Review aide for the preliminary examination

1. Definition and examples of topological spaces
   - how to compare topologies
   - continuous functions and homeomorphisms
2. Methods for constructing topological spaces
   - basis and subbasis
   - induced topology
   - product topology
   - metric topology and continuous functions on metric spaces
   - quotient topology
   - manifolds
3. Closed sets, limit points
   - definition and examples of closed sets
   - closure of a set
   - limit points (the sequence lemma, continuity and convergence of sequences)
   - definition of Hausdorff spaces.
4. Connected, locally connected, path connected, locally path connected spaces
   - examples and conterexamples
   - basic properties (e.g. continuous functions map connected spaces to connected spaces)
   - connected spaces of the real line, the intermediate value property with applications
   - connected, path connected components
5. Compact spaces
   - examples and counterexamples
   - basic properties (e.g. every continuous function has a maximum, behaviour under continuous maps)
   - tube lemmma with proof
   - compactness of product spaces (the case of finitely many spaces)
   - Lebesgue's number lemma
   - compact spaces in $\mathbf{R}^n$ with applications
   - in metric spaces compact is equivalent to sequentially compact with proof
   - local compactness, the one point compactification theorem
6. The separation axioms

- basic properties of regular and normal spaces and some criteria for a space to be regular or normal
- the Uryson lemma
- the Tietze extension theorem

Problems

1. Show that the standard topology has a subbasis given by all the sets of the form $\{x > a,\ a \in \mathbb{Q}\}$ and $\{x < a,\ a \in \mathbb{R}\backslash\mathbb{Q}\}$.
2. Describe a basis of a topology on $C([0,1])$, the continuous functions on the unit interval, such that if $f_n \to f$ then $f_n(x) \to f(x)$ for all x.
3. Explain which of the two is a basis of the standard topology of the plane: (i) $\{(x,y)\,|\,a \leq x \leq b, c \leq y \leq b,\ a,b,c,d \in \mathbb{R}\}$, (ii) $\{(x,y)\,|\,a < x < b, c < y < d,\ a,b,c,d \in \mathbb{Q}\}$.
4. Let $A \subset X$ and $B \subset Y$. Show that in $X \times Y$,
$$\overline{A \times B} = \overline{A} \times \overline{B}$$
5. Let $\mathbb{R}_l$ be the space $\mathbb{R}$ endowed with the topology having the basis $\{[a,b)\,|\,a,b \in \mathbb{R}\}$. What are the continuous functions $f : \mathbb{R} \to \mathbb{R}_l$, where the first space is with the standard topology?
6. Show that X is Hausdorff if and only if the diagonal $\Delta = \{(x,x)\,|\,x \in X\}$ is closed in $X \times X$.
7. Let $f, g : X \to Y$ be continuous functions. Show that $h(x) = \min(f(x), g(x))$ is continuous.
8. Show that $(0,1)$ is not homeomorphic to $[0,1]$.
9. Show that the closure of a connected set is connected.
10. Let $Y \subset X$, such that X and Y are connected. Show that if A and B form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected.
11. Show that $\mathbb{R}^2\backslash\mathbb{Q}^2$ is path connected.
12. Prove that every continuous mapping of a circle into a line carries some pair of diametrically opposite points to the same point.
13. Let $f : [a,b] \to [a,b]$ be a continuous function. Prove that f has a fixed point.
14. A cross country runner runs a six-mile course in 30 minutes. Prove that somewhere along the course the runner ran a mile in exactly 5 minutes.
15. Give an example of a space that is connected but not locally connected. Give an example of a space that is locally connected but not connected. Give an example of a space that is connected but not path connected.

16. Give $[0,1]^{\mathbf{R}}$ the uniform topology. Find an infinite subset of this space that has no limit point.

17. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \geq 64.$$

18. Show that if $f : X \to Y$ is a closed, continous, surjection with X locally compact and each $f^{-1}(y)$ compact, then Y is locally compact.

19. Prove that the cube $[0,1]^n$ is compact.

20. If $f : X_1 \to X_2$ is a homeomorphism between locally compact Hausdorff spaces, show that f extends to a homeomorphism of their one-point compactification.

21. Let X have a countable basis, and let A be an uncountable subset of X. Show that uncountably many points of A are limit points of A.

22. Show that if X has a countable dense subset, every collection of disjoint open sets in X is countable.

23. Show that if X and Y are regular, then the product $X \times Y$ is regular.

24. Show that every compact Hausdorff space is normal.

25. Prove that if X is second countable, then every base for the topology of X has a countable subcollection which is a base for that topology.