Topics for the Preliminary Examination

Results for which you should know both statement and proof:

1. Holomorphic maps preserve angles. (Theorem 4)
2. Cauchy's theorem. (Theorem 7)
3. The Cauchy-Pompeiu formula. (Theorem 8)
4. Morera's theorem. (Theorem 12)
5. Goursat's theorem (Theorem 13)
6. The fundamental theorem of algebra (Theorem 19)
7. The argument principle. (Theorem 22)
8. Rouche's theorem. (Theorem 23)
9. Laurent series development (Theorem 27)
10. The Casorati-Weierstrass theorem (Theorem 28)
11. The maximum modulus theorem (Theorem 30)
12. The Schwartz's lemma (Theorem 31)
13. The fact that $H(X, Y)$ is closed in $C(X, Y)$ (Theorem 37)
14. Hurwitz's theorem (Theorem 40)
15. Convergence of theta functions (Theorem 41)
16. Montel's theorem (Theorem 49)
17. The Riemann mapping theorem (Theorem 51)
18. Runge's approximation theorem I (Theorem 52)
19. The Mittag-Leffler theorem (Theorem 55)
20. Schwarz reflection principle (Theorem 56)
21. The properties of the Weierstrass p-function (Propositions 25,26,27)

Topics for which you should know statements only

1. The Weierstrass factorization theorem. (Theorem 44)
2. Definition of the gamma function.
3. Definition of Riemann's zeta function and the Riemann hypothesis.
4. The existence of the universal covering space. (Theorem 62)
5. The uniformization theorem. (Theorem 63)

Techniques you should know:

1. How to find the radius of convergence of a power series.
2. How to construct Möbius transformations between given domains.
3. How to classify isolated singularities of a holomorphic function and the behaviour of a function near a singularity.
4. How to use residues to compute integrals.
5. How to use Rouche's theorem to locate zeros.
6. How to prove the uniform convergence of a sequence/series of functions.
7. How to construct biholomorphic functions between simply connected domains in the plane.
8. How to prove that the Riemann sphere and a one-dimensional abelian variety is a Riemann surface.
9. How to check that a curve in $\mathbb{C}^2$ is a Riemann surface.
10. How to check that a projective curve is a Riemann surface.
11. How to check that a function is harmonic and how to find its harmonic conjugates.

1. Find the radius of convergence of the power series expansion of the function

$$f(z) = \sqrt{z} + \frac{1}{\sin z} + \frac{1}{z-3}$$

at $z = 2$.

2. Classify the singularities of the function

$$f(z) = \frac{1}{z} + \frac{1}{\sin \frac{1}{z}} + \frac{\sin \pi z}{z-2}.$$

3. Find the Möbius transformations that are automorphisms of the upper half-plane $\{z \mid \text{Im } z > 0\}$.

4. How many zeros of the polynomial

$$p(z) = z^6 + 3z^5 + z^4 + 1$$

are inside the square with vertices $4 + 4i,\ 4 - 4i,\ -4 + 4i,\ -4 - 4i$ outside the square with vertices $2, 2i, -2, -2i$?

5. Compute the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + x + 1}.$$

6. Prove that the series

$$\sum_{k=1}^{\infty} \frac{k}{(z-k)^k}$$

defines a meromorphic function on $\mathbb{C}$, while the series

$$\sum_{k=1}^{\infty} \frac{k}{z^k}$$

does not define a meromorphic function on $\mathbb{C}$.

7. Find a biholomorphic map f between $G_1 = \{z \mid |z-1| < 1\}$ and $G_2 = \{z \mid 3\pi/4 < \arg z < 5\pi/4\}$ such that $f(1) = -1$ and $f'(1) > 0$.

8. Find a conformal map from the first quadrant to itself that maps the point $1 + 2i$ to the point $2 + i$, and whose derivative at $1 + 2i$ is imaginary.

9. Which of the following curves in $\mathbb{C}^2$ are Riemann surfaces?
(a) $\{(z, w) \in \mathbb{C}^2 \mid e^w - z = 0\}$ (have you seen this before?)
(b) $\{(z, w) \in \mathbb{C}^2 \mid w^3 = z^5\}$ (have you seen this before?)
(c) $\{(z, w) \in \mathbb{C}^2 \mid w^2 = z^4 + z^3 + z^2 + z + 1\}$.

10. Which of the following projective curves in $\mathbb{C}P^2$ define Riemann surfaces?
(a) $\{[Z_0, Z_1, Z_2] \in \mathbb{C}P^2 \mid Z_0Z_2^2 - Z_1^3 - Z_0Z_1^2 - Z_0^2Z_1 - Z_0^3 = 0\}$
(b) $\{[Z_0, Z_1, Z_2] \in \mathbb{C}P^2 \mid Z_0Z_2^2 - Z_1^3 = 0\}$.

11. Which of the functions $h_1(x, y) = x^3 - 3xy^2$, $h_2(x, y) = \frac{x}{x^2+y^2}$, $h_3(x, y) = x^2 + y^2$ are harmonic? Find their harmonic conjugates.