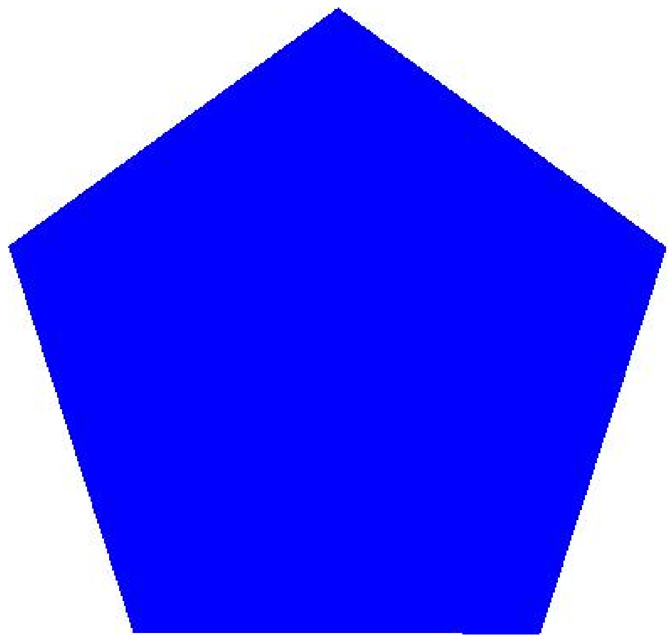


Construct a regular pentagon.



Theorem. (Gauss-Wantzel) *A regular polygon with n sides can be constructed if an only if the odd prime factors of n are distinct Fermat primes.*

This means that

$$n = 2^m(2^{2^{k_1}} + 1)(2^{2^{k_2}} + 1)\cdots(2^{2^{k_t}} + 1),$$

where each $2^{2^{k_i}} + 1$ is prime and the k_i's are distinct.

Examples:

- *regular pentagon* $5 = 2^{2^1} + 1$
- *regular heptadecagon* $17 = 2^{2^2} + 1$
- *regular polygon with* 2570 *sides*

Problem 0. *What regular polygons tesselate the plane?*

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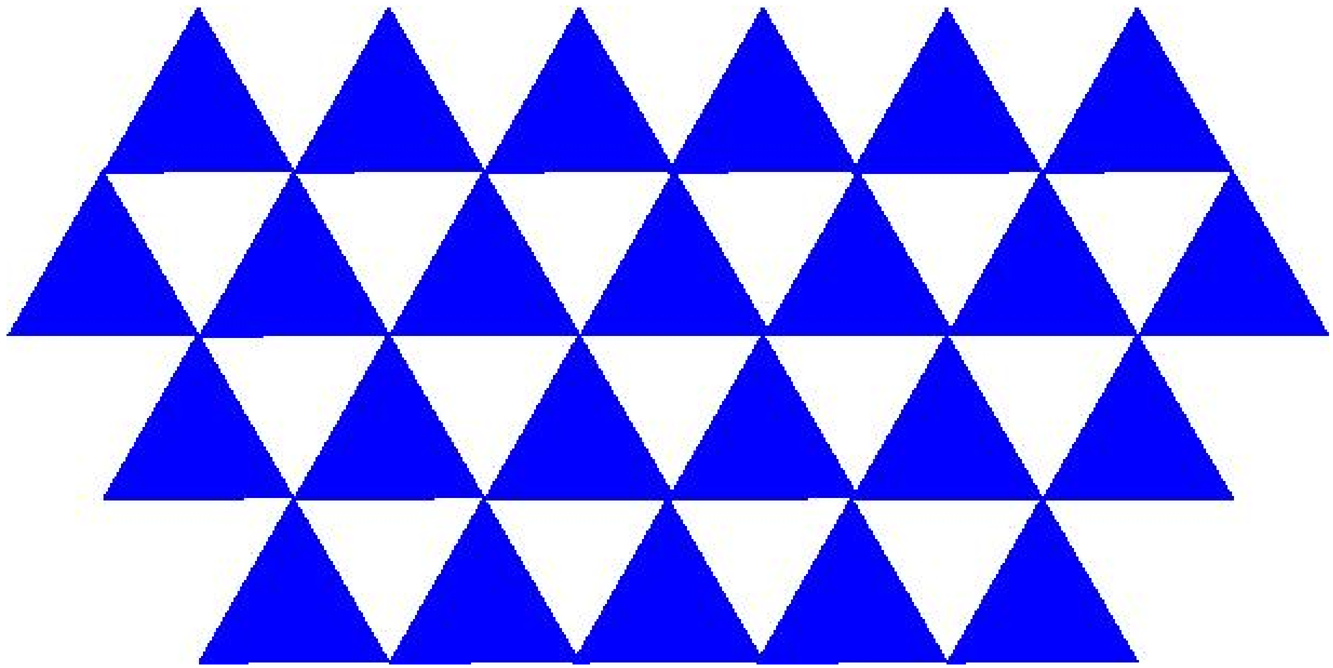

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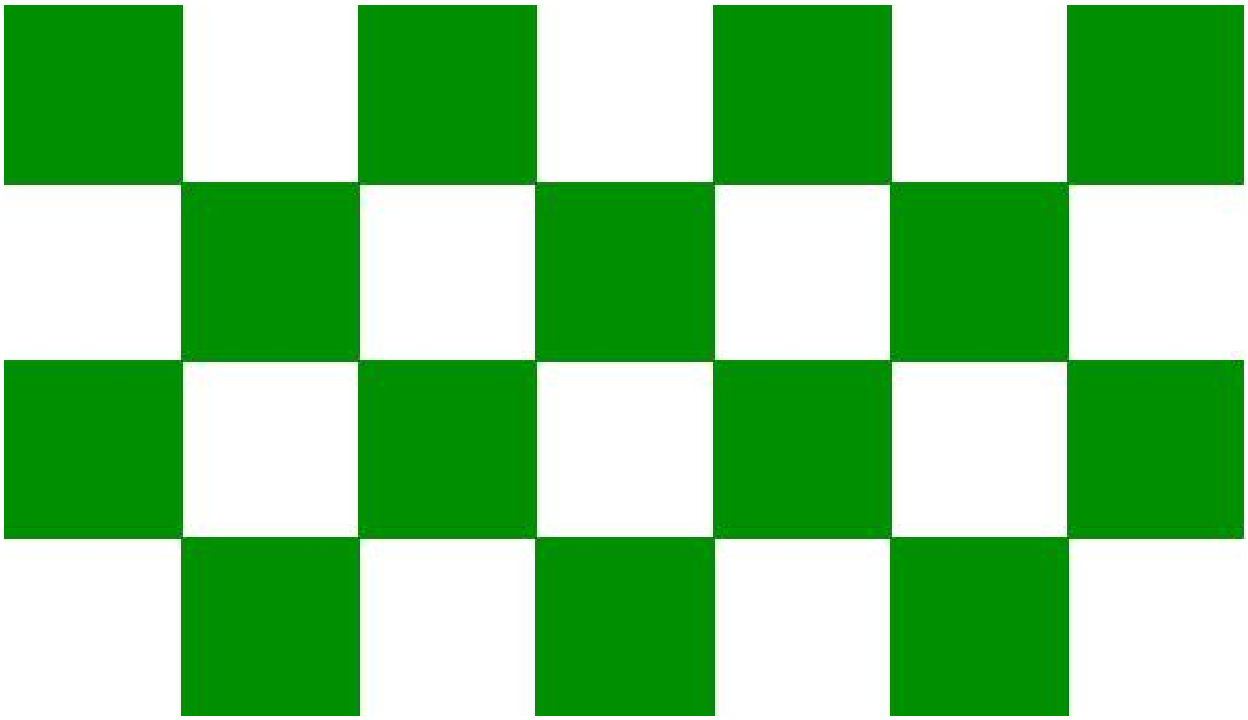
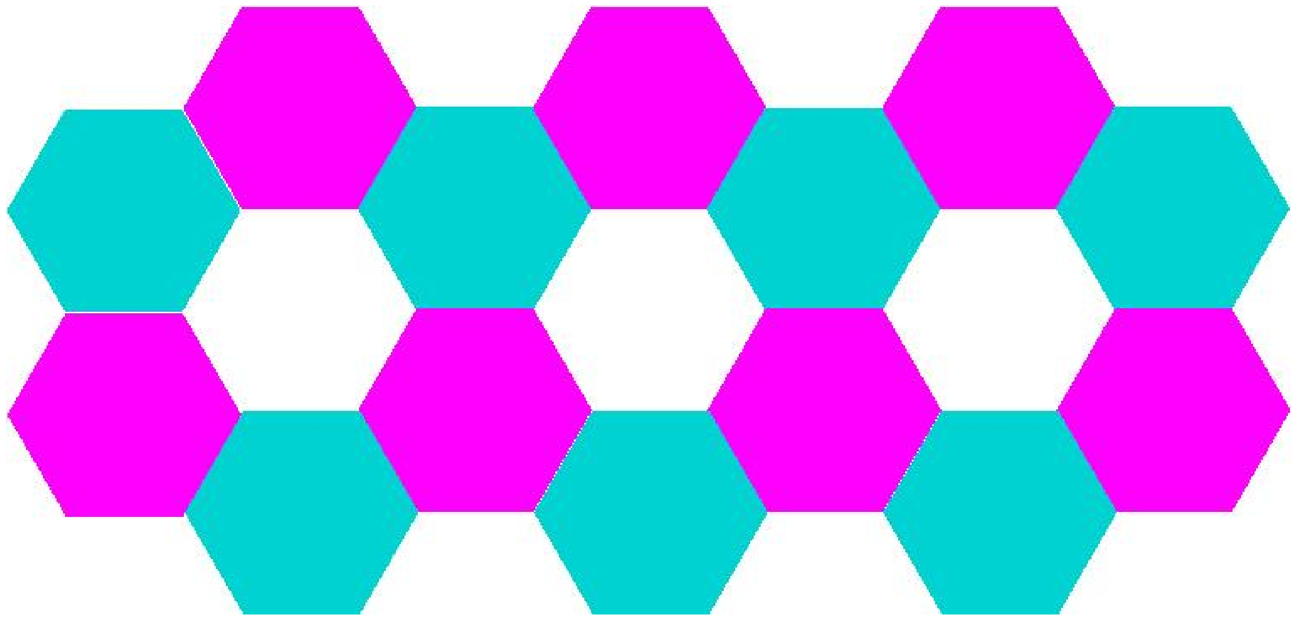


Problem 1. *What regular polygons tesselate the plane?*



Are there others?

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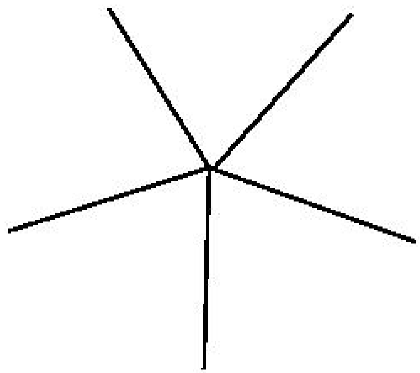
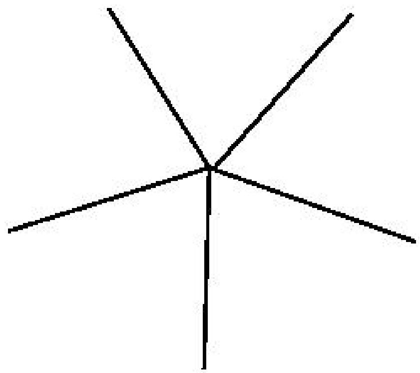


The angles that meet at a point should add up to 360°.

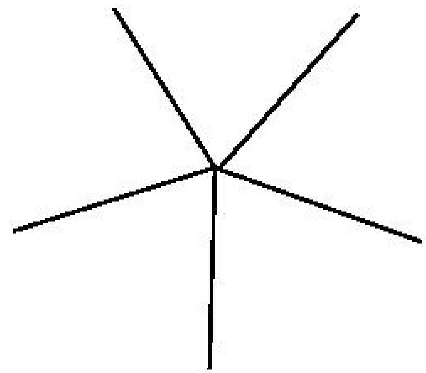
Are there others?



The angles that meet at a point should add up to $360°$.

The angles of a regular n*-gon are equal to* $\frac{n-2}{n} \times 180°$.

Are there others?

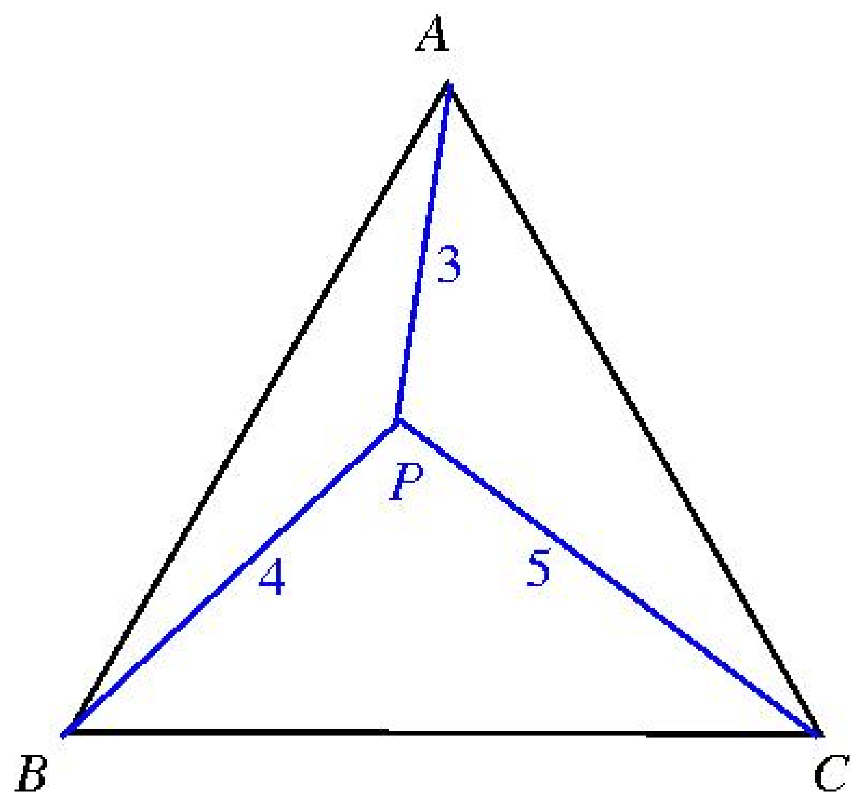


The angles that meet at a point should add up to 360°.

The angles of a regular n*-gon are equal to* $\frac{n-2}{n} \times 180^\circ$.

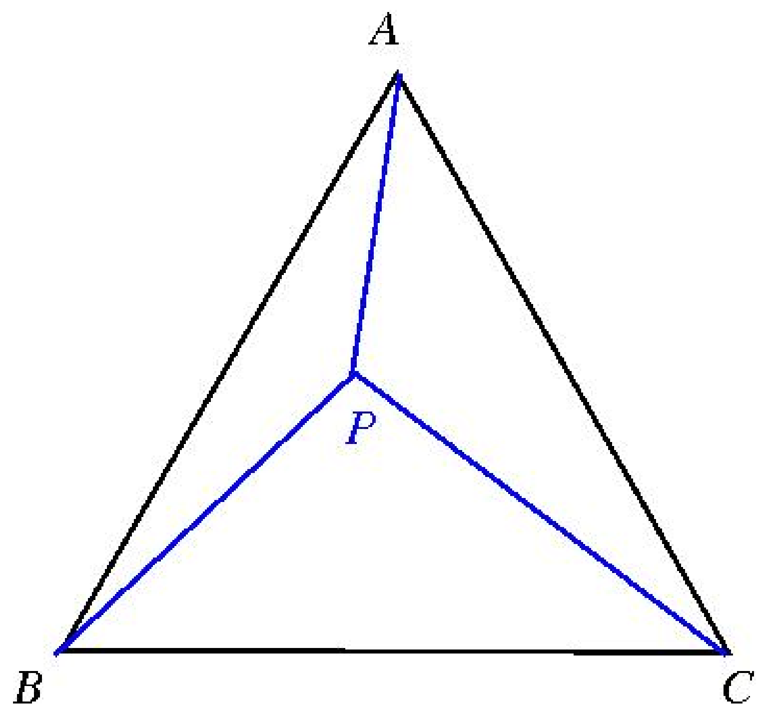
Hence $\frac{n-2}{n}$ *multiplied by some integer should equal* 2. *The equality* $(n-2)k = 2n$ *can only hold for* $n = 3, k = 6$; $n = 4$, $k = 4$; $n = 6, k = 3$.

Problem 2. *Let ABC be an equilateral triangle and P a point in its interior such that $PA = 3$, $PB = 4$, $PC = 5$. Find the side-length of the triangle.*

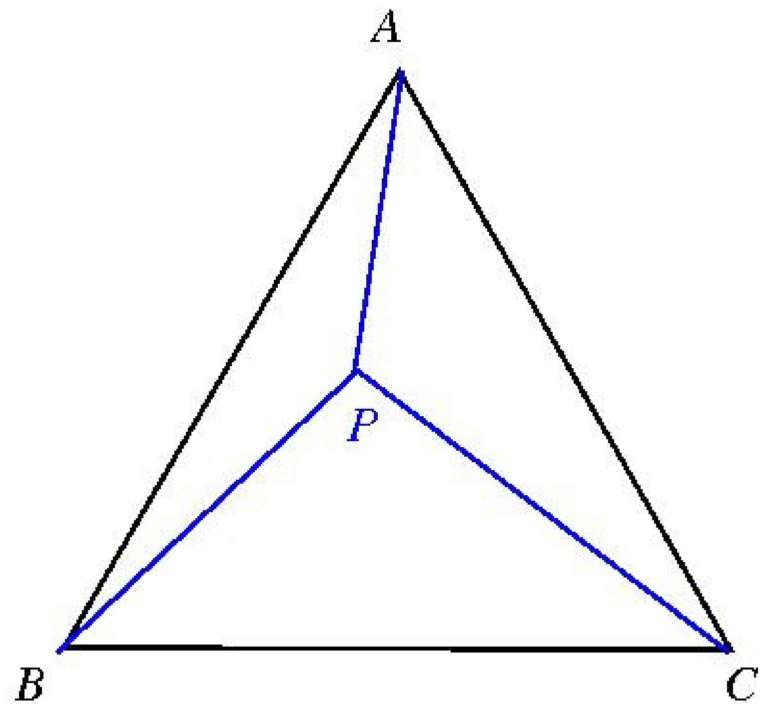


We will use the following result:

Pompeiu's Theorem. *Let ABC be an equilateral triangle and P a point in its plane. Then there is a triangle whose sides are PA, PB, PC.*



Here is the proof:



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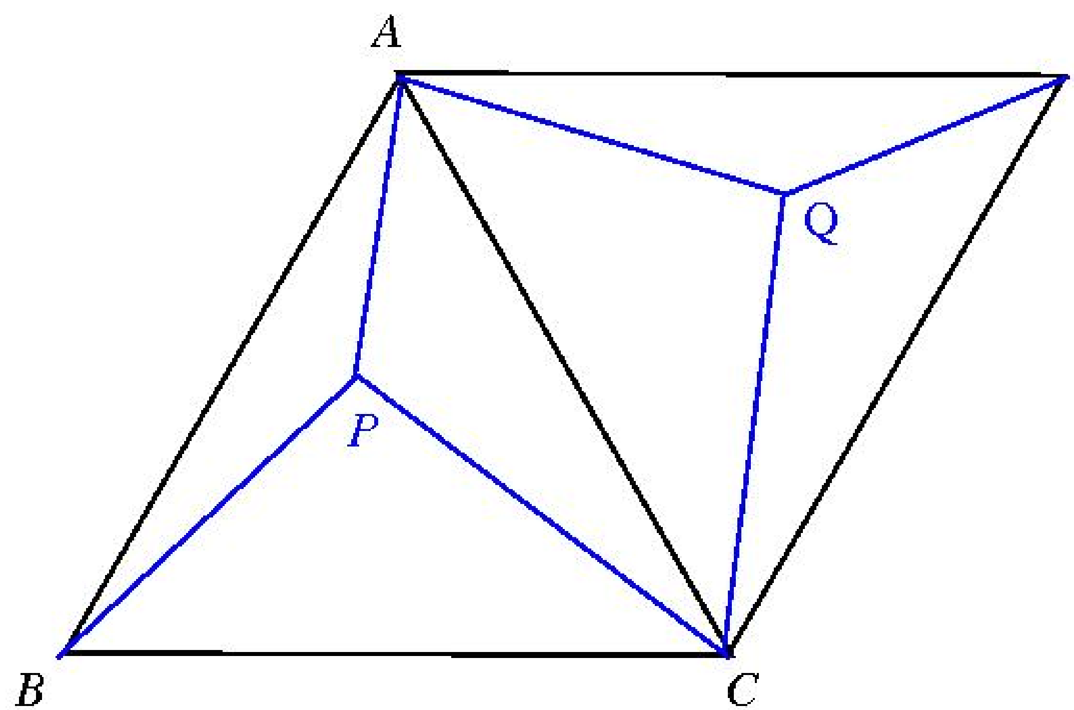

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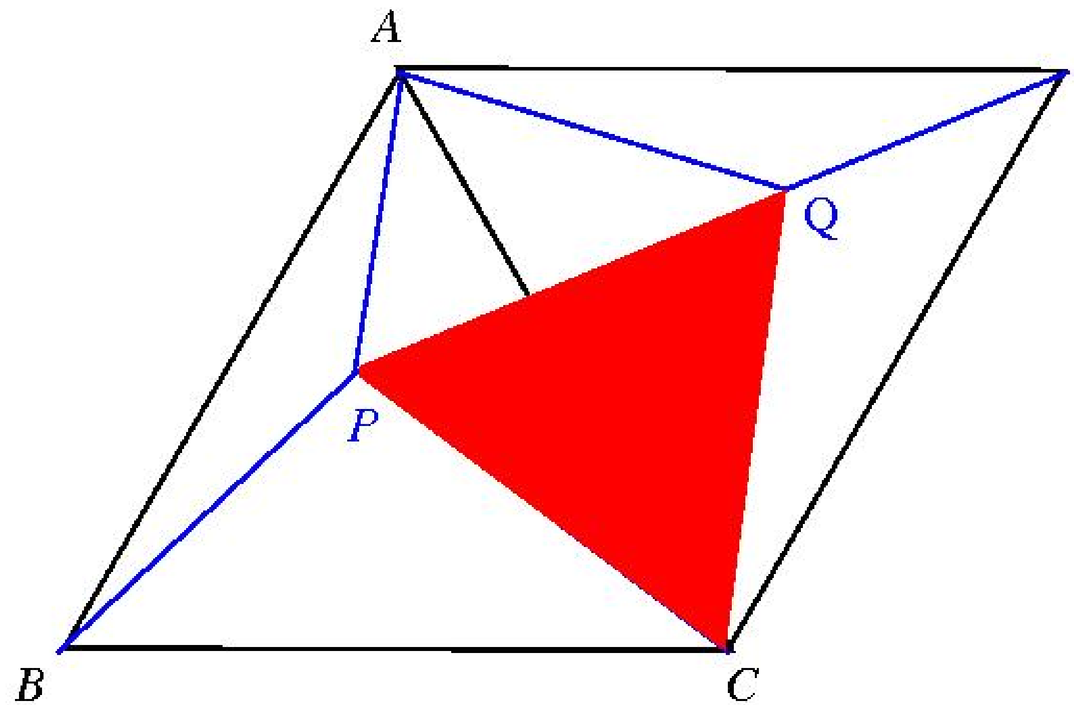

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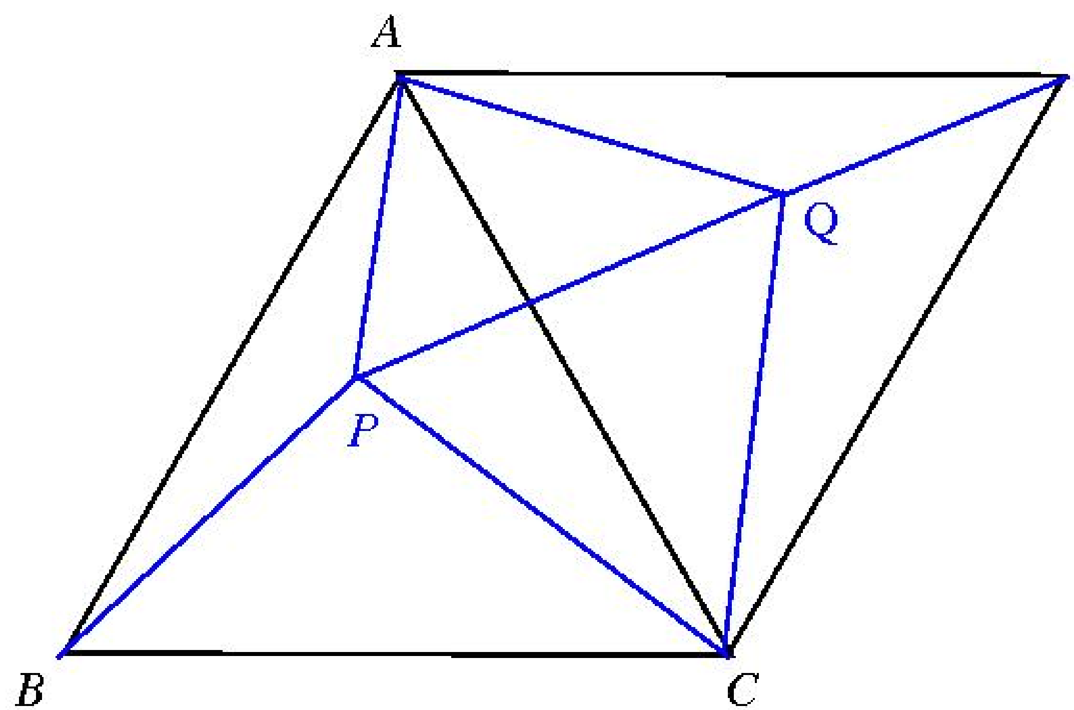

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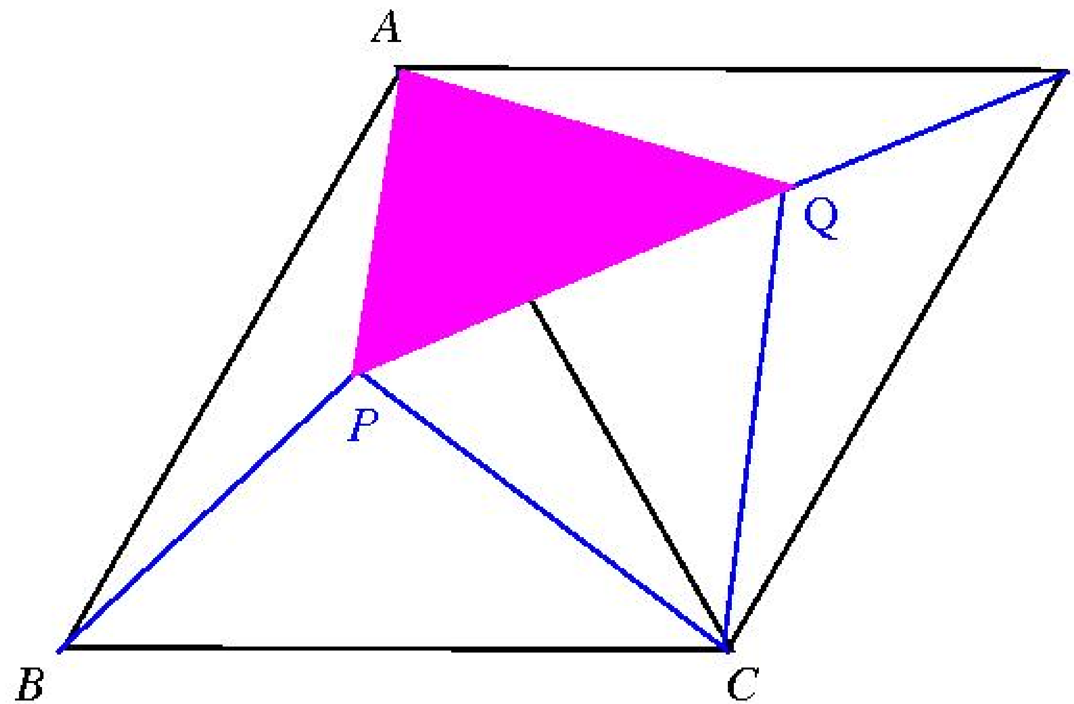
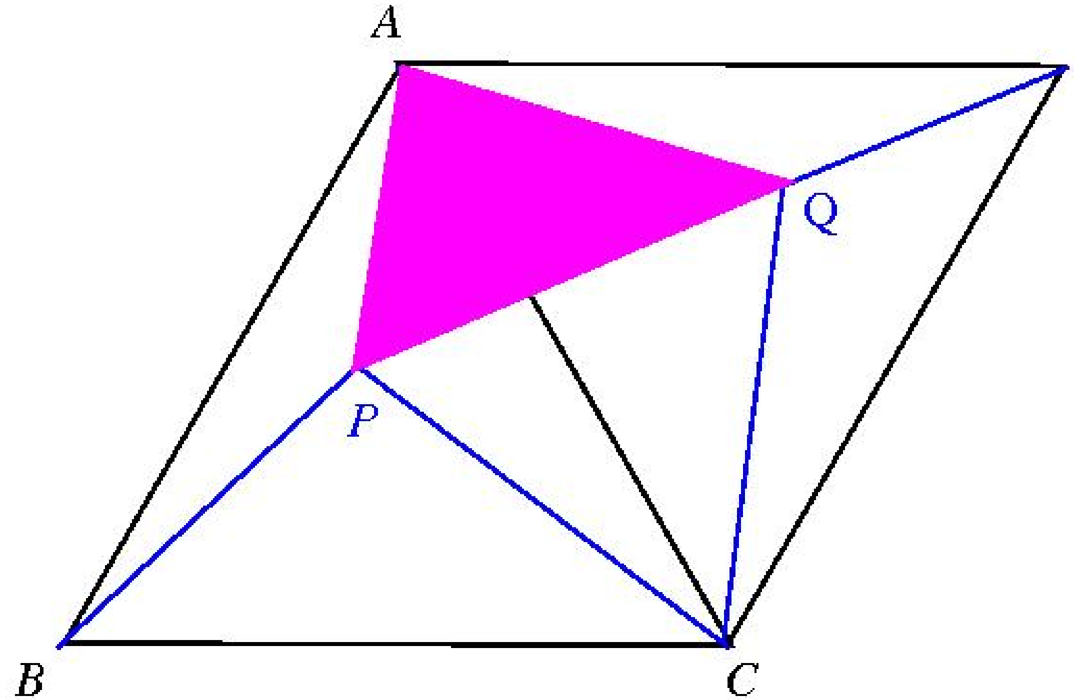


Let us return to the original problem:



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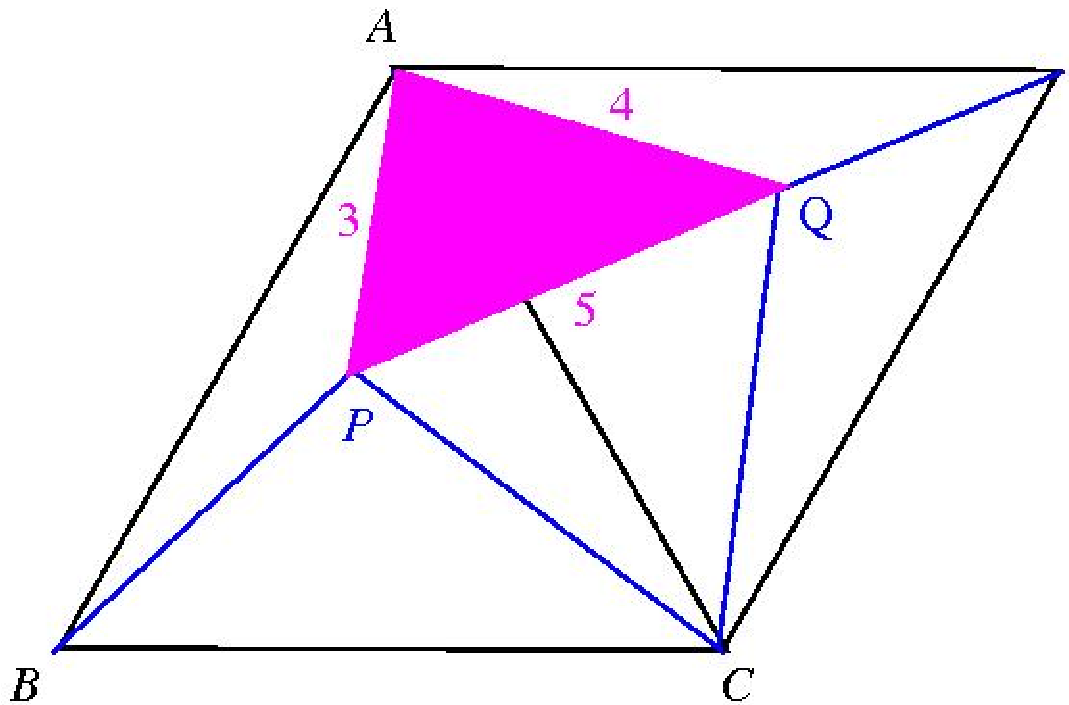

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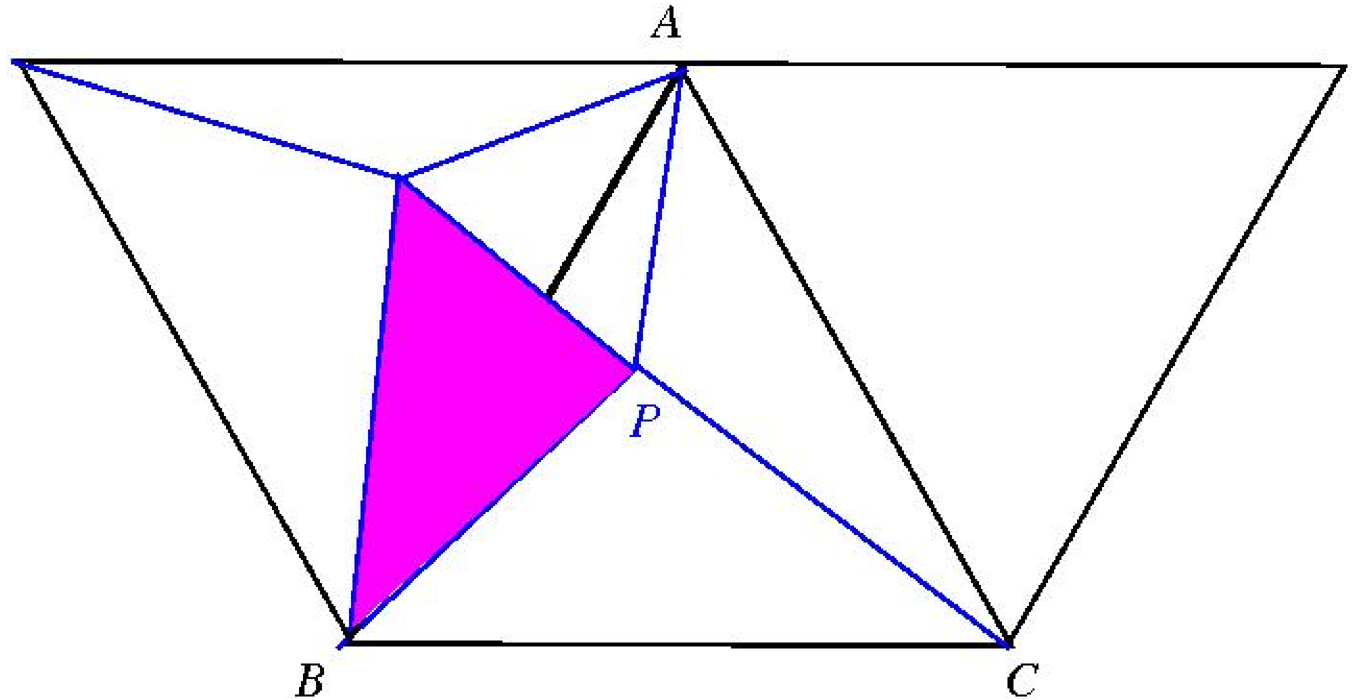

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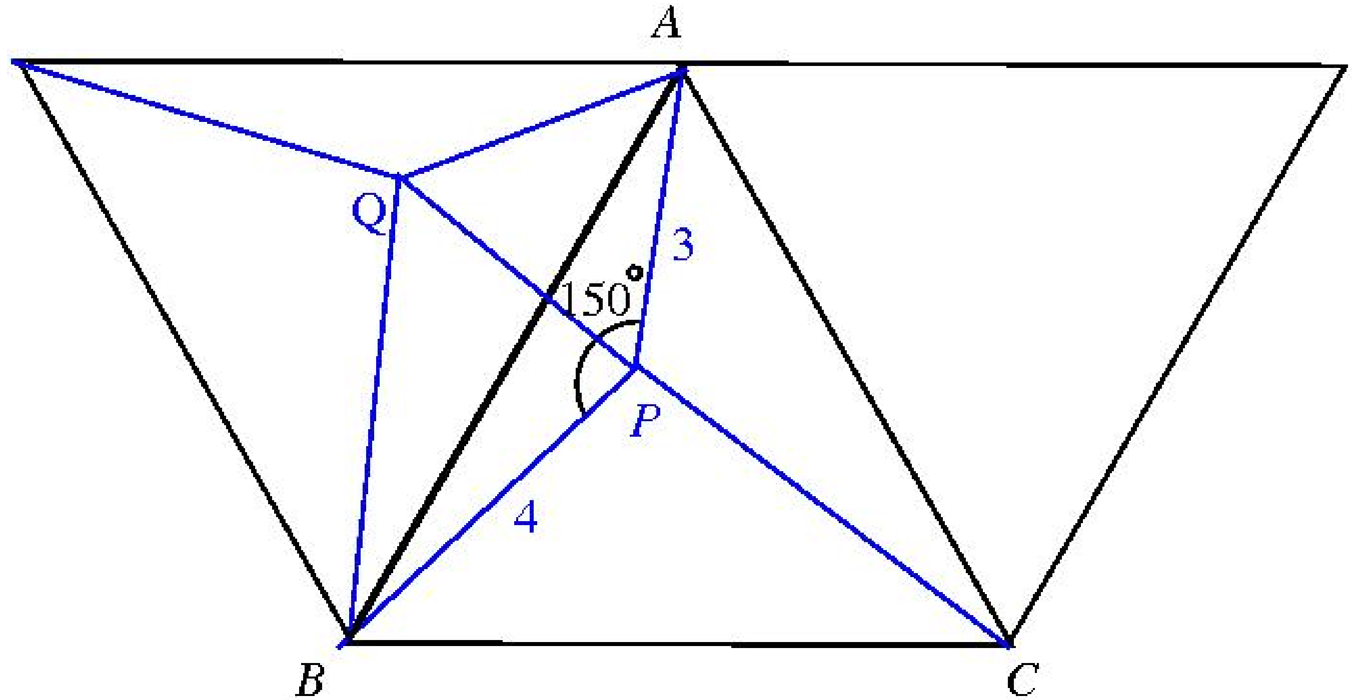

The Law of Cosines gives

$$AB^2 = 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cdot \cos 150^\circ$$

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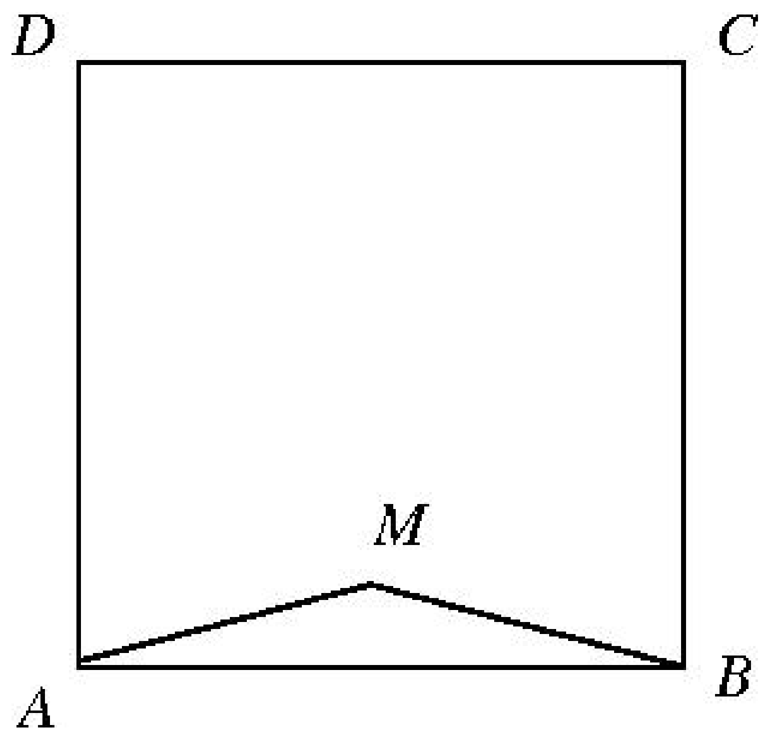
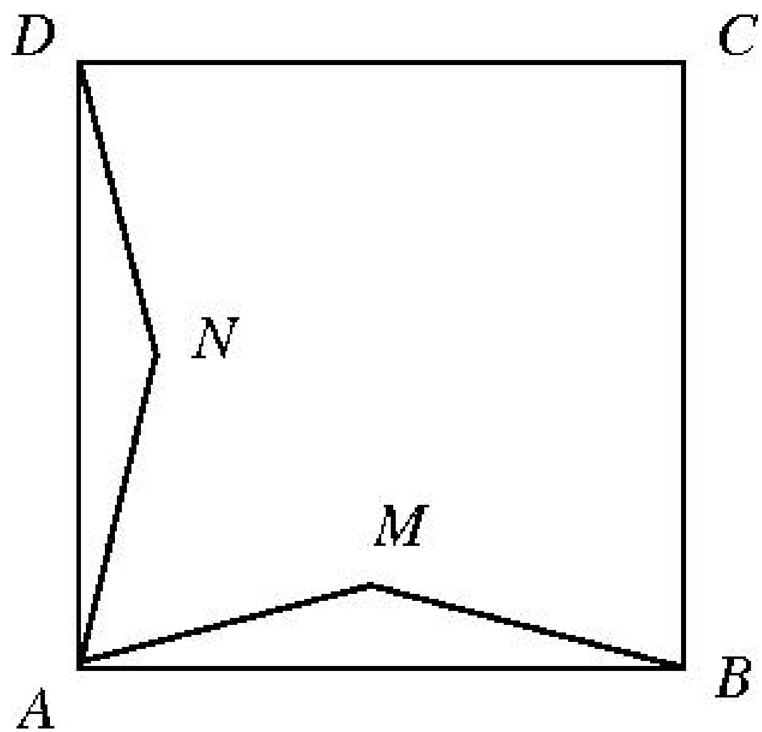
$$\begin{aligned} AB^2 &= 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cdot \cos 150^\circ \\ &= 25 + 12\sqrt{3}, \end{aligned}$$

and hence

$$AB = \sqrt{25 + 12\sqrt{3}}.$$

Problem 3. *Let $ABCD$ be a square and M a point inside it such that $\angle MAB = \angle MBA = 15^\circ$. Find the angle $\angle DMC$.*




D
C
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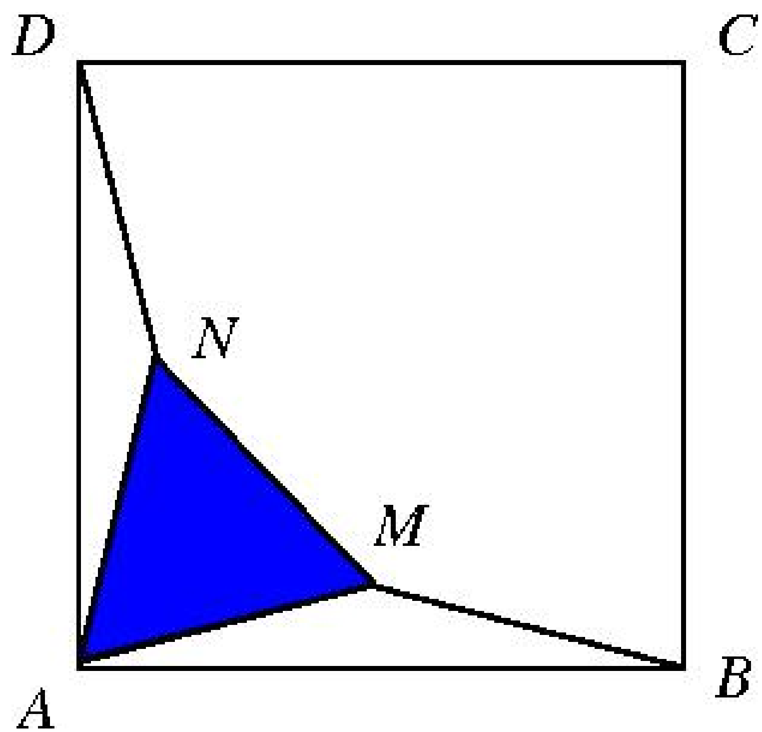
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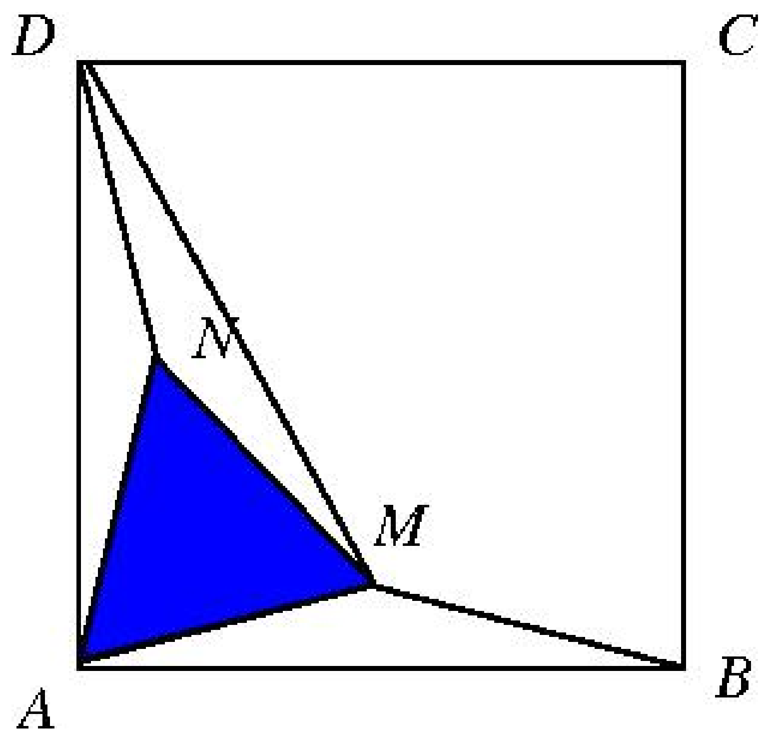
D
C
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M
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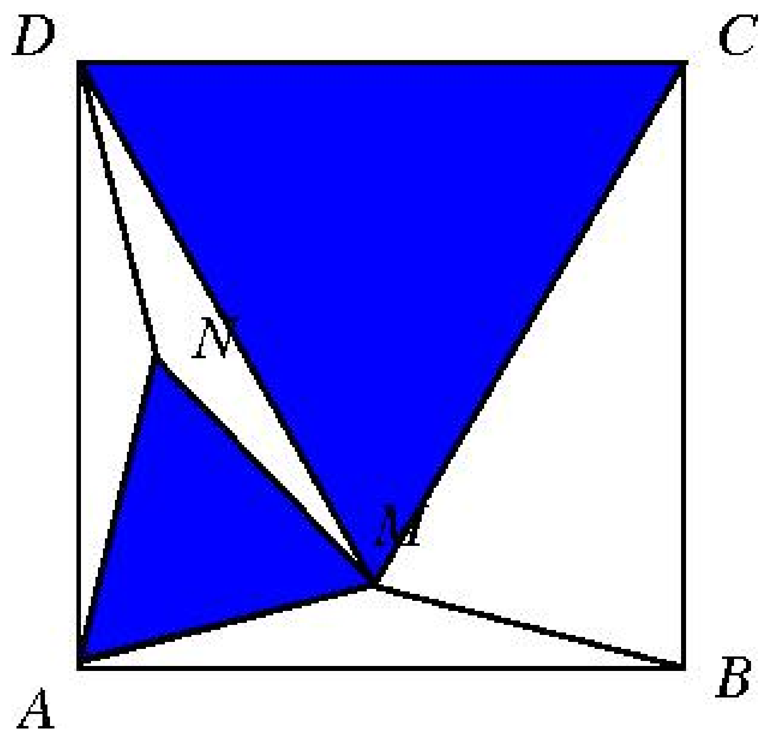
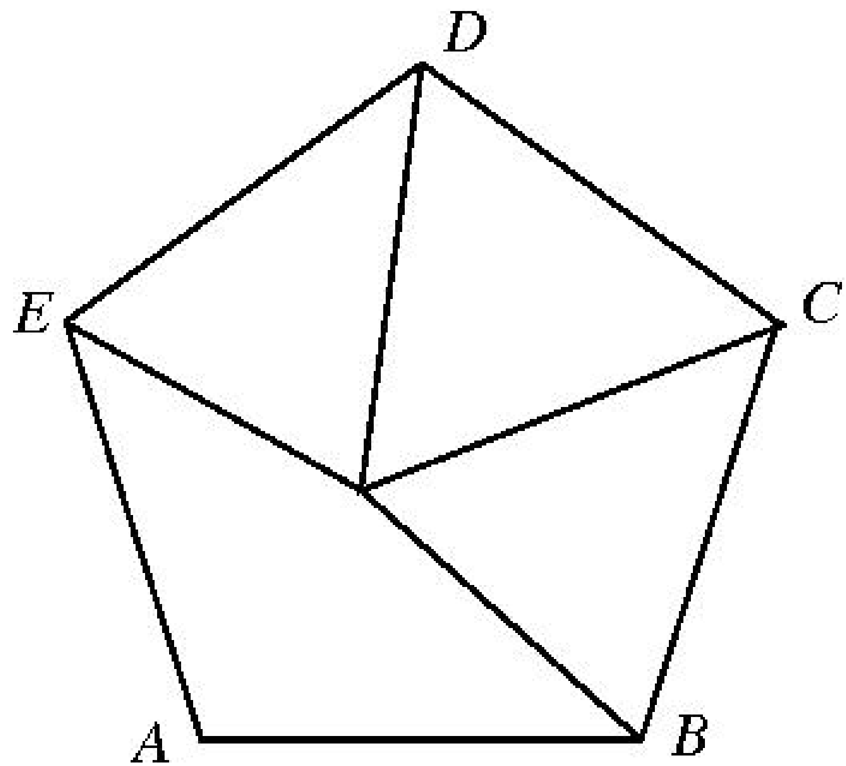
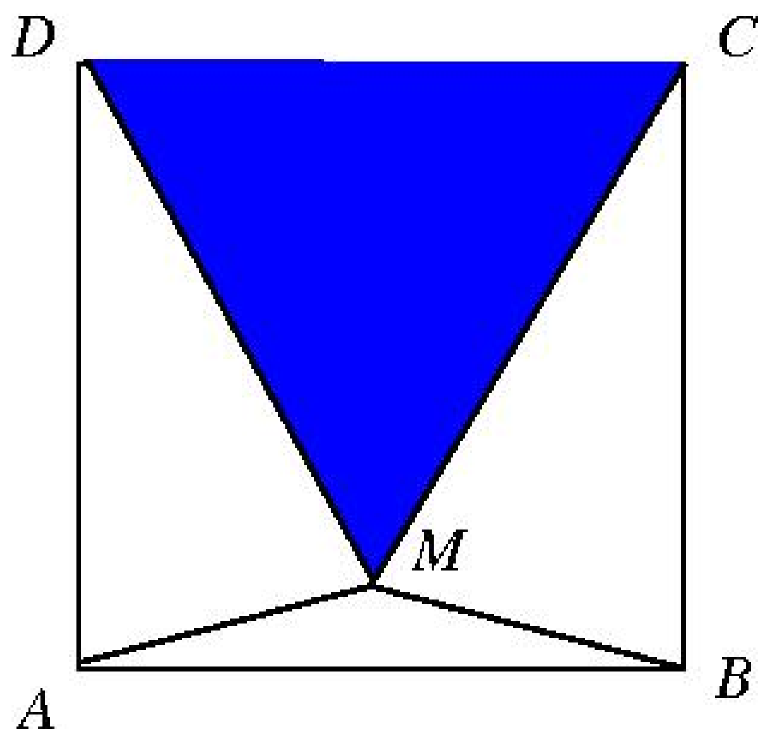

D
C
N
M
A
B

Problem 4. *Let $ABCDE$ be a regular pentagon and M a point in its interior with the property that $\angle MBA = \angle MEA = 42^\circ$. Find $\angle CMD$.*

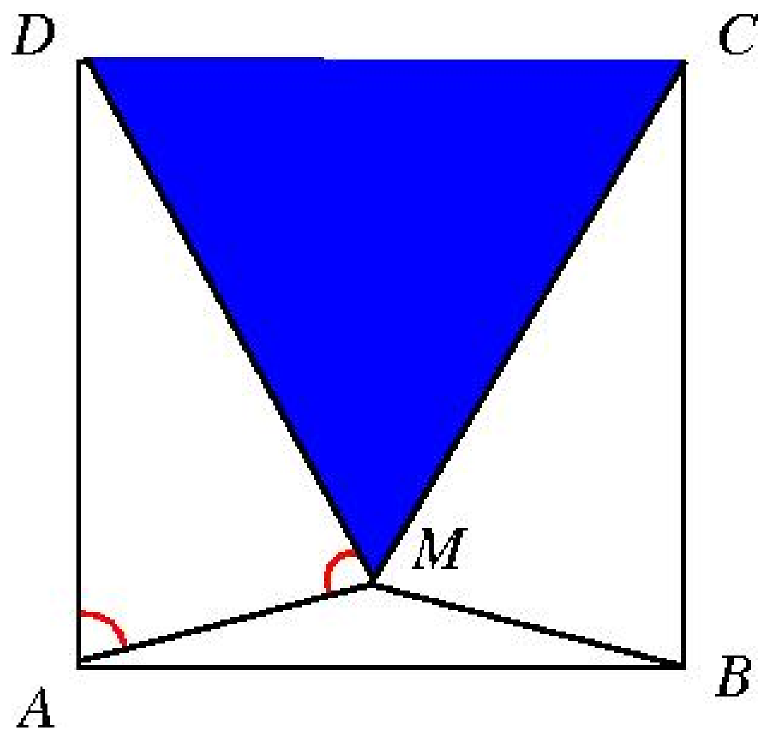


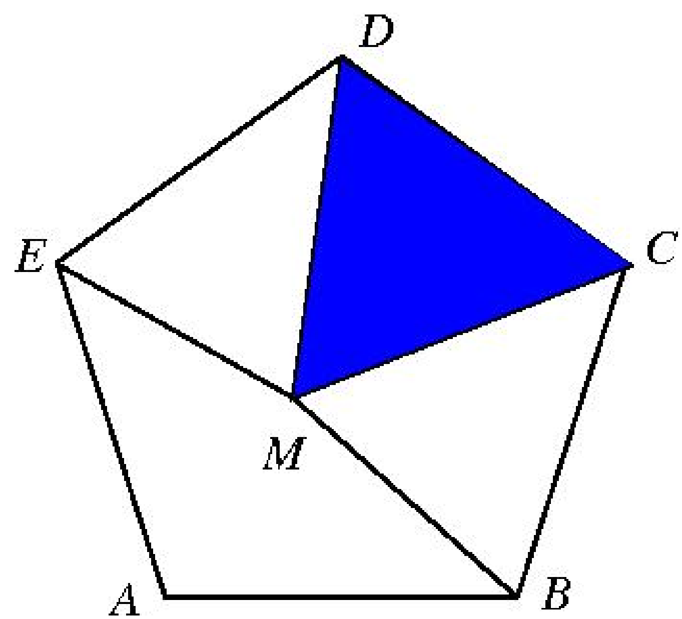
Let us return to the previous problem.

Assume that somehow we guessed *that* $\angle CMD = 60^\circ$. *How can we prove it?*



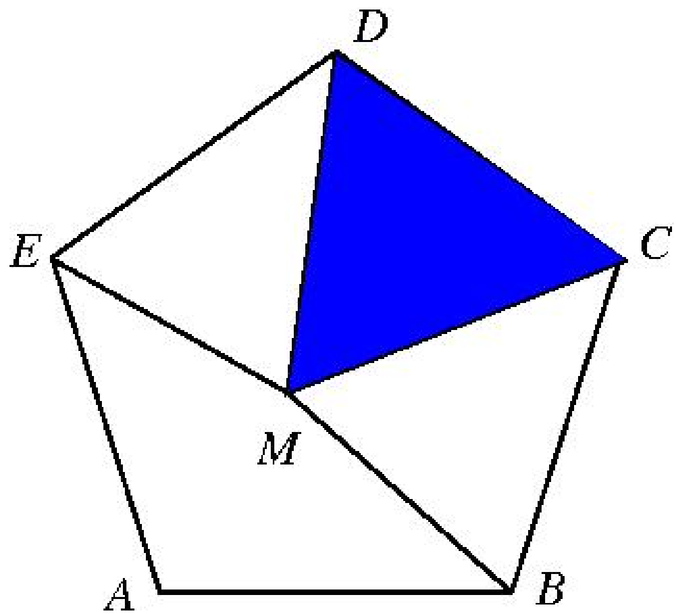
Construct instead M *such that the triangle* DMC *is equilateral. Then* $DA = DM$ *and* $CB = CM$*. So the triangles* DAM *and* CBM *are isosceles. It follows that* $\angle DAM = \angle DMA = 75^\circ$*, so* M *is the point from the statement of the problem.*





Now let us return to the problem with the regular pentagon. Construct instead the point M such that the triangle CMD is equilateral. Then triangle DEM is isosceles, and

$$\angle EDM = 108^\circ - 60^\circ = 48^\circ.$$



Thus

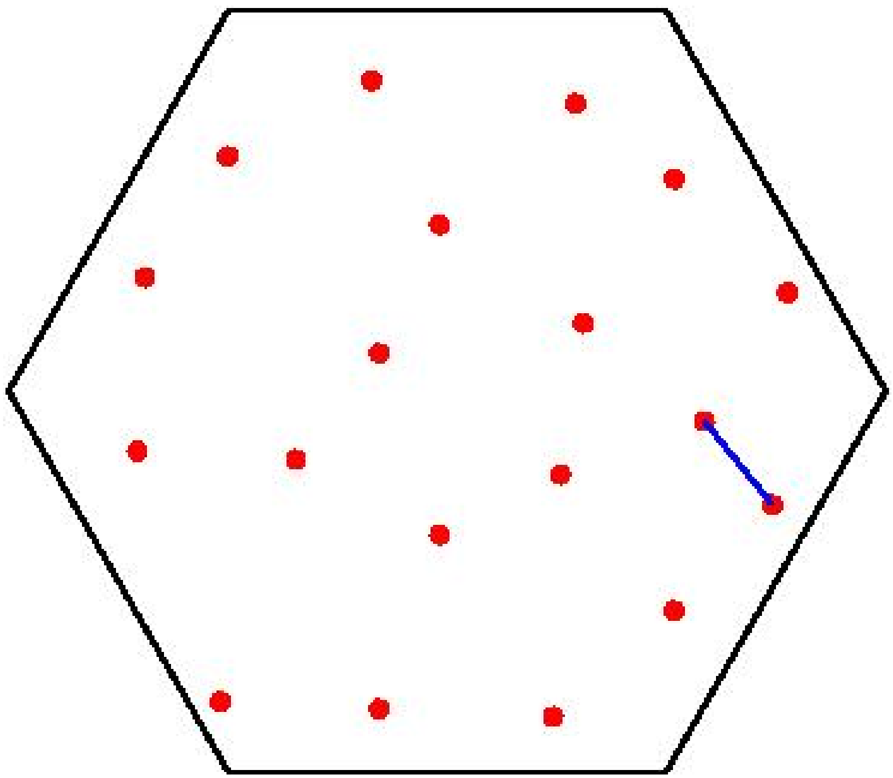
$$\angle DEM = \frac{1}{2}(180^\circ - 48^\circ) = 66^\circ.$$

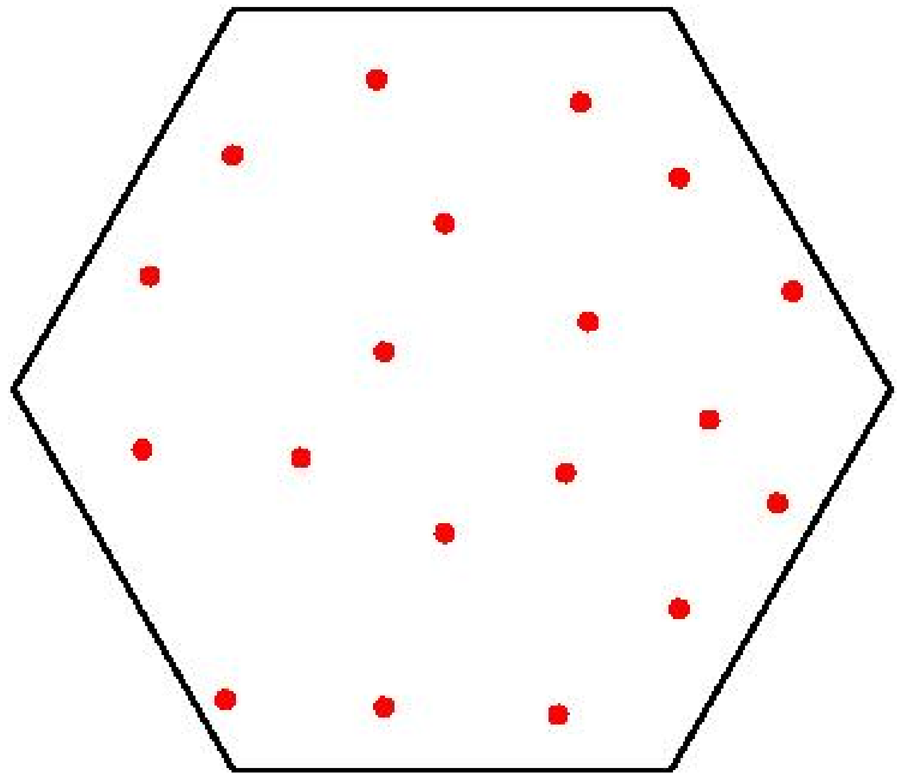
We get

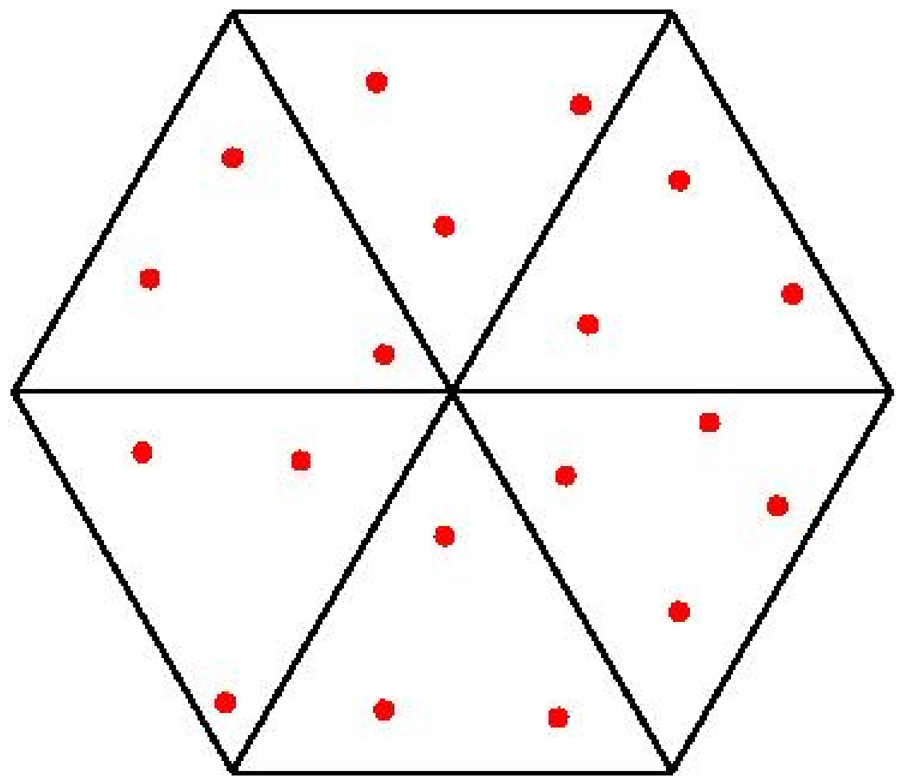
$$\angle AEM = 180^\circ - 66^\circ = 42^\circ.$$

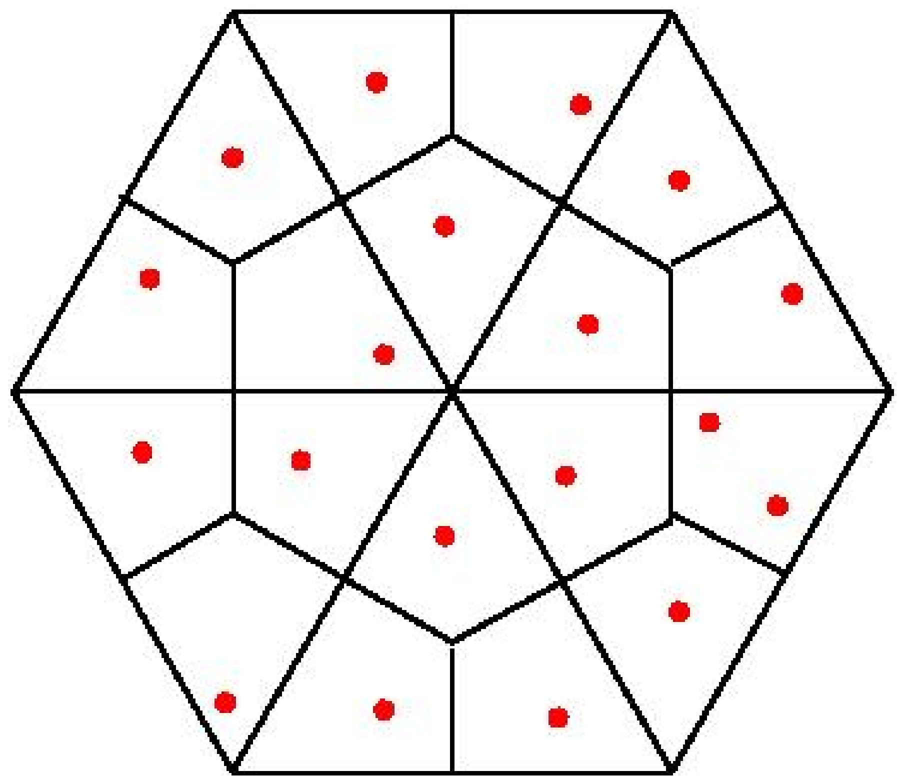
Similarly $\angle MBA = 42^\circ$ *and thus* M *is the point from the statement of the problem.*

Problem 5. *Nineteen darts hit a target which is a regular hexagon of side-length* 1*. Show that two of the darts are at distance at most* $\sqrt{3}/3$ *from each other.*



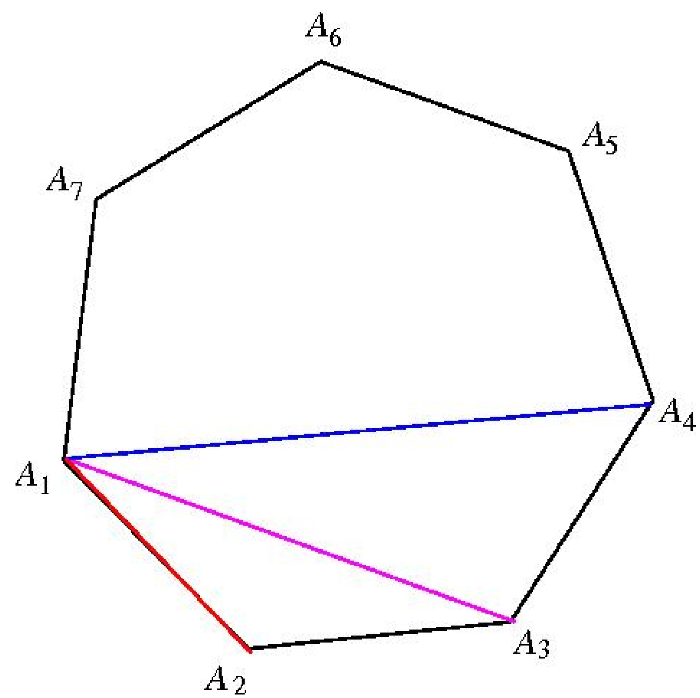


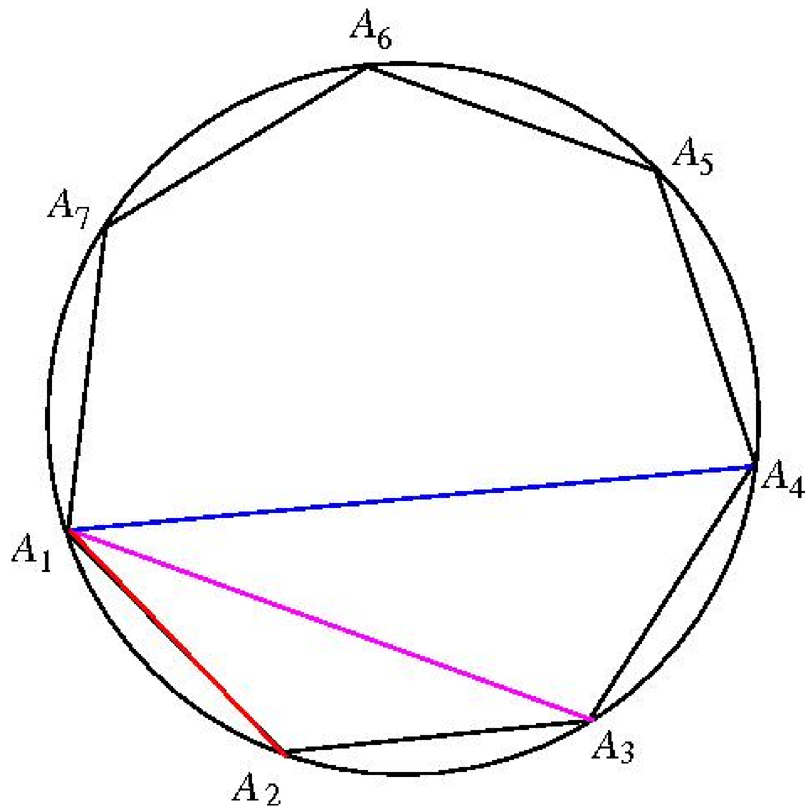




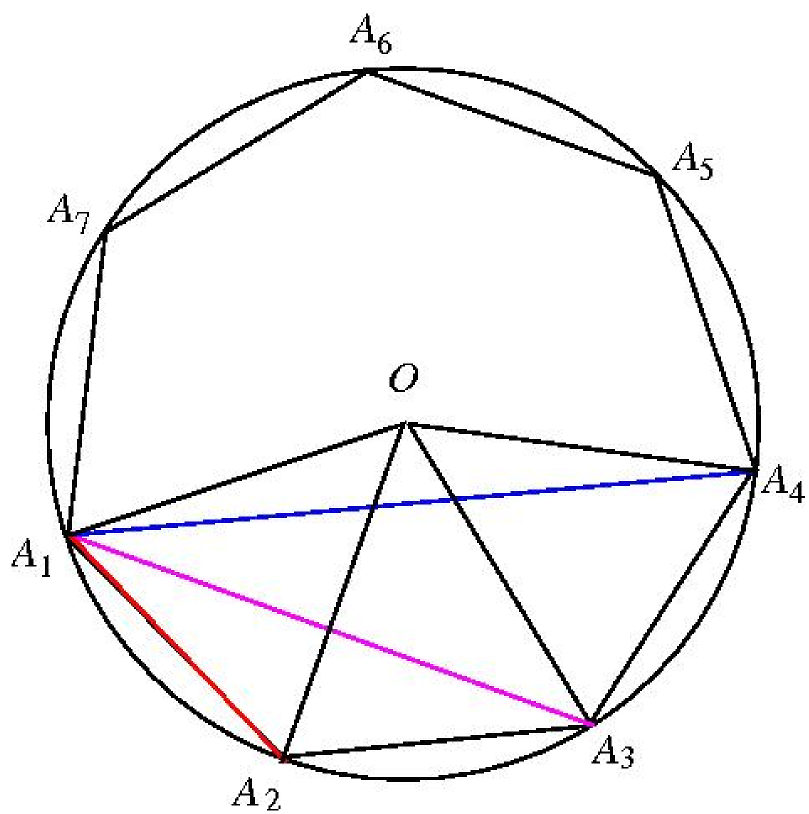
Problem 6. *Let $A_1A_2A_3A_4A_5A_6A_7$ be a regular heptagon. Prove that*

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}.$$

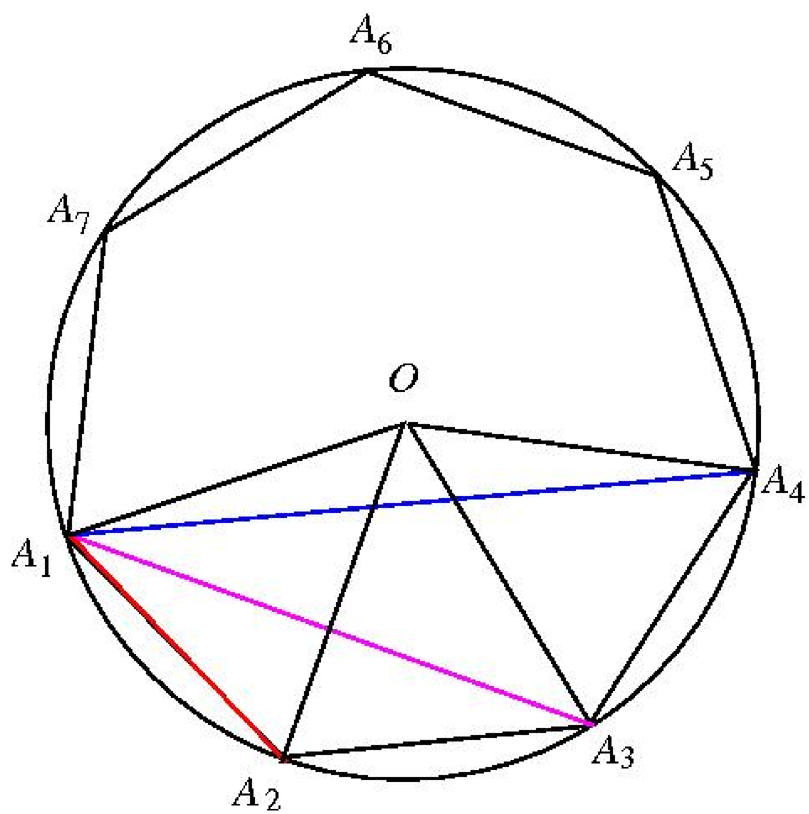




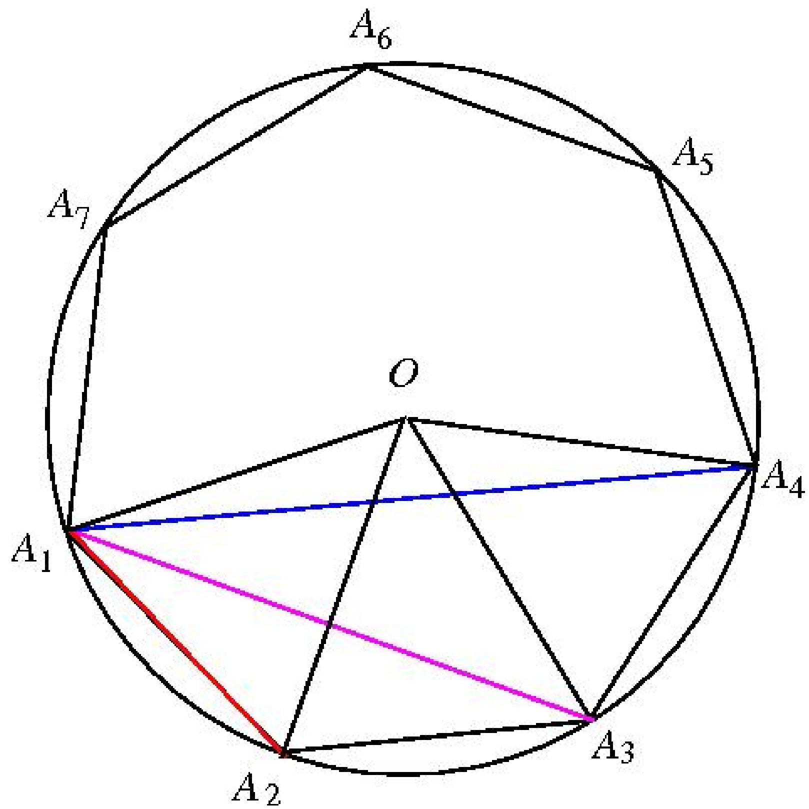
A_6
A_5
A_7
A_4
A_1
A_3
A_2



A_6
A_5
A_7
O
A_4
A_1
A_3
A_2



$$\angle A_1OA_2 = \frac{360^\circ}{7}, \quad \angle A_1OA_3 = \frac{720^\circ}{7}, \quad \angle A_1OA_4 = \frac{1080^\circ}{7}.$$



$$A_1A_2 = 2R\sin\frac{180^\circ}{7}, \quad A_1A_3 = 2R\sin\frac{360^\circ}{7}, \quad A_1A_4 = 2R\sin\frac{540^\circ}{7}.$$

So we have to prove that

$$\frac{1}{\sin\frac{180^\circ}{7}} = \frac{1}{\sin\frac{360^\circ}{7}} + \frac{1}{\sin\frac{540^\circ}{7}}.$$

Rewrite as

$$\sin\frac{360^\circ}{7}\sin\frac{540^\circ}{7} = \sin\frac{180^\circ}{7}\sin\frac{360^\circ}{7} + \sin\frac{180^\circ}{7}\sin\frac{540^\circ}{7}.$$

Now we use the formula

$$2 \sin a \sin b = \cos(a - b) - \cos(a + b).$$

So we have to prove that

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Rewrite as

$$\sin\frac{360^\circ}{7}\sin\frac{540^\circ}{7} = \sin\frac{180^\circ}{7}\sin\frac{360^\circ}{7} + \sin\frac{180^\circ}{7}\sin\frac{540^\circ}{7}.$$

... to write this as

$$-\cos\frac{900^\circ}{7} + \cos\frac{180^\circ}{7} = \cos\frac{180^\circ}{7} - \cos\frac{540^\circ}{7} + \cos\frac{360^\circ}{7} - \cos\frac{720^\circ}{7}.$$

We are left with showing that

$$\cos\frac{540^\circ}{7}+\cos\frac{720^\circ}{7}-\cos\frac{900^\circ}{7}-\cos\frac{360^\circ}{7}=0.$$

Note that $7\times 180^\circ = 1260^\circ$ *and* $\cos(180^\circ - x) = -\cos x$. *Hence the left-hand side is zero, as desired.*

There is a more elegant way to write this, which makes the solution more natural.

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Use $180^\circ = \pi$*.*

$$\frac{1}{\sin\frac{\pi}{7}} = \frac{1}{\sin\frac{2\pi}{7}} + \frac{1}{\sin\frac{3\pi}{7}}$$

$$\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \sin\frac{\pi}{7}\sin\frac{2\pi}{7} + \sin\frac{\pi}{7}\sin\frac{3\pi}{7}$$

$$-\cos\frac{5\pi}{7} + \cos\frac{\pi}{7} = \cos\frac{\pi}{7} - \cos\frac{3\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{4\pi}{7}$$

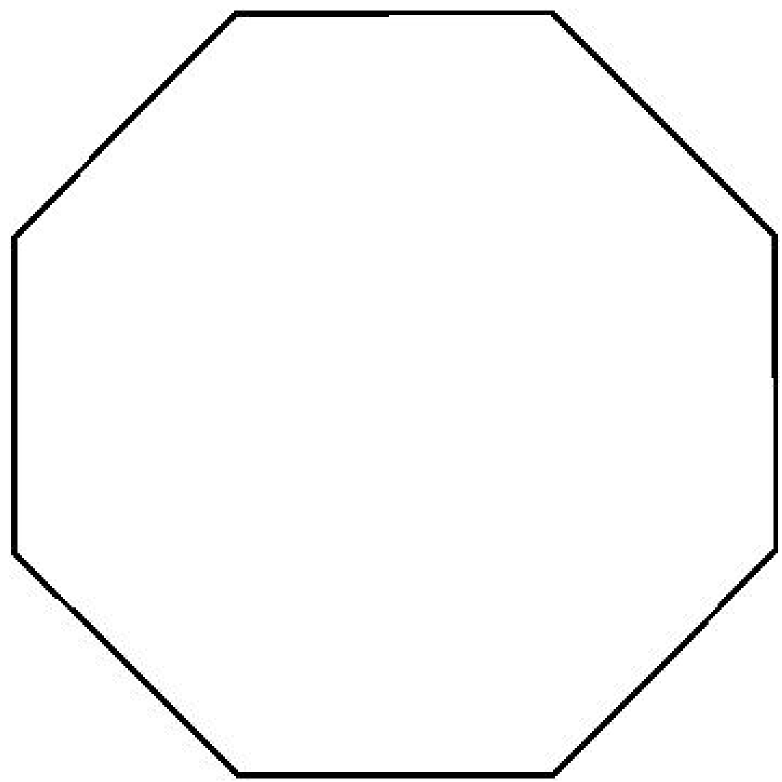
$$\cos\frac{3\pi}{7} + \cos\frac{4\pi}{7} - \cos\frac{5\pi}{7} - \cos\frac{2\pi}{7} = 0$$

This is the same as

$$\cos\frac{3\pi}{7} + \cos\left(\pi - \frac{3\pi}{7}\right) - \cos\frac{5\pi}{7} - \cos\left(\pi - \frac{5\pi}{7}\right) = 0$$

Now use $\cos(\pi - x) = -\cos x$ *to conclude that this is true.*

Problem 7. *A regular octagon of side-length* 1 *is dissected into parallelograms. Find the sum of the areas of the rectangles in the dissection.*



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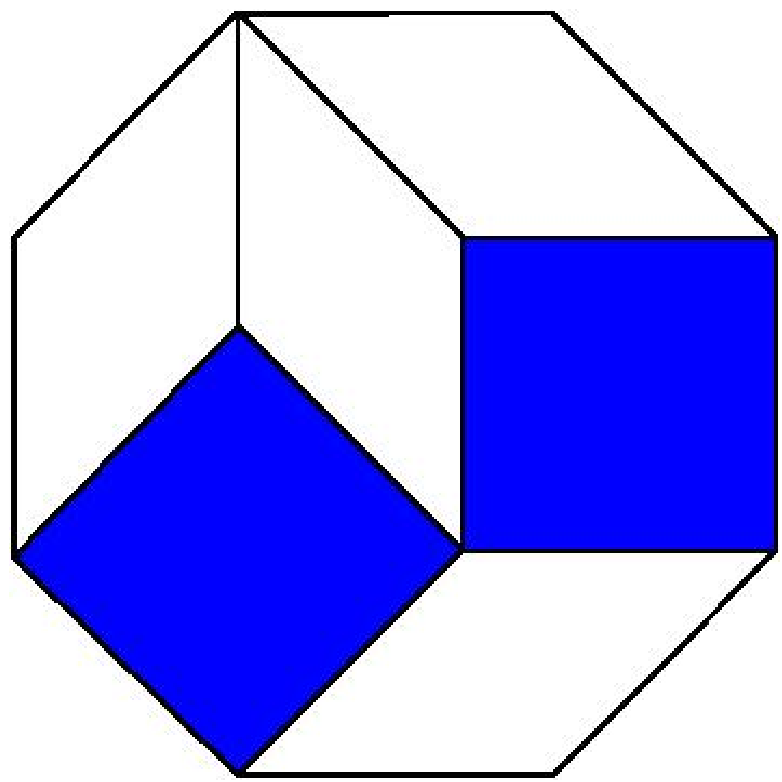
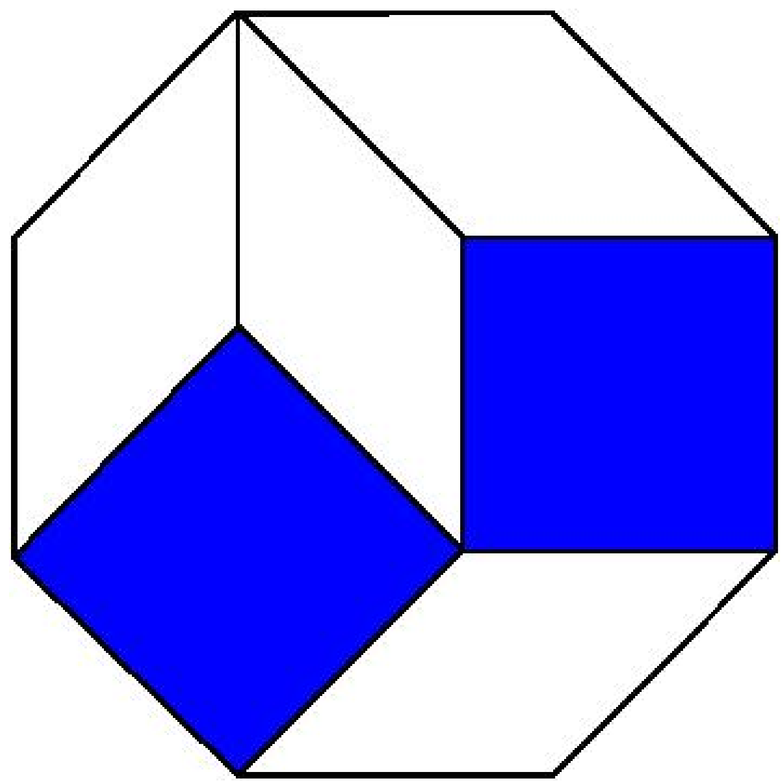
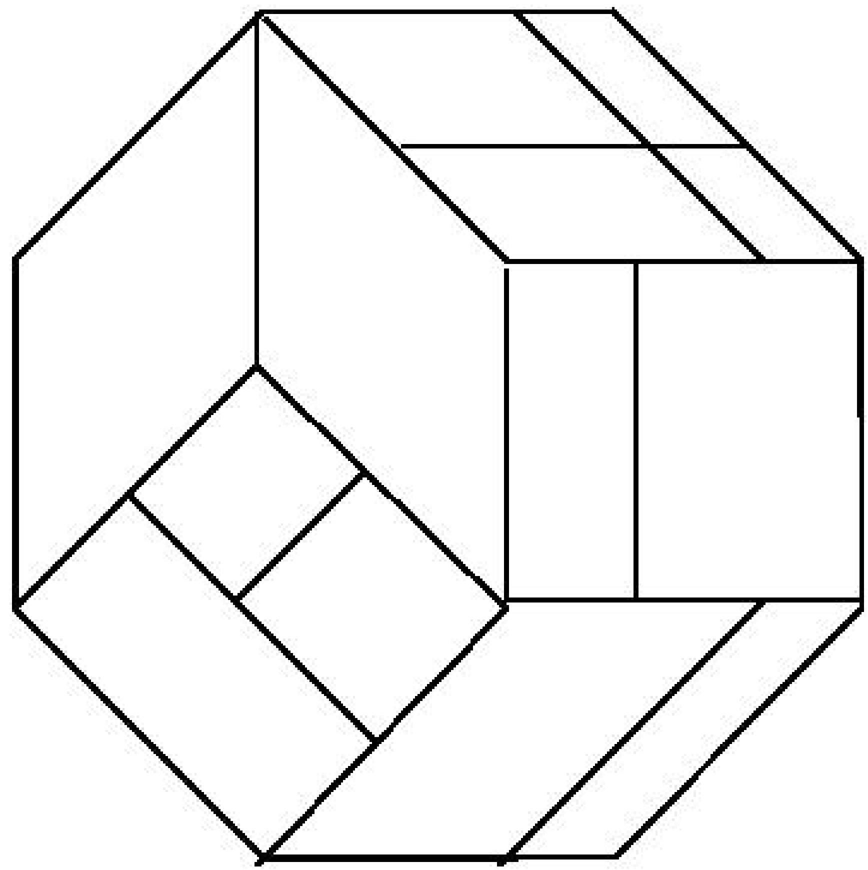
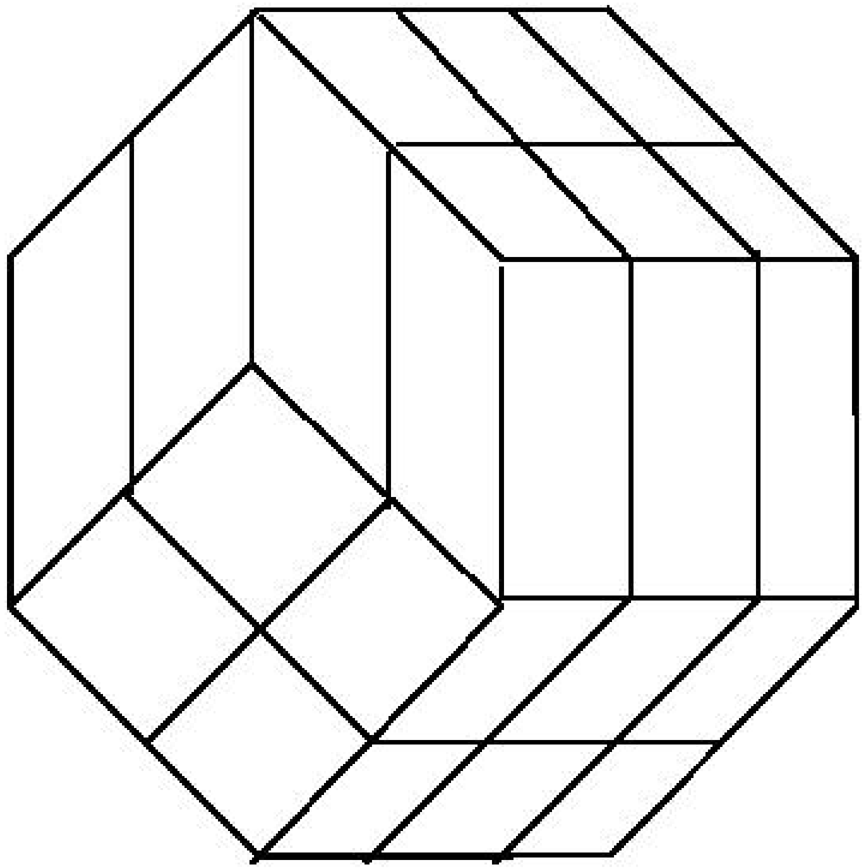


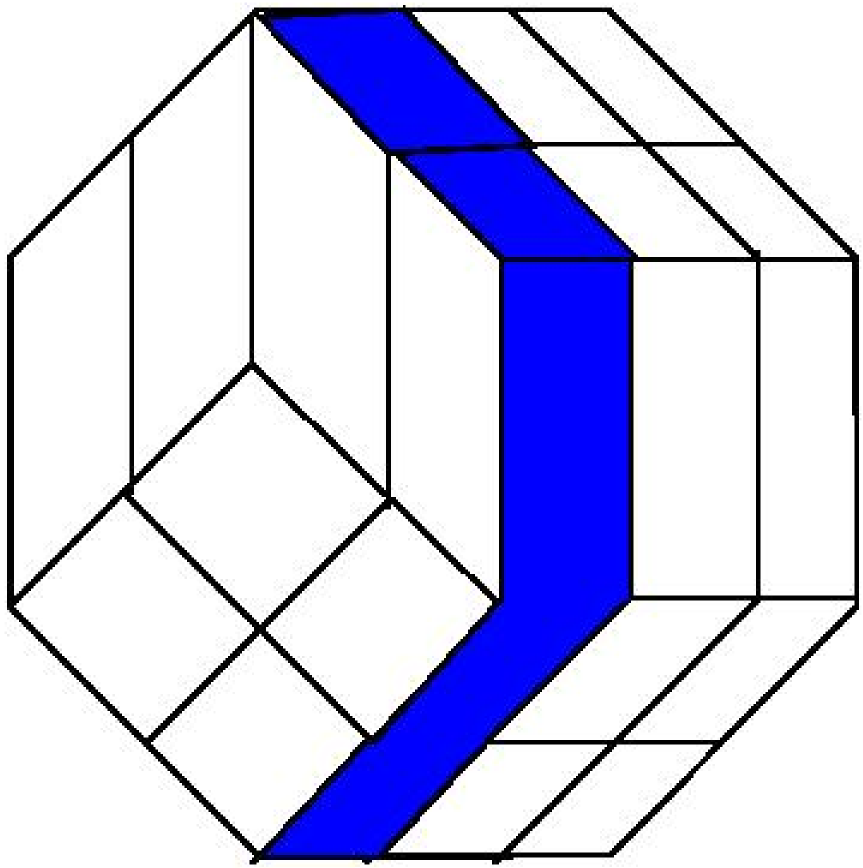
Problem 7. *A regular octagon of side-length* 1 *is dissected into parallelograms. Find the sum of the areas of the rectangles in the dissection.*

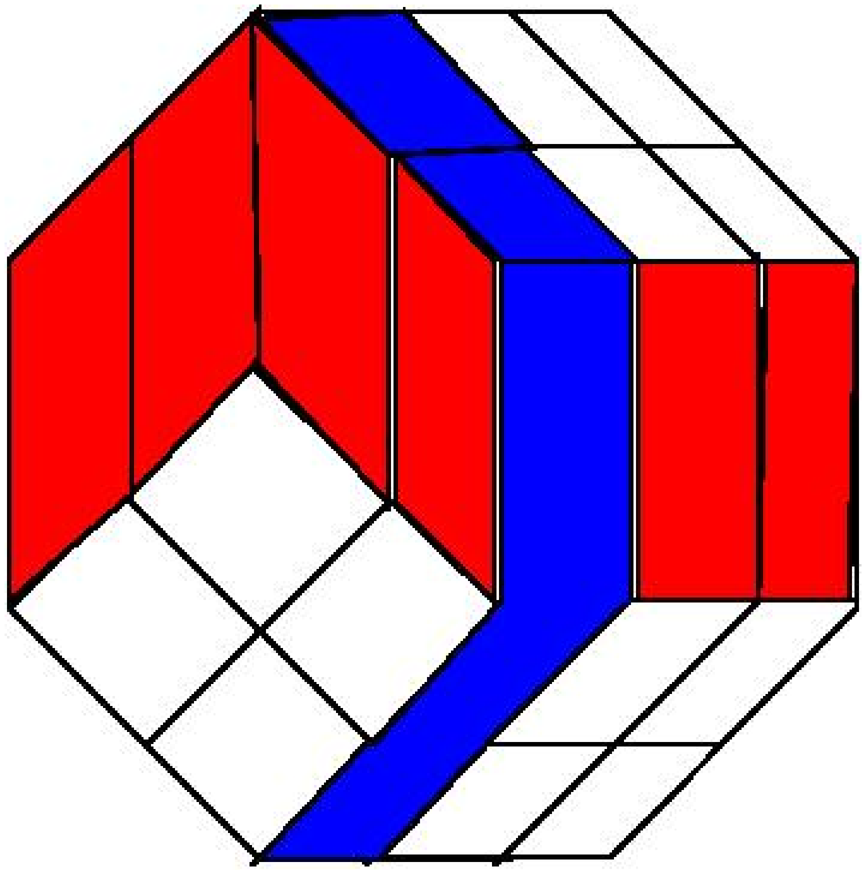


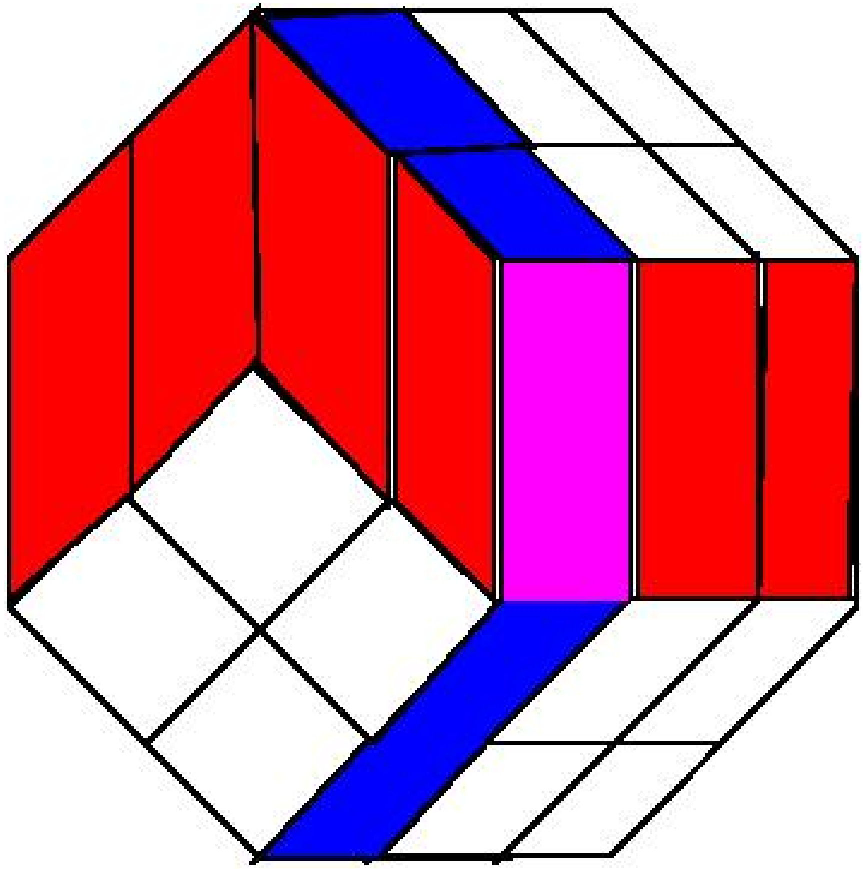
Answer: 2.



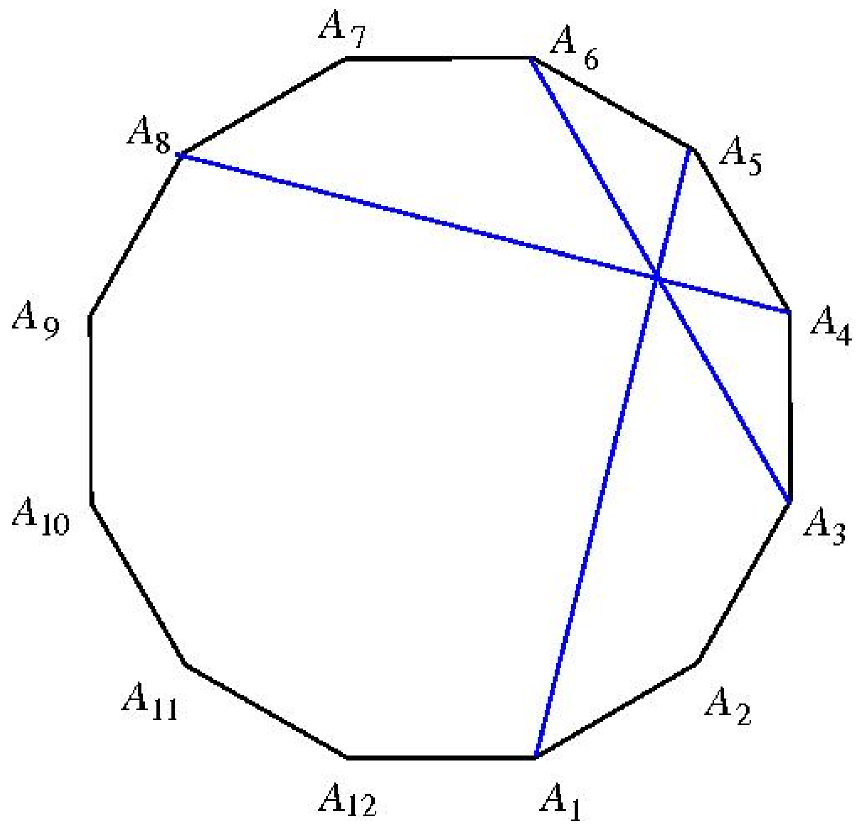




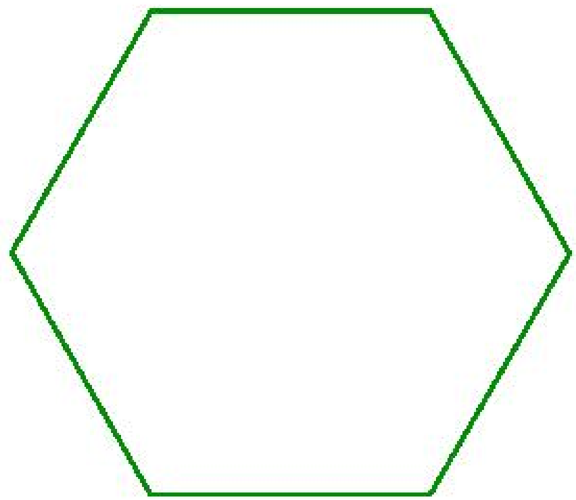


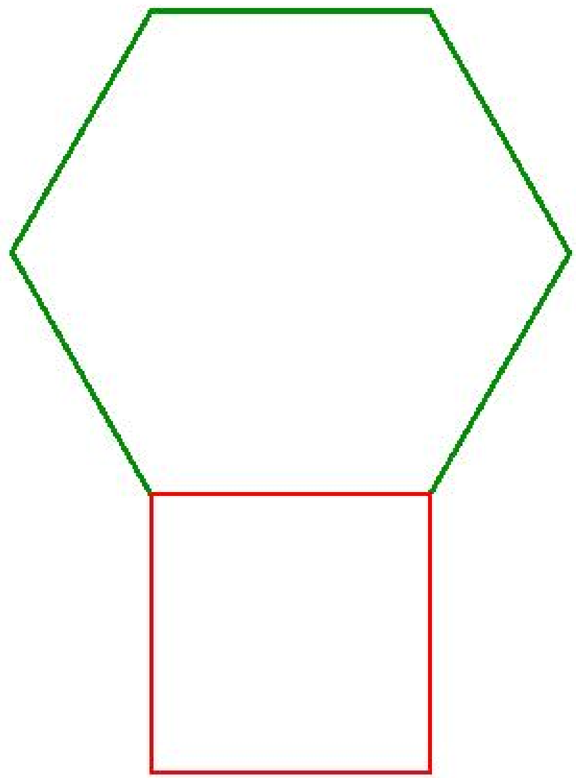


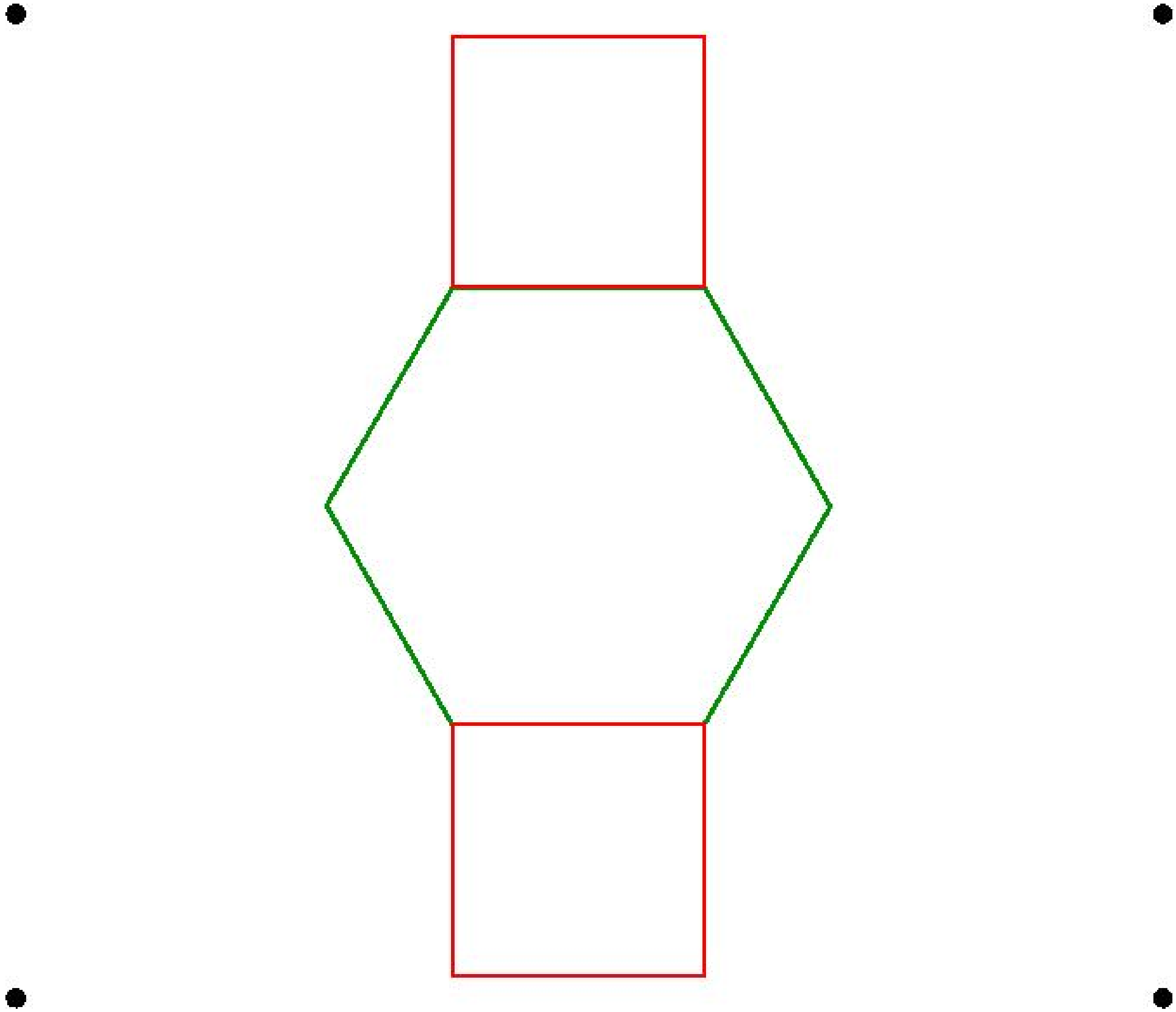
Problem 8. *Let* $A_1A_2A_3\ldots A_{12}$ *be a regular dodecagon. Prove that* A_1A_5, A_4A_8, *and* A_3A_6 *intersect at one point.*

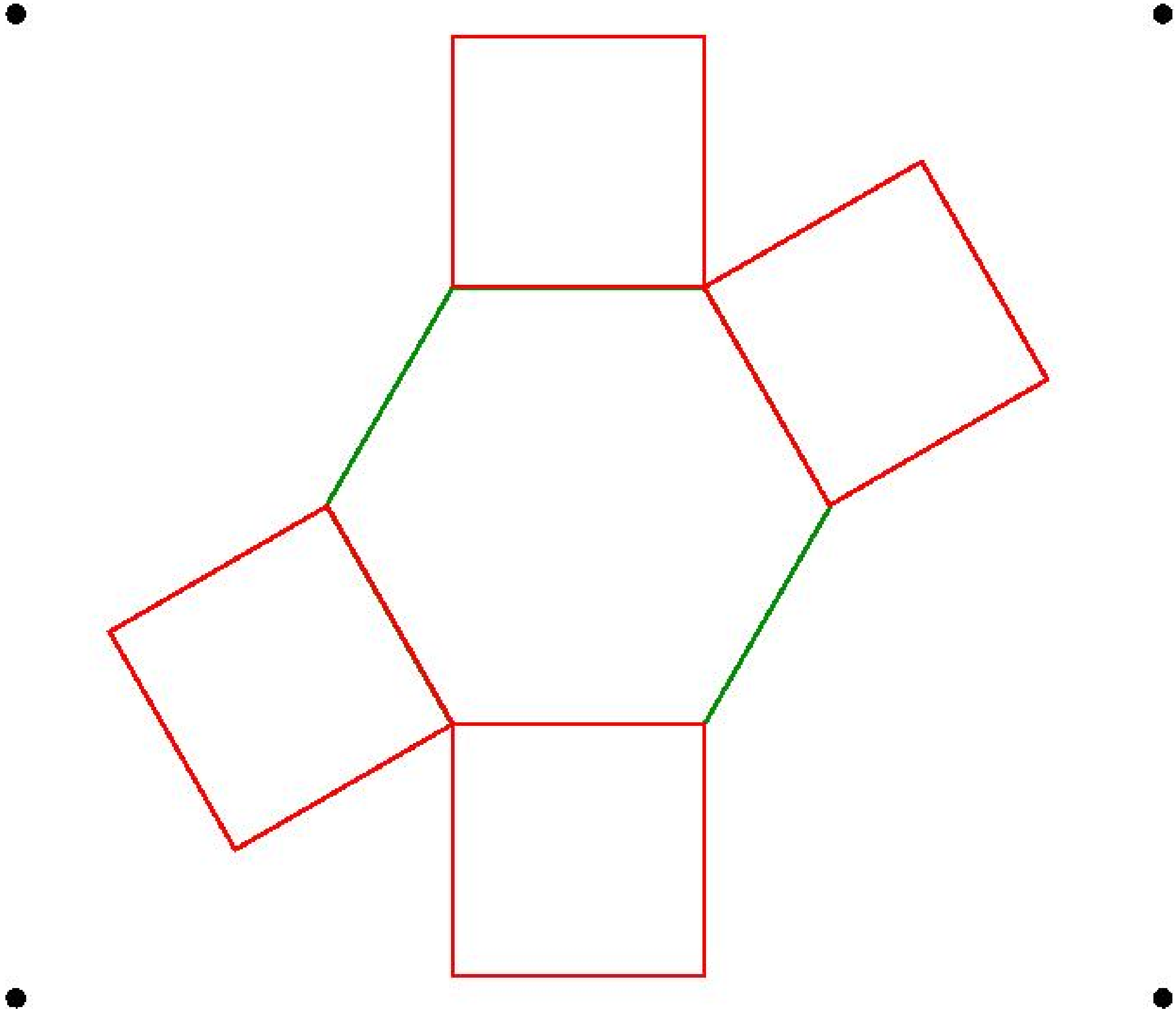


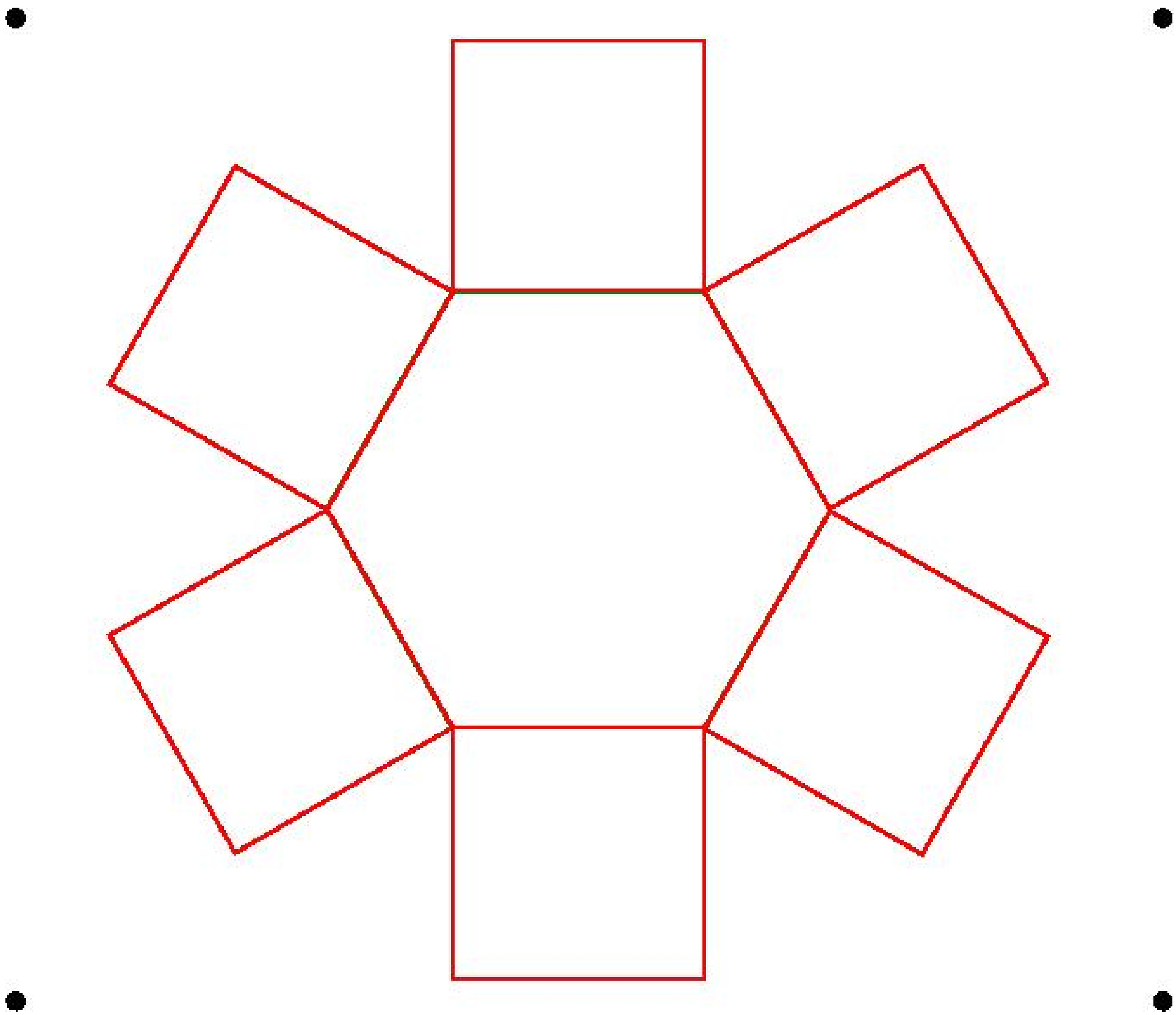
First let us recall the construction of a regular dodecagon.

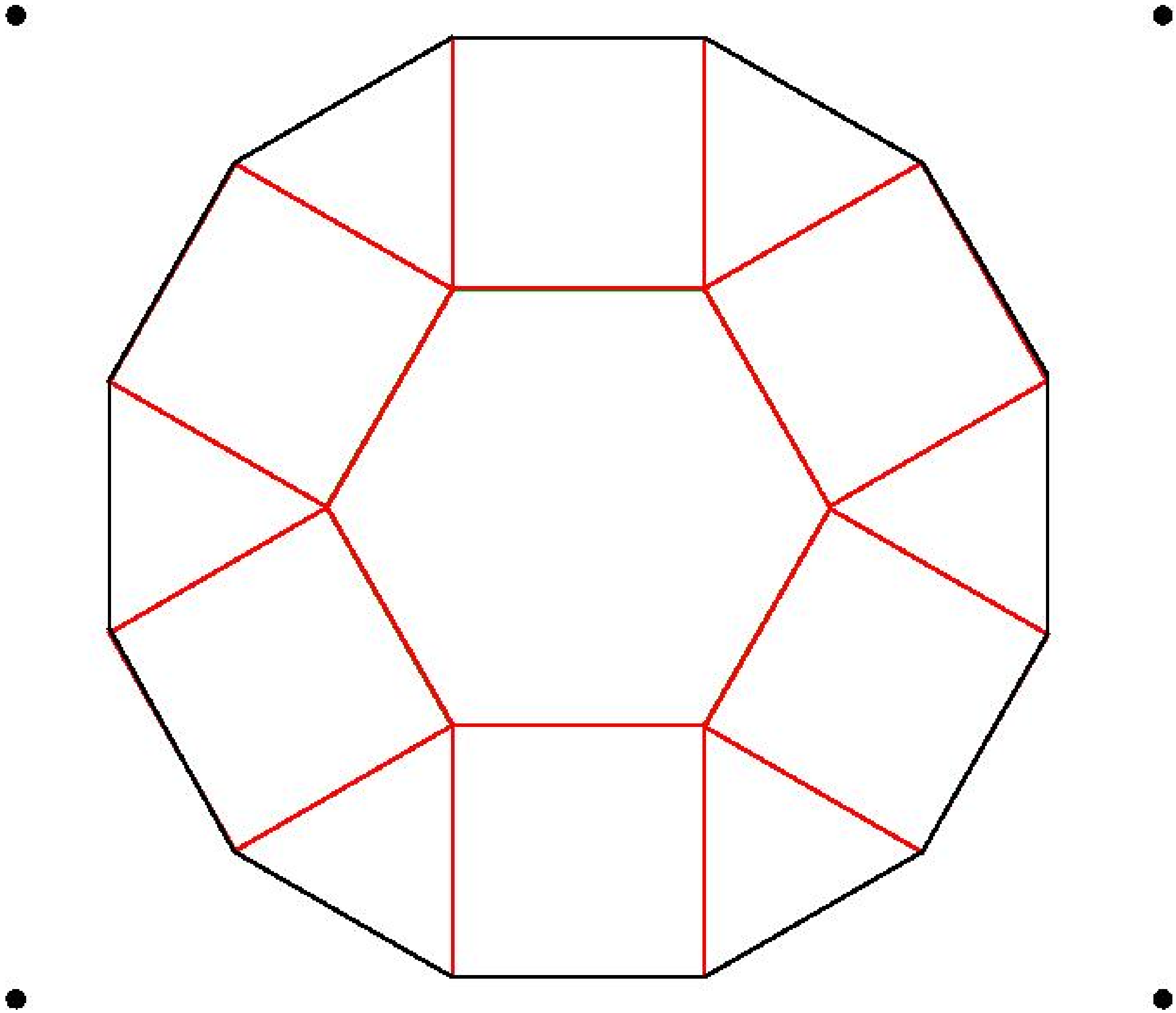


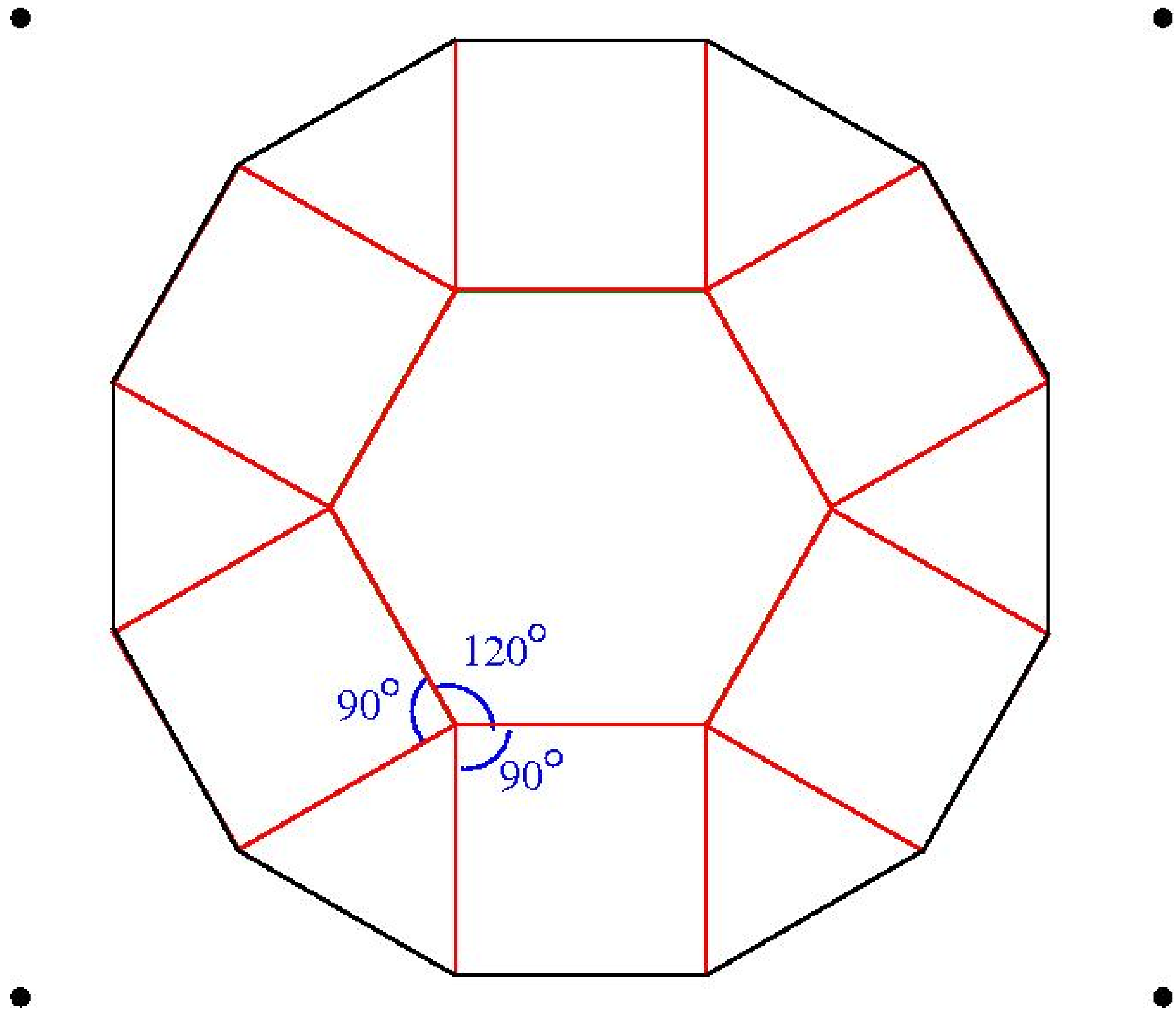


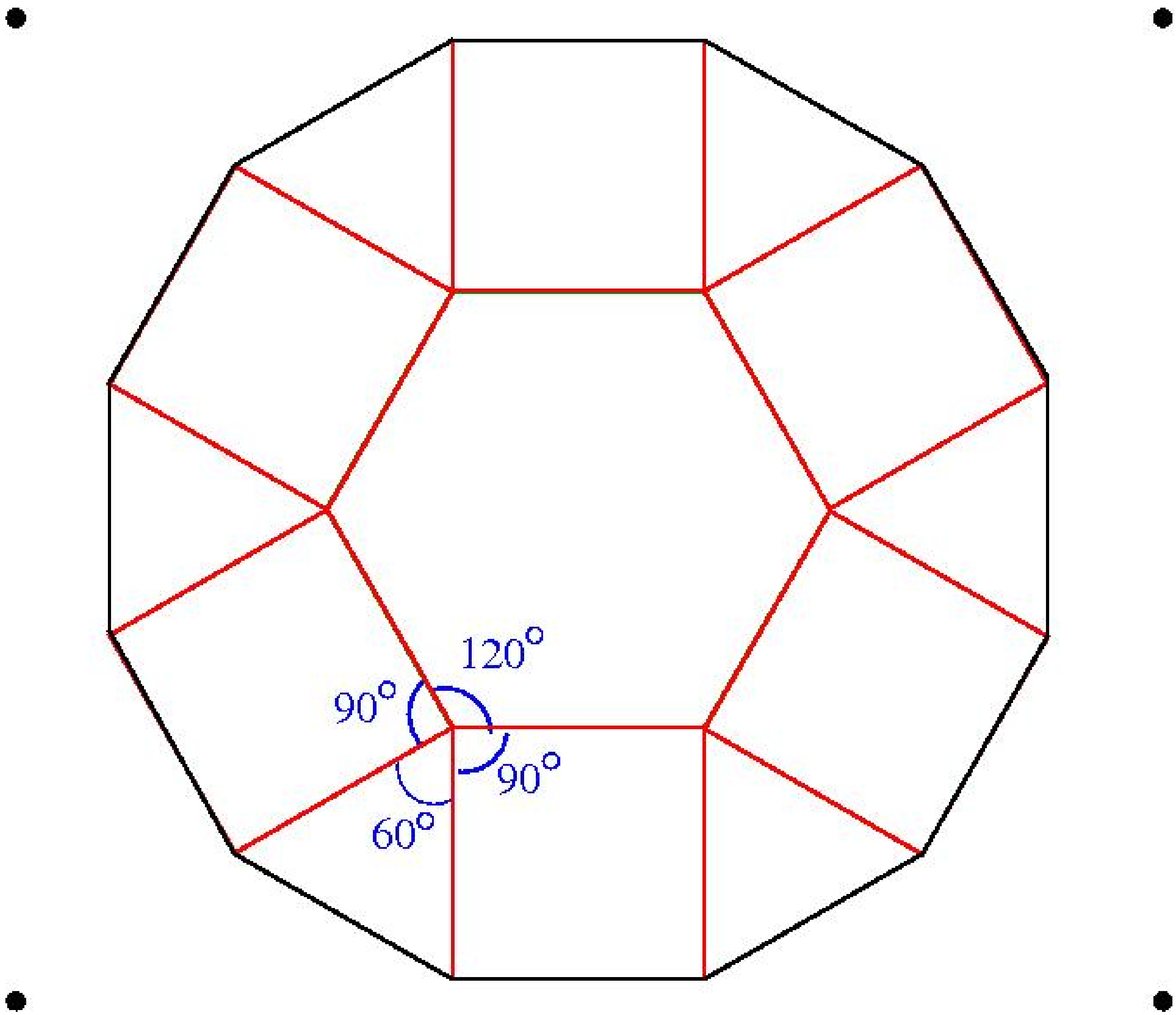


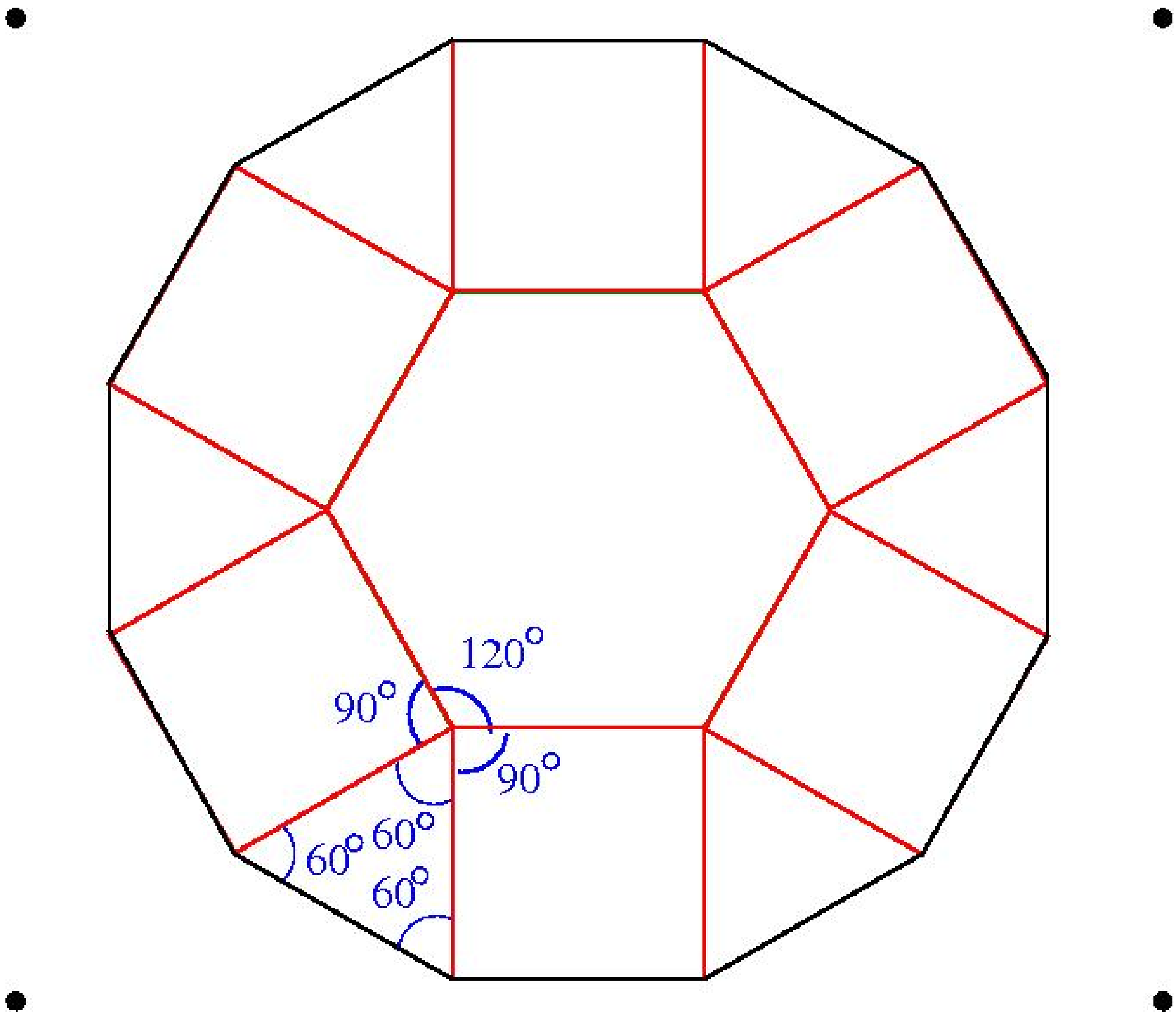


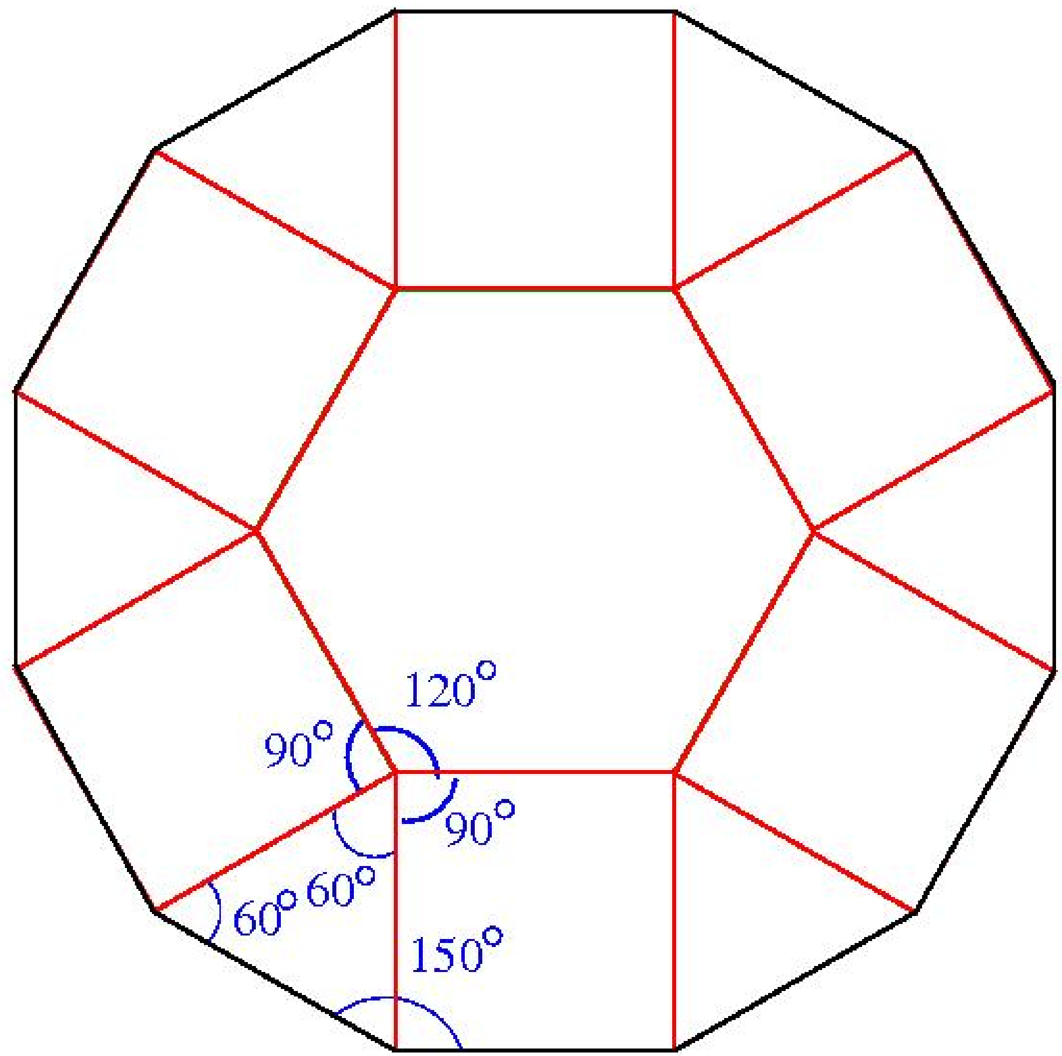


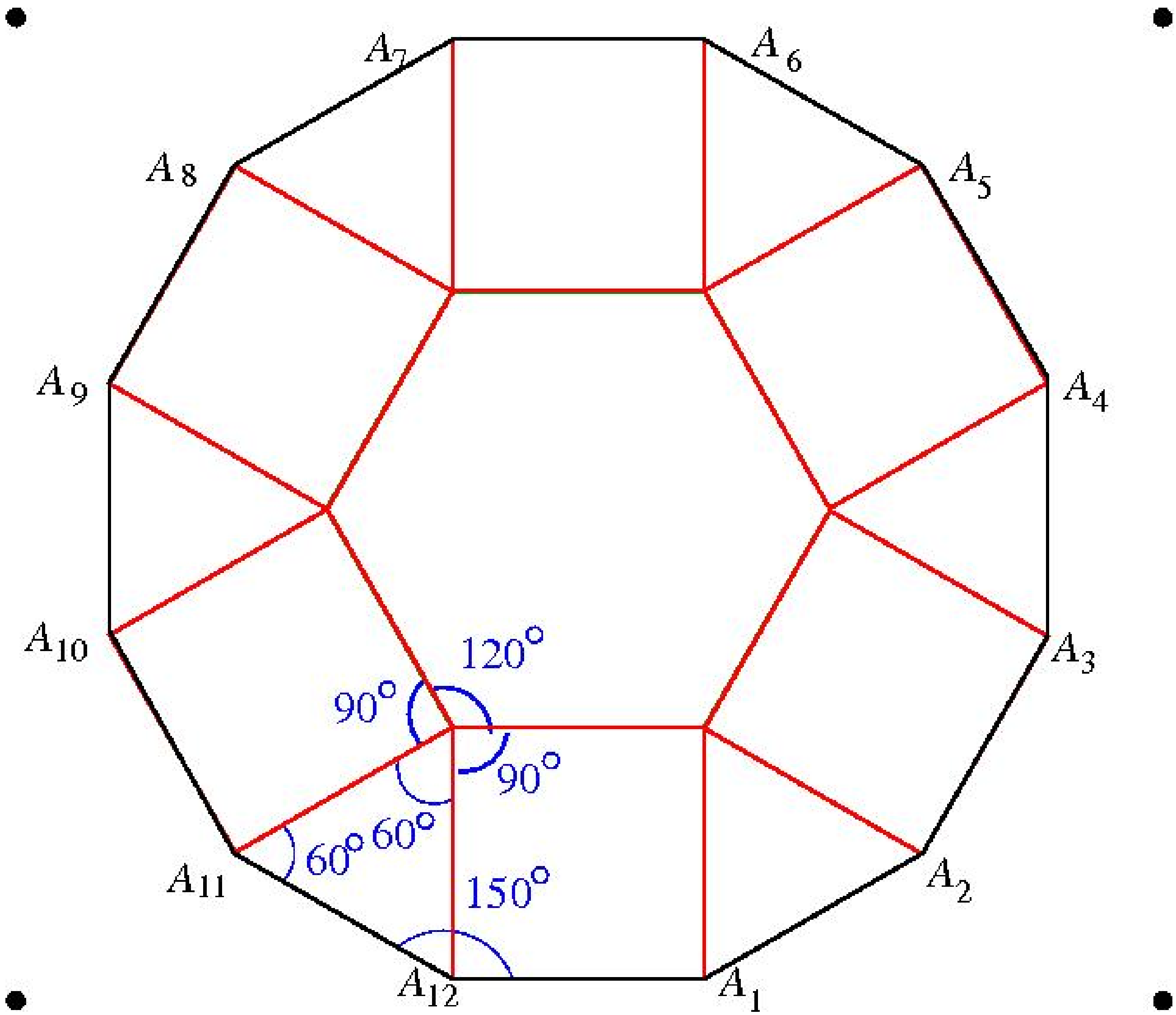



120°
90°
90°

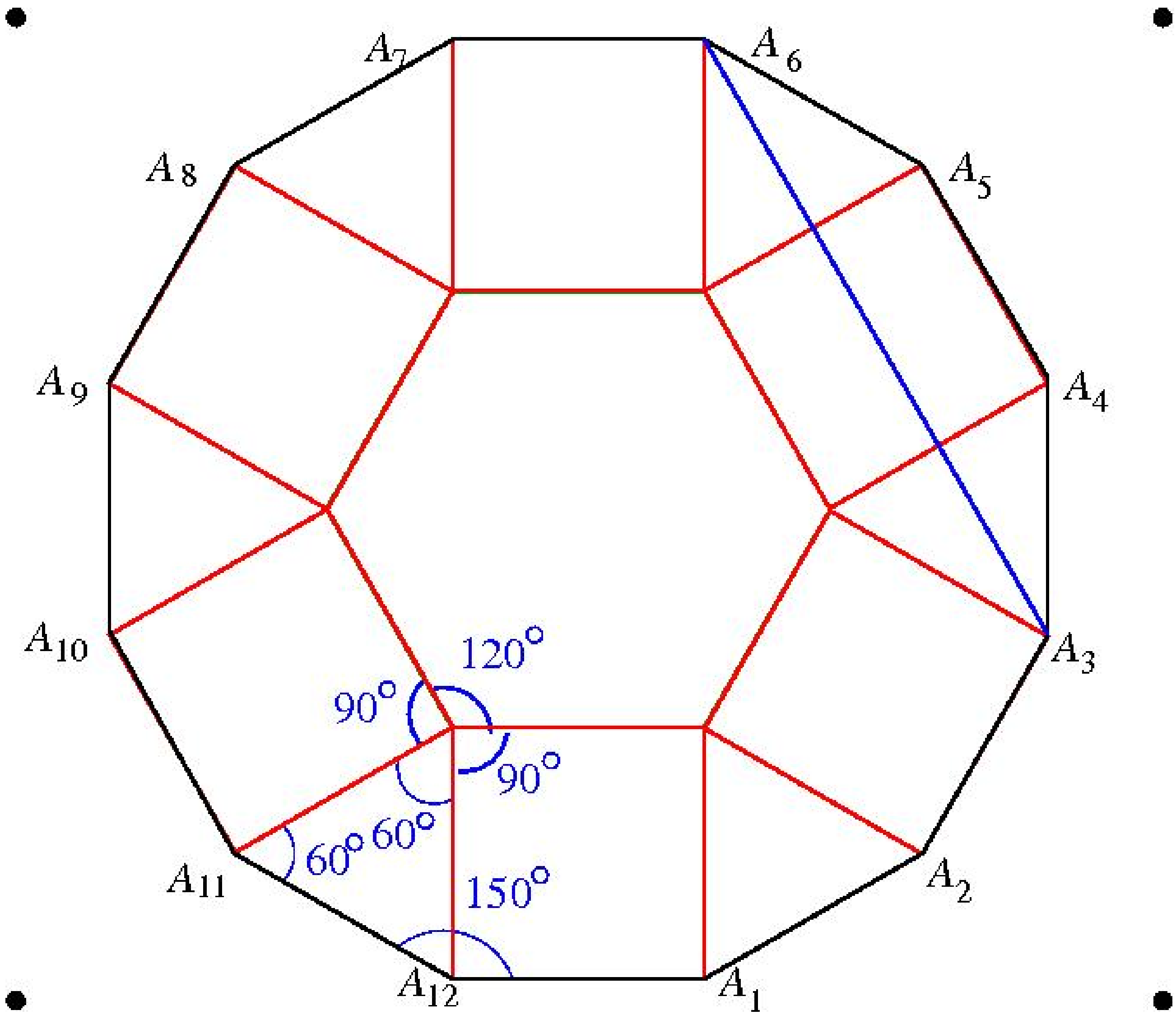

120°
90°
90°
60°


120°
90°
90°
60°
60°
60°

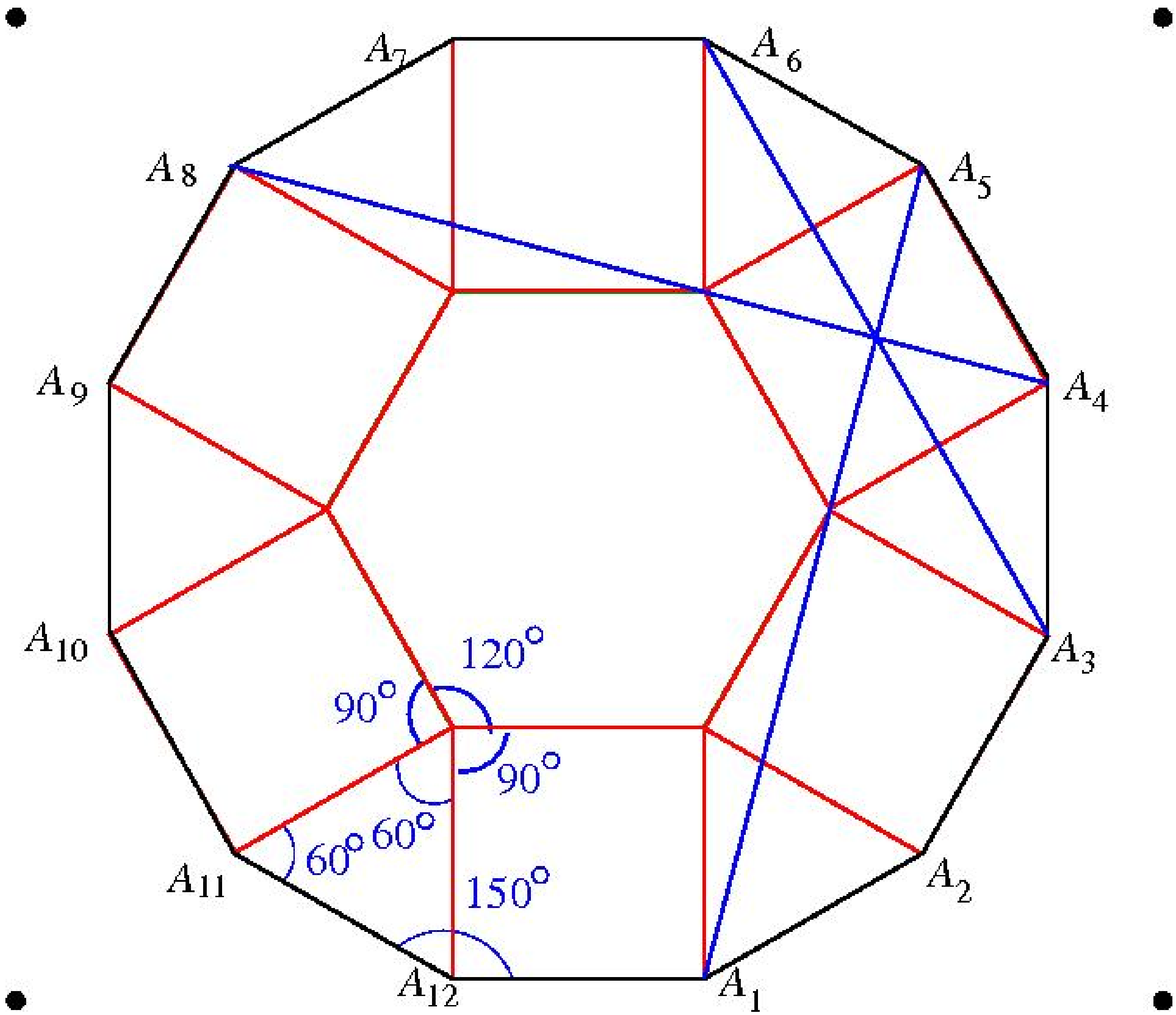

120°
90°
90°
60°
60°
150°



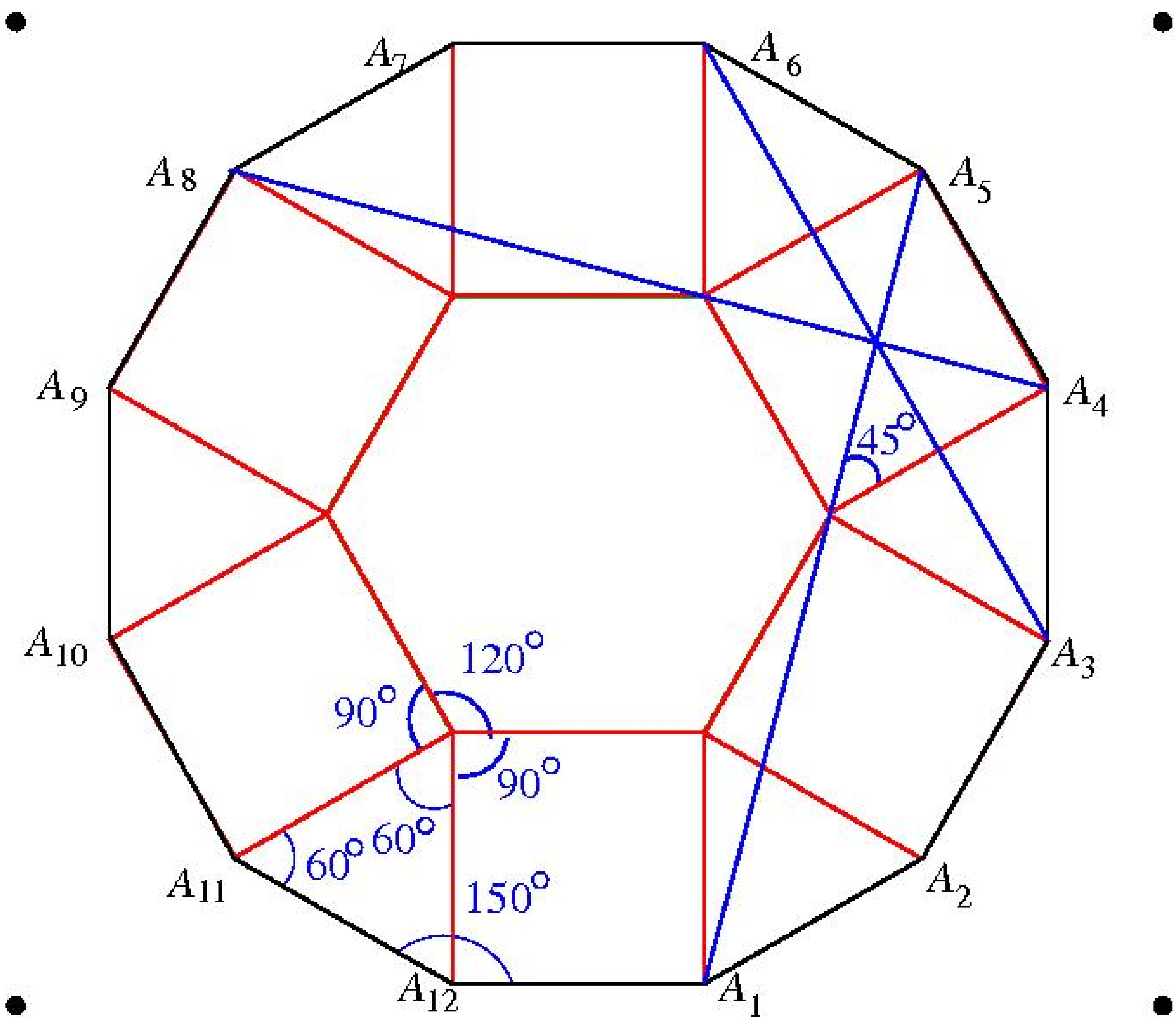
A_1
A_2
A_3
A_4
A_5
A_6
A_7
A_8
A_9
A_{10}
A_{11}
A_{12}
120°
90°
90°
60°
60°
150°



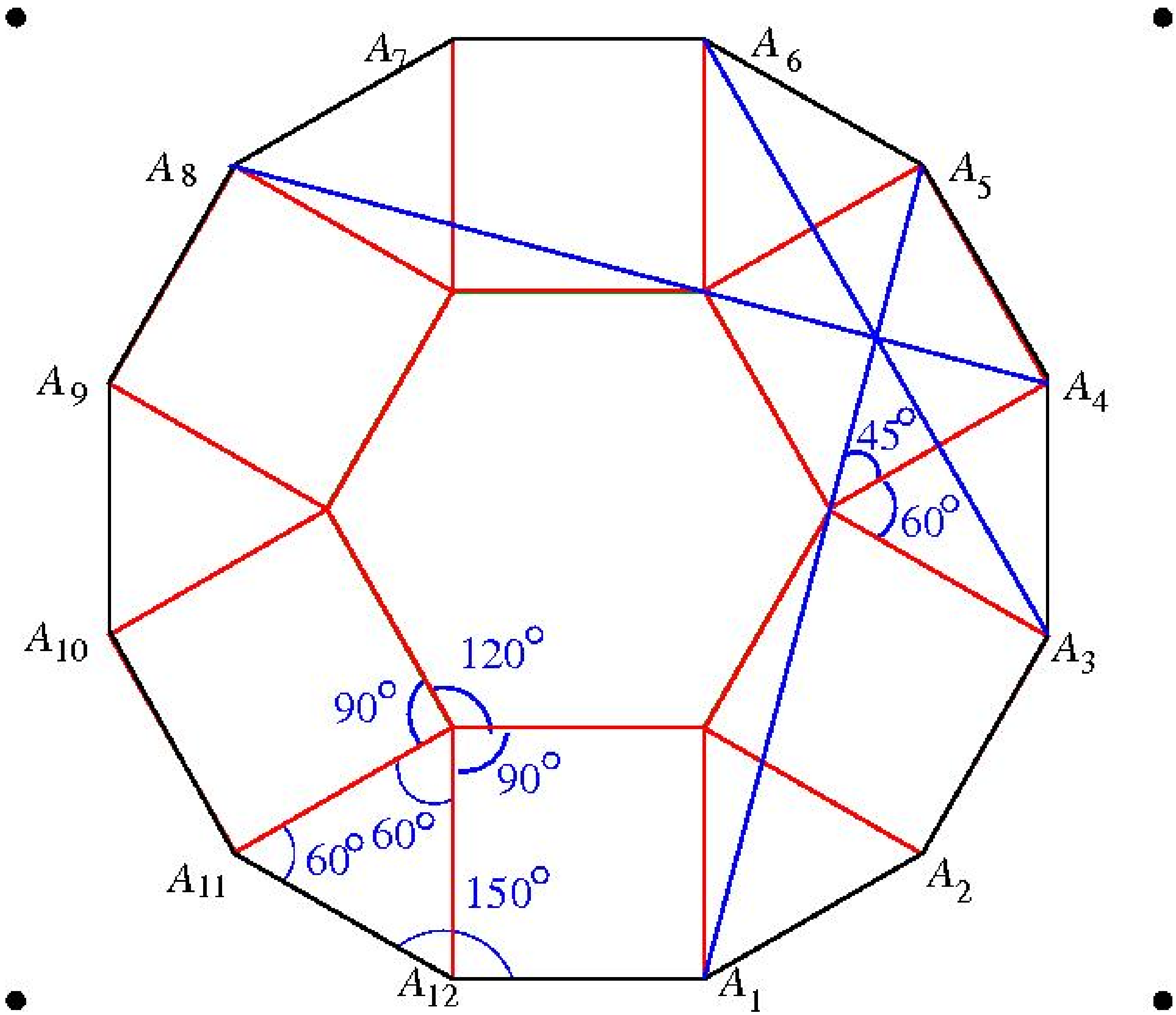
A_1
A_2
A_3
A_4
A_5
A_6
A_7
A_8
A_9
A_{10}
A_{11}
A_{12}
120°
90°
90°
60°
60°
150°



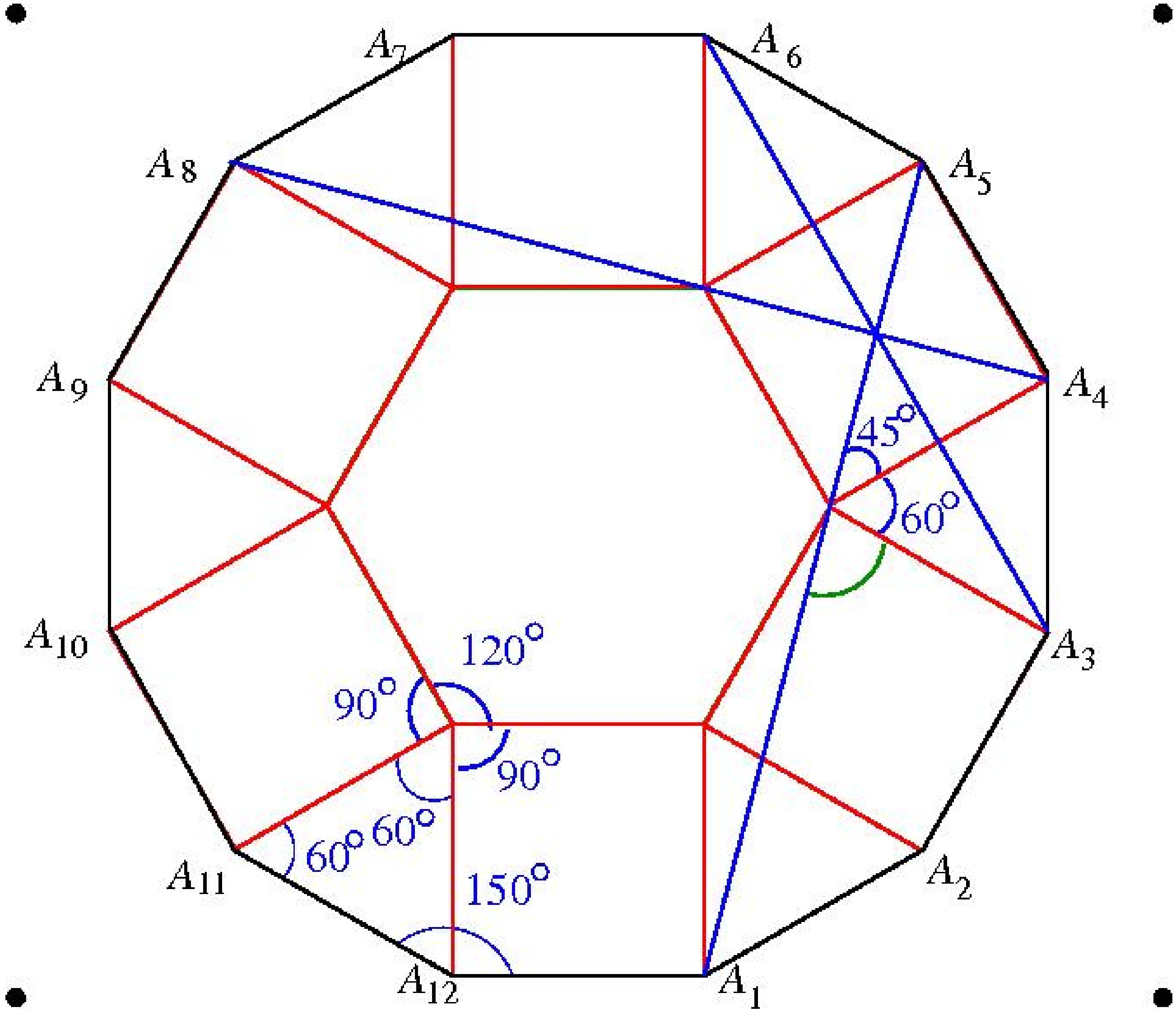
A_1
A_2
A_3
A_4
A_5
A_6
A_7
A_8
A_9
A_{10}
A_{11}
A_{12}
120°
90°
90°
60°
60°
150°



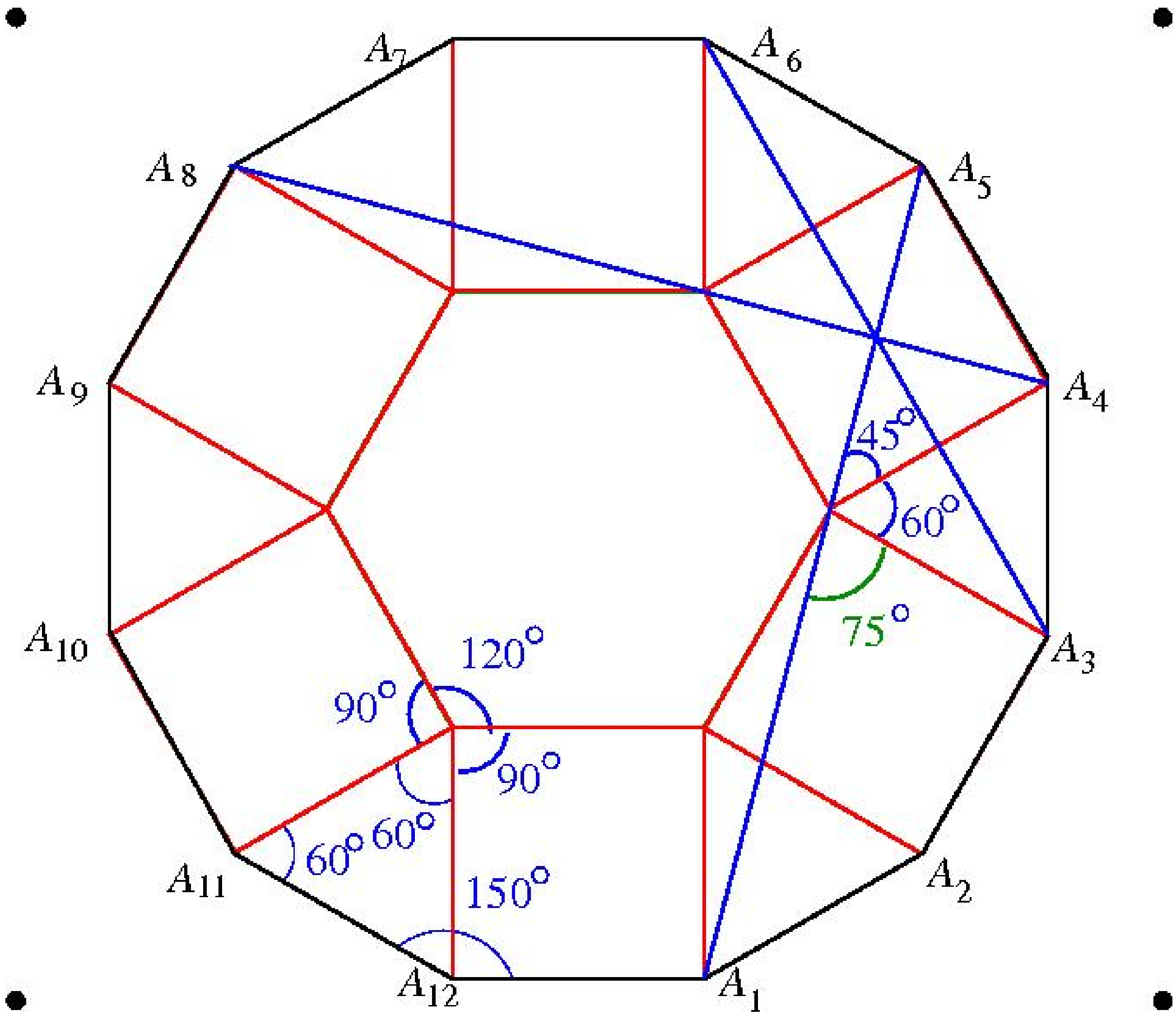
A_7
A_6
A_8
A_5
A_9
A_4
$45°$
A_{10}
A_3
$120°$
$90°$
$90°$
$60°$
$60°$
A_{11}
A_2
$150°$
A_{12}
A_1



A_7
A_6
A_8
A_5
A_9
A_4
45°
60°
A_{10}
A_3
120°
90°
90°
60°
60°
A_{11}
150°
A_2
A_{12}
A_1



A_1
A_2
A_3
A_4
A_5
A_6
A_7
A_8
A_9
A_{10}
A_{11}
A_{12}
45°
60°
120°
90°
90°
60°
60°
150°

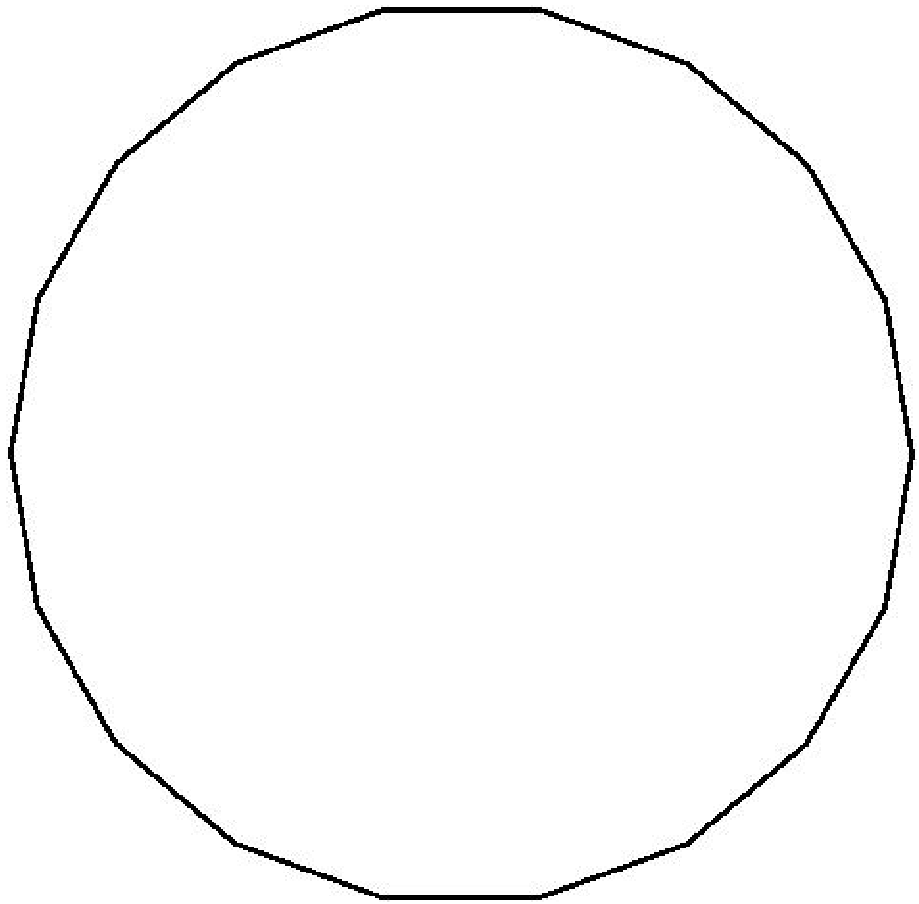


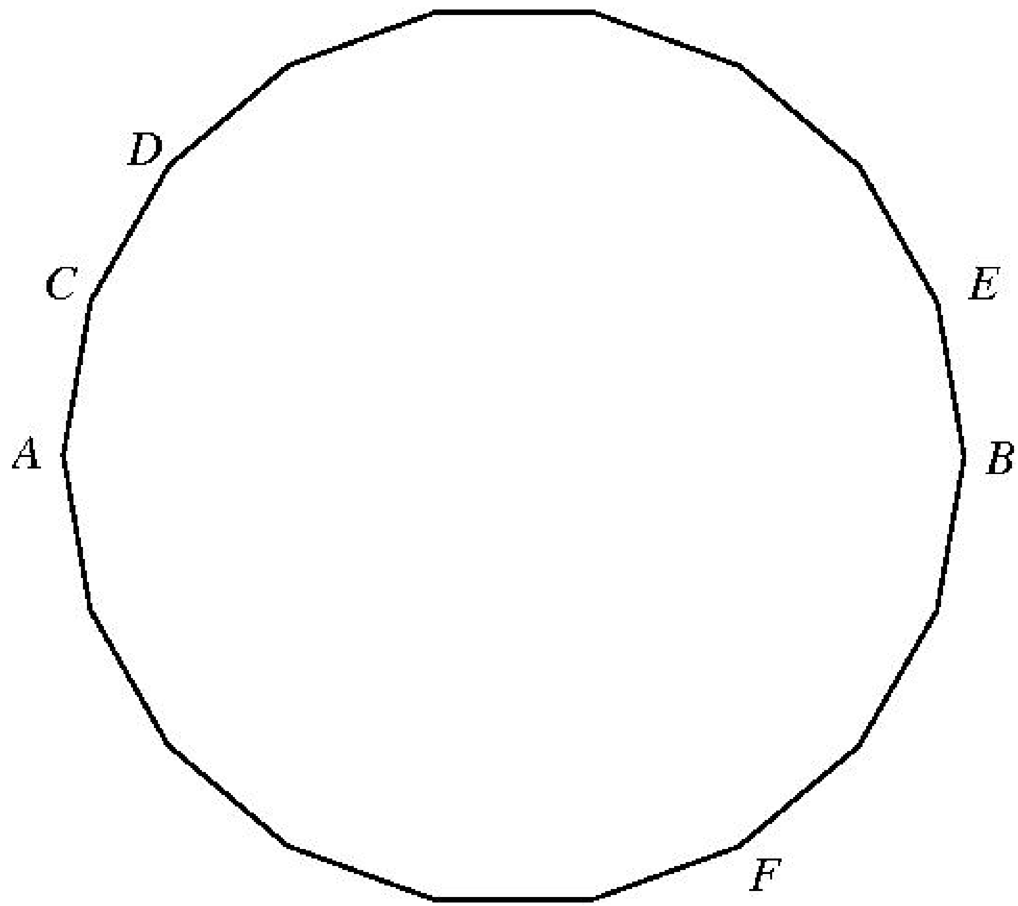
A_1
A_2
A_3
A_4
A_5
A_6
A_7
A_8
A_9
A_{10}
A_{11}
A_{12}
45°
60°
75°
120°
90°
90°
60°
60°
150°

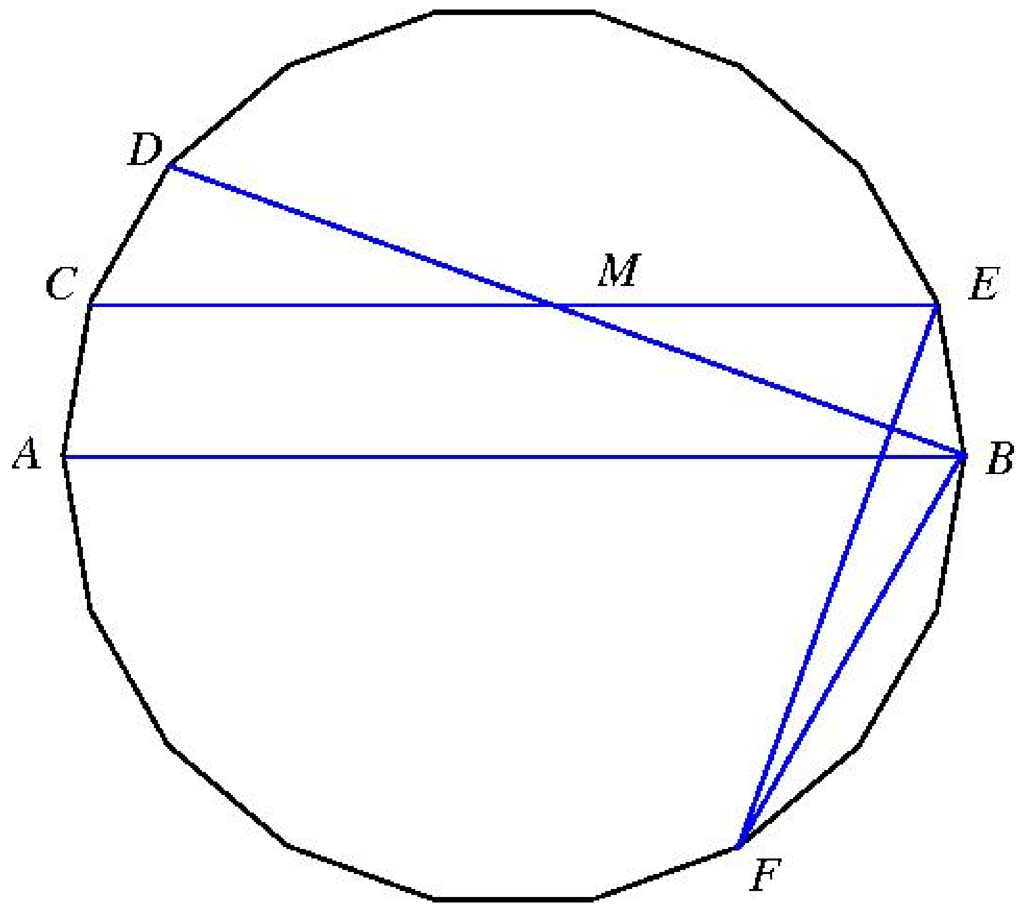
Problem 9. *On a circle of diameter AB choose points C, D, E on one side of AB and F on the other side such that $\overset{\frown}{AC} = \overset{\frown}{CD} = \overset{\frown}{BE} = 20^\circ$ and $\overset{\frown}{BF} = 60^\circ$. Prove that $FM = FE$.*

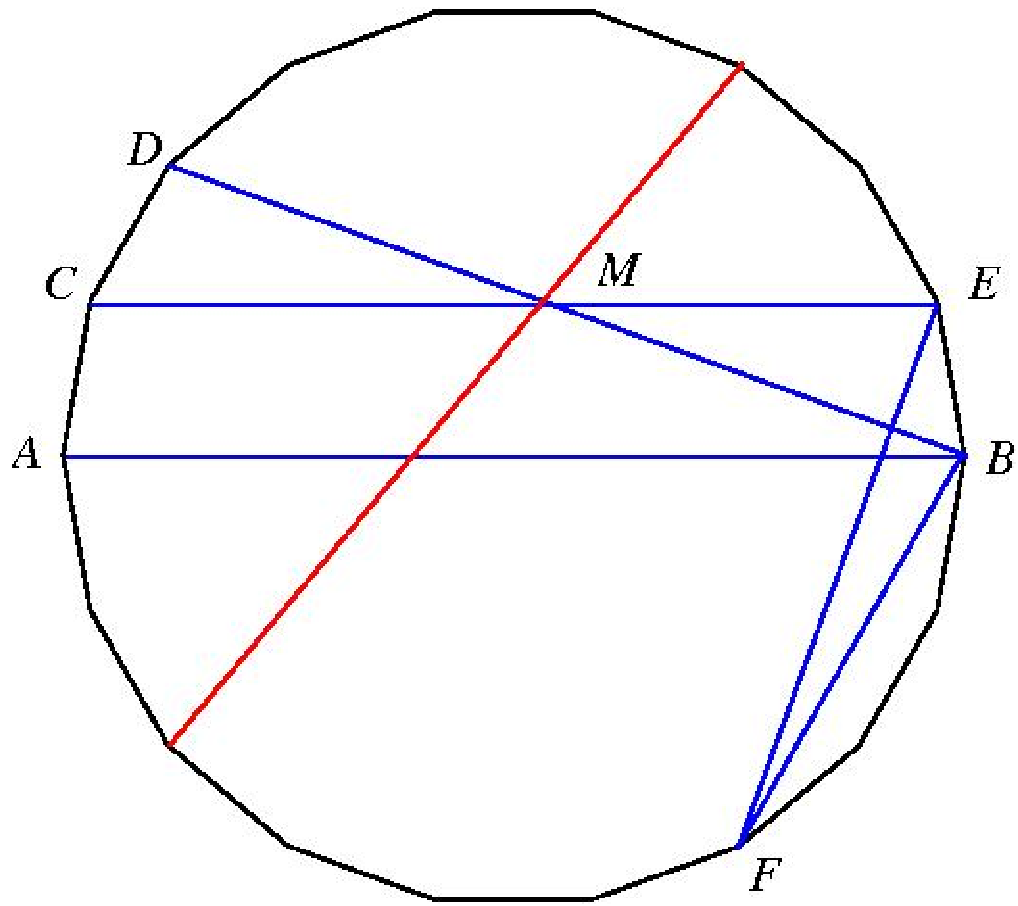
Problem 9. *On a circle of diameter AB choose points C, D, E on one side of AB and F on the other side such that $\overset{\frown}{AC}=\overset{\frown}{CD}=\overset{\frown}{BE}=20^\circ$ and $\overset{\frown}{BF}=60^\circ$. Prove that $FM = FE$.*

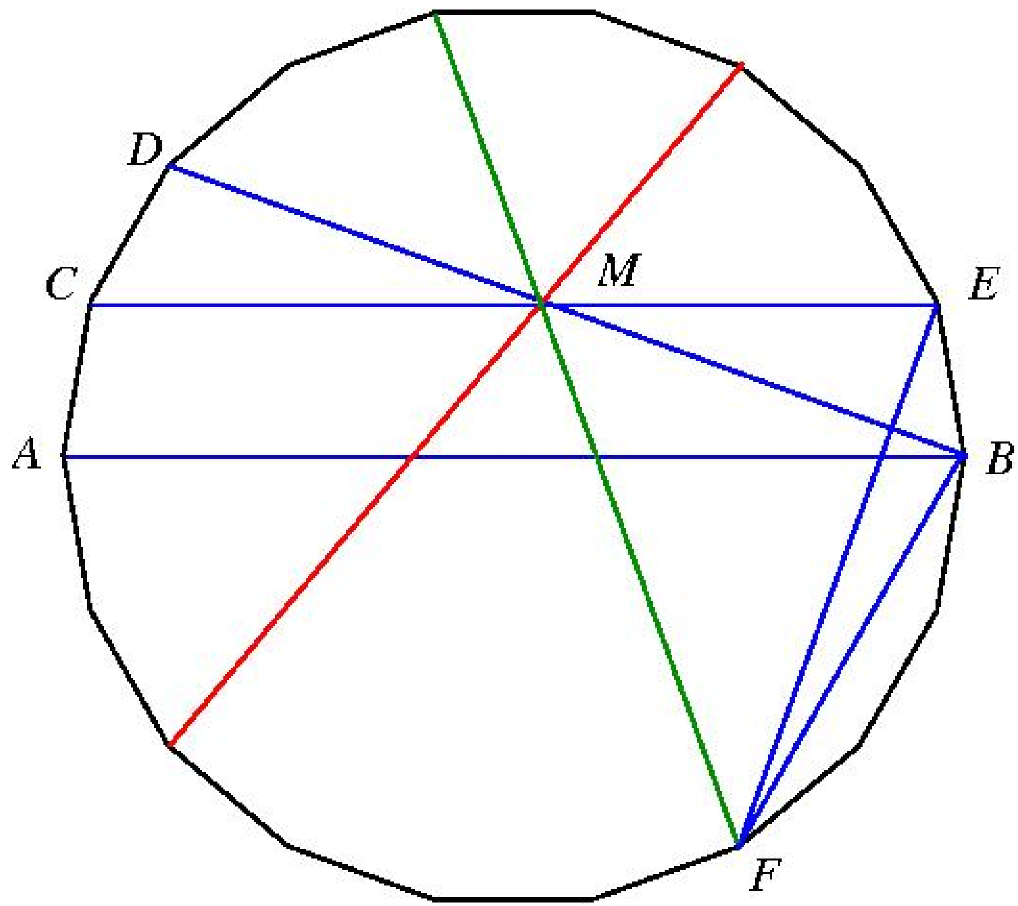
Where is the regular polygon in this problem???

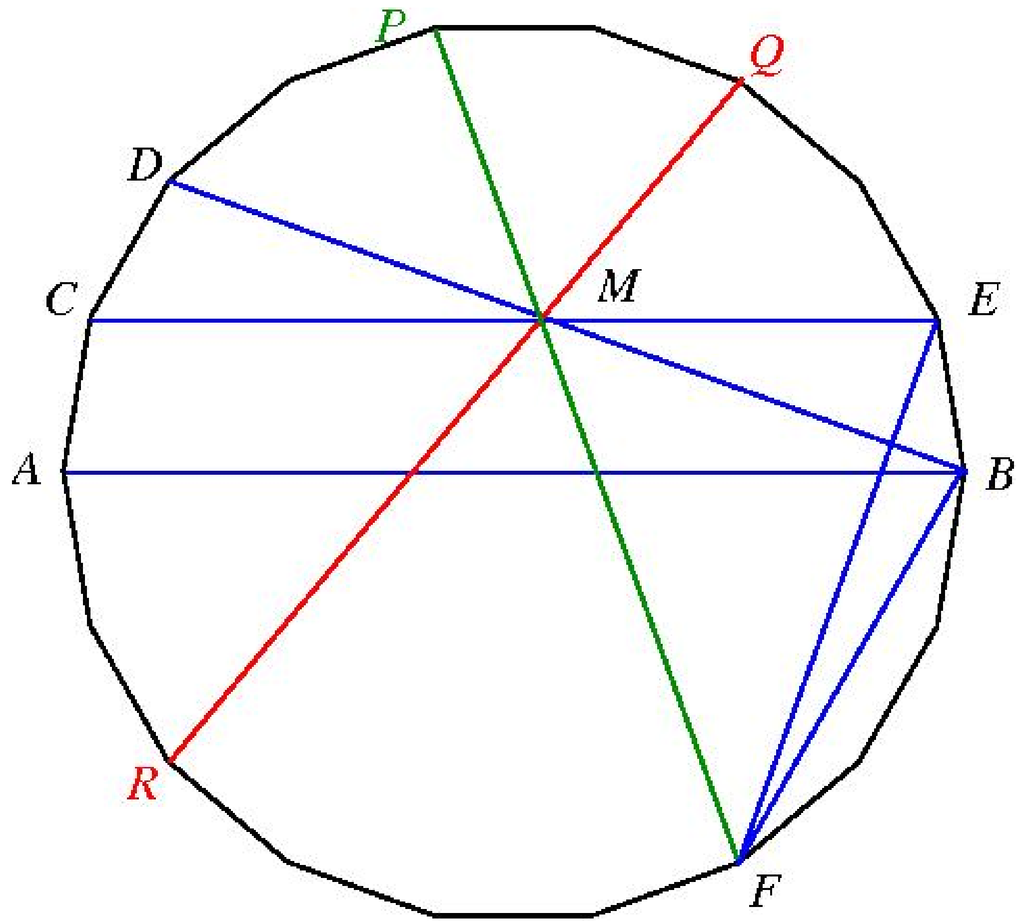


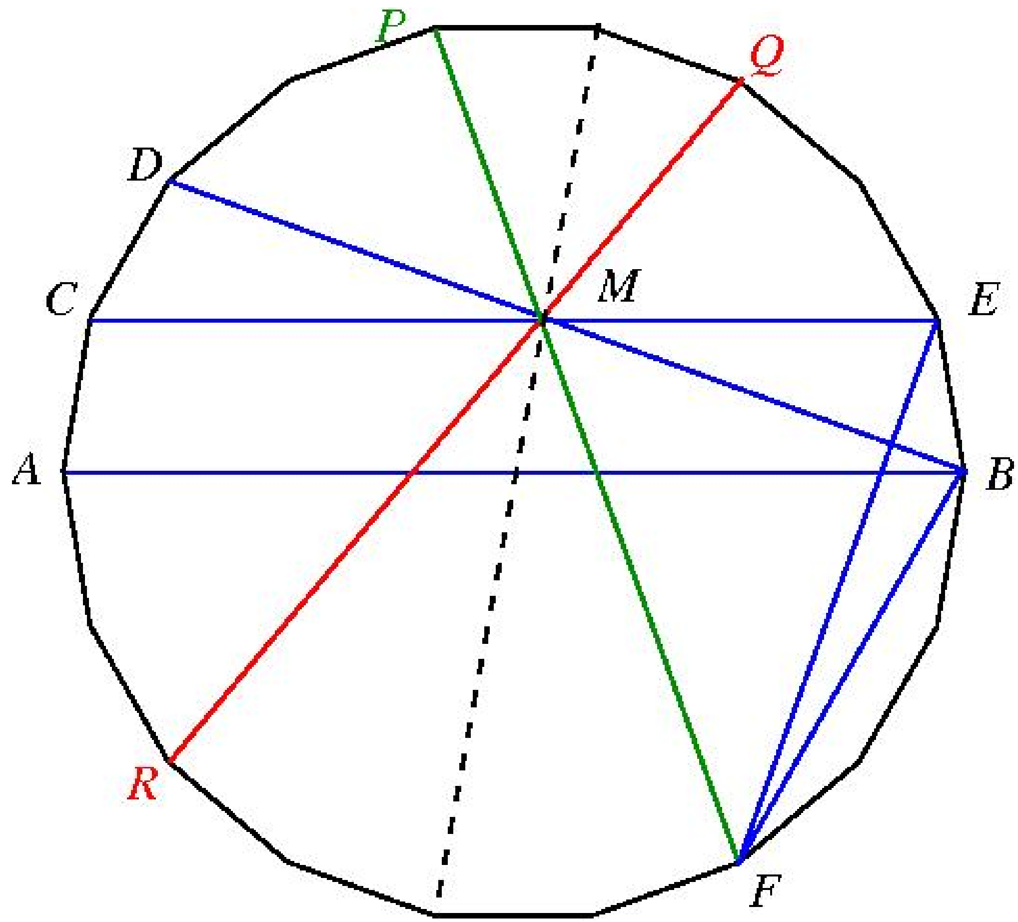

D
C
E
A
B
F


D
C
M
E
A
B
F


D
C
M
E
A
B
F


D
C
M
E
A
B
F


P
Q
D
M
C
E
A
B
R
F


P
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C
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A
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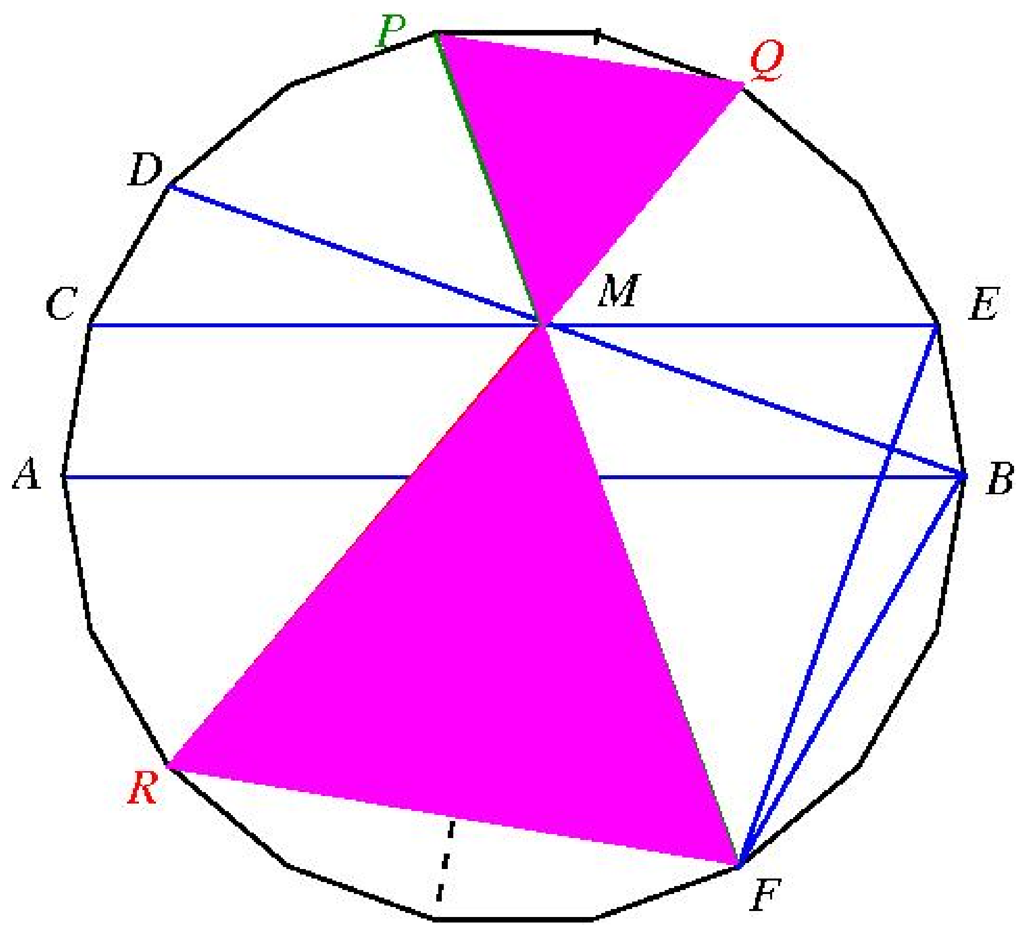
R
F


P
Q
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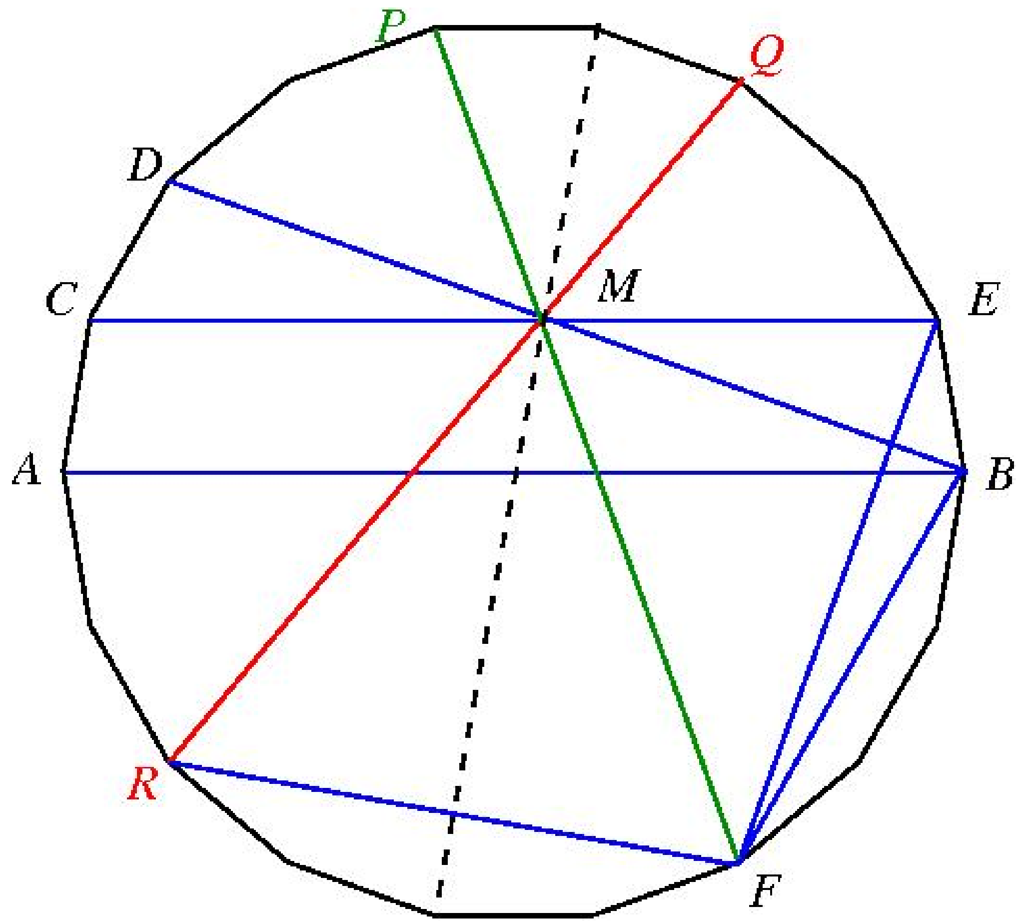
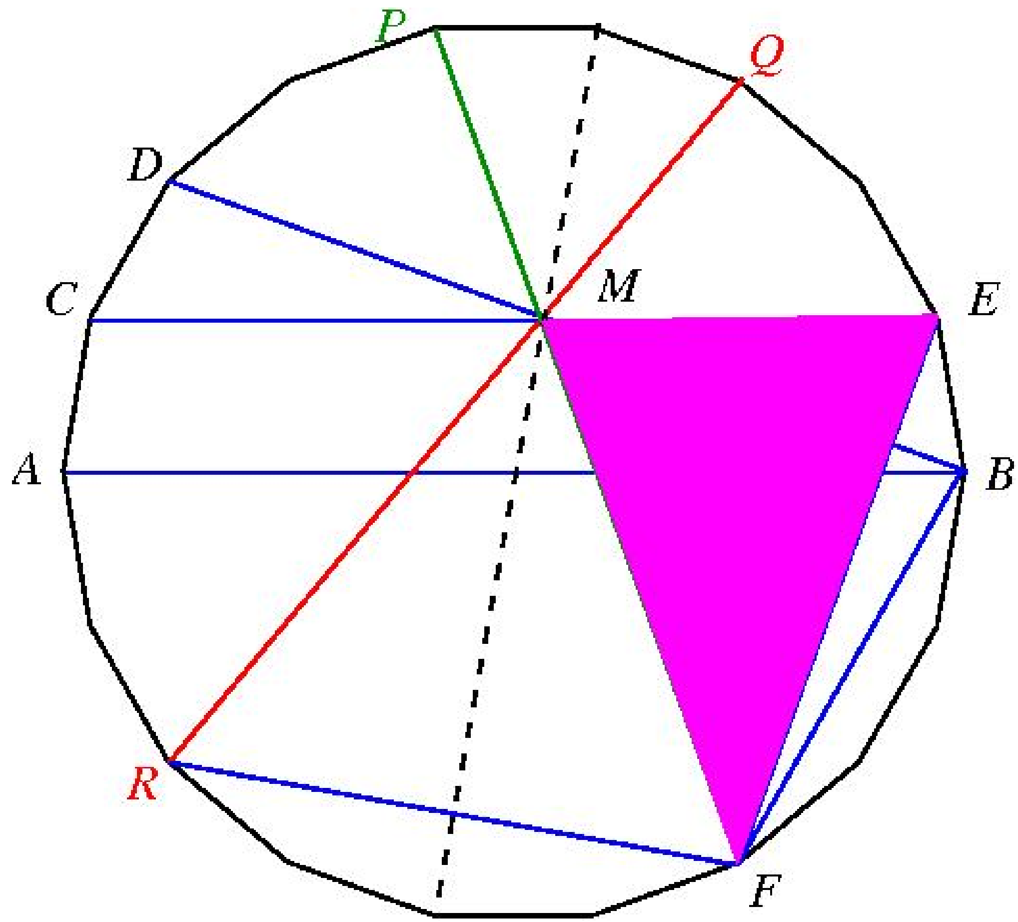
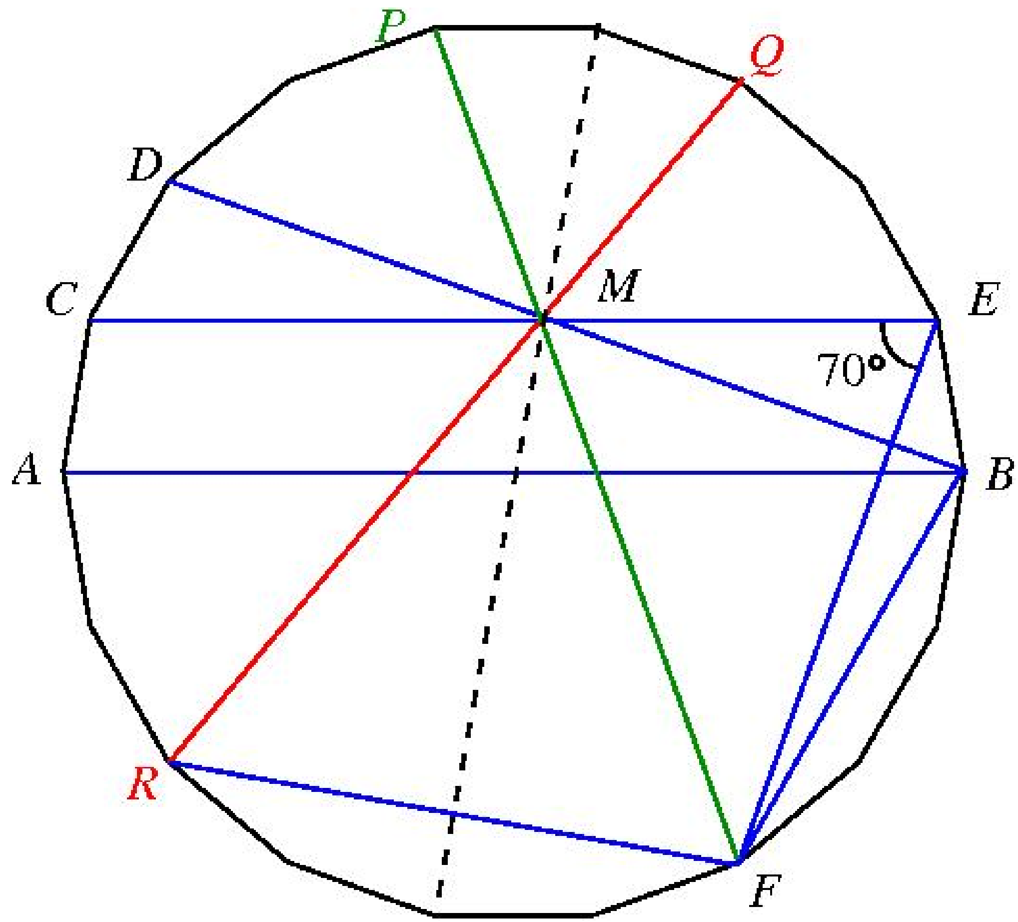

P
Q
D
M
C
E
A
B
R
F


P
Q
D
M
C
E
A
B
R
F


P
Q
D
M
C
E
70°
A
B
R
F

Problem 10. *For what n does there exists an n-gon in the plane all of whose vertices have integer coordinates?*

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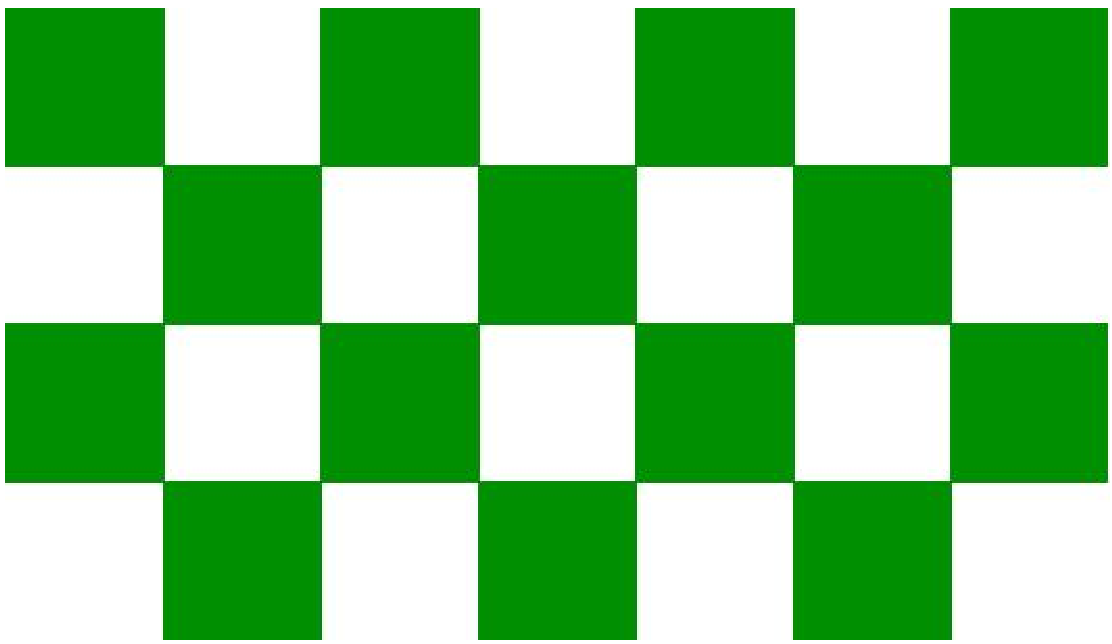
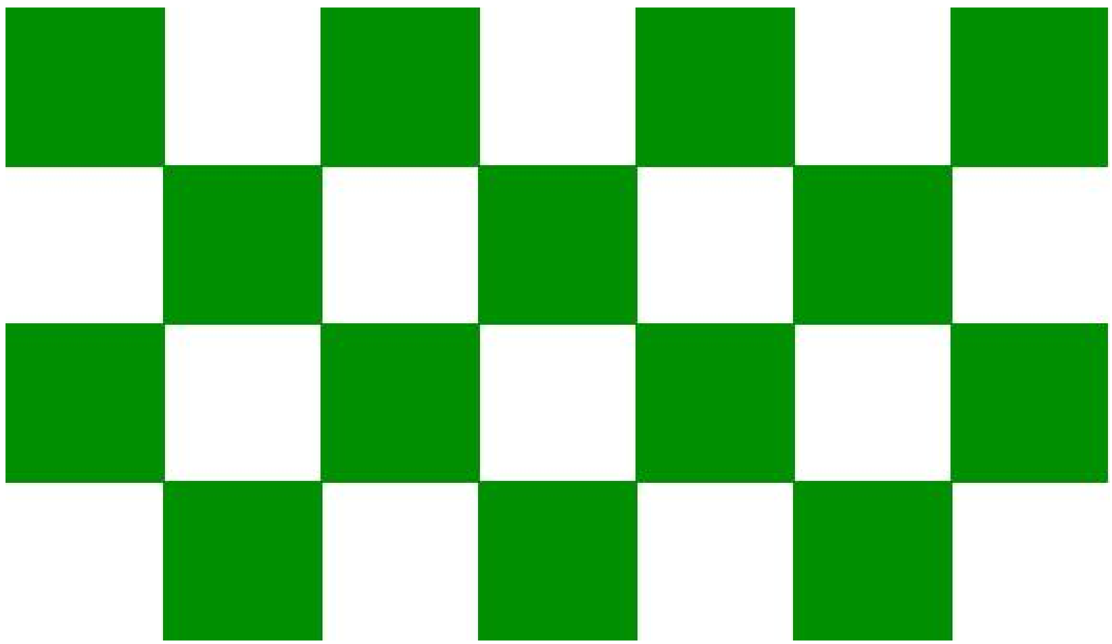
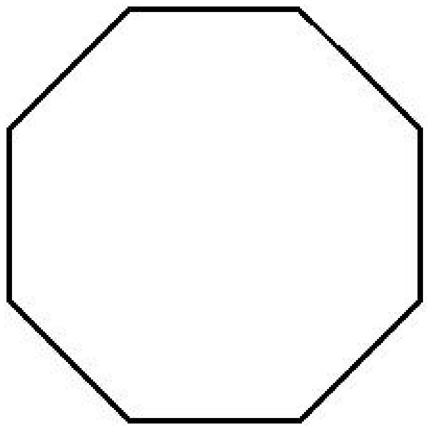


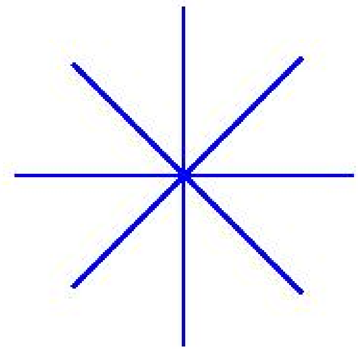
Problem 10. *For what n does there exists an n-gon in the plane all of whose vertices have integer coordinates?*



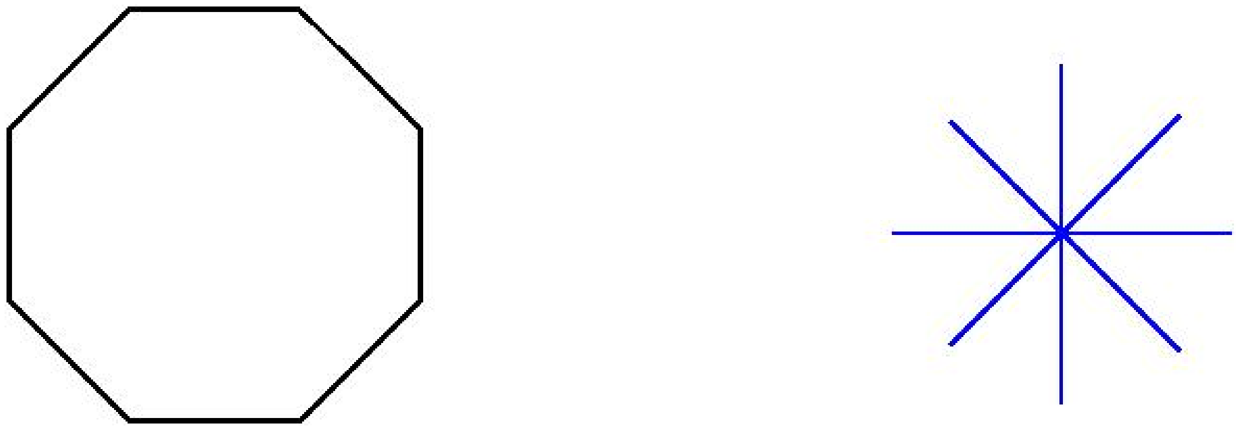
Are there other regular polygons besides the square?

For $n > 6$, start with the smallest such polygon...





and produce a smaller one.



If there is a regular n-gon with vertices of integer coordinates, the center has rational coordinates.

By changing the scale we can assume that the center has integer coordinates as well.

Several 90° rotations around the center produce a regular polygon with 12 or 20 sides with vertices of integer coordinates, which we know cannot exits.