Review aide for the Preliminary Examination and for the final Topology Examination

1. The definition of the fundamental group, the fundamental group of the circle, of the n-dimensional sphere, of a product of two spaces. The theorem about the fundamental groups of homotopically equivalent spaces with proof.

2. Applications of the fundamental group to proving

   – the Brower fixed-point theorem for the disk
   – the fundamental theorem of algebra
   – the 2-dimensional Borsuk-Ulam theorem

3. Covering spaces

   – path lifting theorem, homotopy lifting theorem, the general lifting theorem
   – the universal covering space,
   – examples, fundamental groups computed using covering spaces
   – deck trasformations.

4. Applications of the Seifert-van Kampen Theorem to the computation of various topological spaces:

   – graphs,
   – surfaces,
   – n-fold dunce cap.

5. Homology groups with integer and real coefficients defined using Δ-complexes.

   – computation of homology groups
   – computation of the homomorphisms in homology induced by continuous maps between Δ-complexes.

6. Euler characteristic. Know the proof that for (finite) Δ-complexes the Euler characteristic is the alternated sum of the numbers of simplexes in each dimension.

Problems

1. Compute the fundamental group of the sphere with n crosscaps.
2. Compute the fundamental group of a solid handlebody.
3. Compute the fundamental group of the torus.
4. Show that if $h : S^1 \to S^1$ is nullhomotopic, then h has a fixed point and h maps some point x to its antipode.
5. Let A be a subspace of $\mathbf{R}^n$; let $h : (A, a_0) \to (Y, y_0)$. Show that if h is extendable to a continuous map of $\mathbf{R}^n$ into Y then h_* is the trivial homomorphism.
6. Find topological spaces whose fundamental groups are isomorphic to $\mathbf{Z}_n \times \mathbf{Z}_m$ and $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_3$.
7. Show that every continuous map $f : S^3 \to S^1$ is null-homotopic.
8. What is the universal covering space of the wedge of two circles? What is the group of deck transformations?
9. Compute the homology with integer coefficients of the torus.
10. Compute the homology with integer and real coefficients of a Klein bottle.
11. Compute $H_3(S^3, \mathbb{R})$, where S^3 is the 3-dimensional sphere.
12. Compute the homology groups of a genus 2 surface with two punctures.
13. Compute the Euler characteristic of the 2-dimensional, respectively 3-dimensional sphere.
14. Compute the homology with integer coefficients and the Euler characteristic of the following 1-dimensional complex.