REPRESENTATIONS OF THE KAUFFMAN BRACKET SKEIN ALGEBRA OF THE PUNCTURED TORUS

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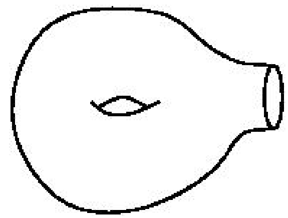
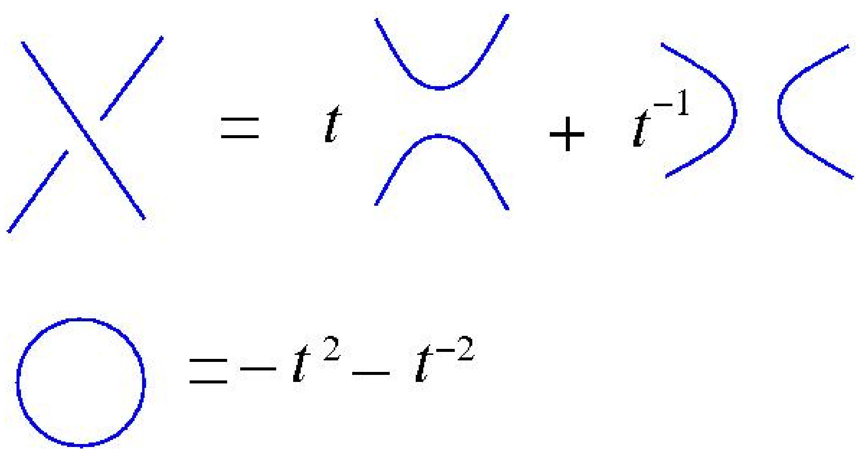
WE STUDY THE REPRESENTATIONS OF THE KAUFFMAN BRACKET SKEIN ALGEBRA OF THE PUNCTURED TORUS ON SKEIN MODULES DEFINED IN THE SOLID TORUS. WE SHOW HOW TO CALCULATE THE MATRICES OF THE RESHETIKHIN-TURAEV REPRESENTATION OF THE MAPPING CLASS GROUP FROM THESE REPRESENTATIONS.

Let $\Sigma_{1,1}$ *be the* *punctured torus*, *namely the torus with one open disk removed.*

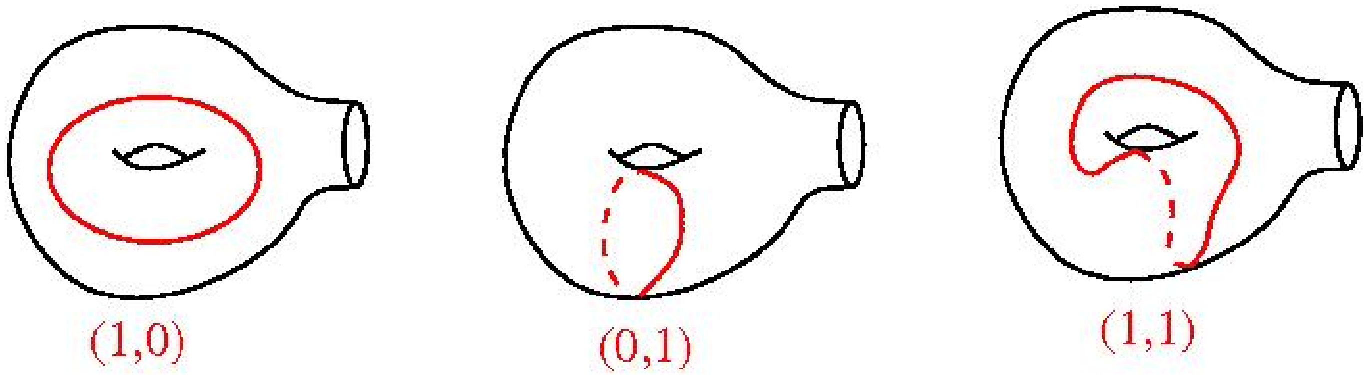


The *Kauffman bracket skein algebra of the punctured torus*, $K_t(\Sigma_{1,1} \times [0,1])$, *is the quotient of the free* $\mathbb{C}[t, t^{-1}]$*-module with basis the set of isotopy classes of framed links in* $\Sigma_{1,1} \times [0,1]$ *by the Kauffman bracket skein relations:*



Multiplication is defined by gluing two copies of $\Sigma_{1,1} \times [0,1]$ *along a* $\Sigma_{1,1}$.

Bullock and Przytycki showed that $K_t(\Sigma_{1,1} \times [0,1])$ *is generated by the curves*

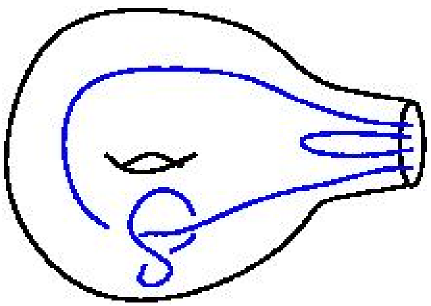


subject to the relations

$$t(1,0)(0,1) - t^{-1}(0,1)(1,0) = (t^2 - t^{-2})(1,1)$$
$$t(0,1)(1,1) - t^{-1}(1,1)(0,1) = (t^2 - t^{-2})(1,0)$$
$$t(1,1)(1,0) - t^{-1}(1,0)(1,1) = (t^2 - t^{-2})(0,1).$$

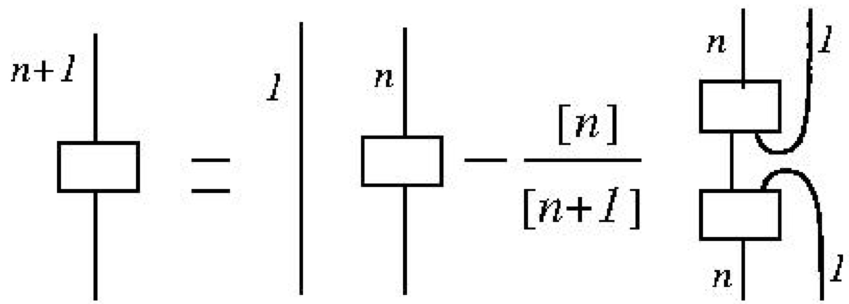
Our goal is to study representations of this algebra related to the problem of quantizing the moduli space of flat $SU(2)$-connections on the punctured torus with fixed holonomy on the boundary.

Fix $2n$ points inside the puncturing disk and consider the Kauffman bracket skein module of the solid torus with $2n$ points on the boundary, $K_t(S^1 \times \mathbb{D}^2, 2n)$, in which skeins consist of framed links and n framed arcs with end-points the $2n$ points.



Fix a positive integer r and let $t = e^{\frac{i\pi}{2r}}$. Factor $K_t(S^1 \times \mathbb{D}^2, 2n)$ by an additional skein relation obtained by setting the $r-1$st Jones-Wenzl idempotent equal to zero.

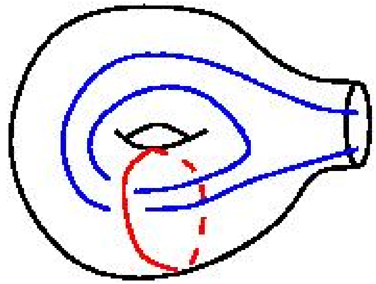
Recall the recursive definition of Jones-Wenzl idempotents



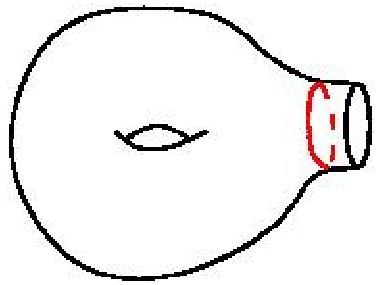
where $[n] = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}$.

The result is denoted by $K_{t,r}(S^1 \times \mathbb{D}^2, 2n)$ *and is called the reduced Kauffman bracket skein module.*

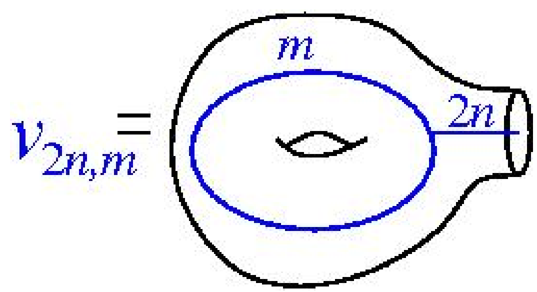
The operation of gluing an $\Sigma_{1,1} \times [0,1]$ *to the part of the boundary of* $S^1 \times \mathbb{D}^2$ *that lies outside of the puncturing disk induces a representation of* $K_t(\Sigma_{1,1} \times [0,1])$ *on* $K_{t,r}(S^1 \times \mathbb{D}^2, 2n)$.

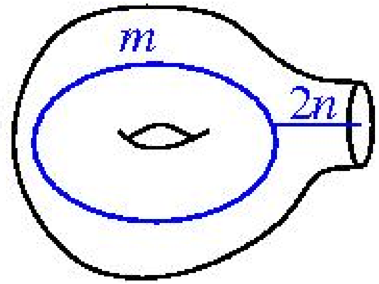


The boundary curve is a central element of $K_t(\Sigma_{1,1}\times[0,1])$ so its eigenspaces are invariant subspaces of the representation.

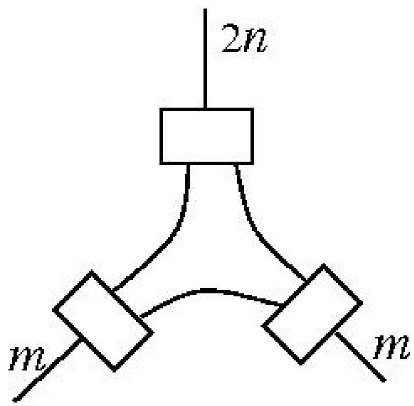


Let $V_{r,n}$ be the subspace of the eigenvalue $-t^{4n+2}-t^{-4n-2}$. Then $V_{r,n}$ is an irreducible representation with basis $v_{2n,m}$, $n \leq m \leq r-2-n$ given by



$v_{2n,m} =$ 

Here the strands are colored by the m*th and* $2n$*th Jones-Wenzl idempotents and the trivalent vertex is a Kauffman triad*



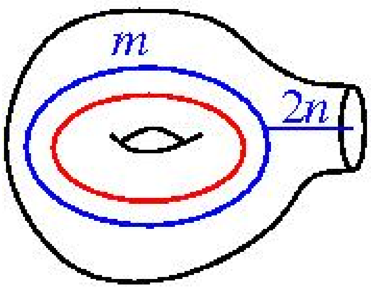
THEOREM. *Let* n *be an integer such that* $0 \le n \le \frac{r-2}{2}$. *The representation of* $K_t(\Sigma_{1,1} \times [0,1])$ *on* $V_{r,n}$ *is given by*

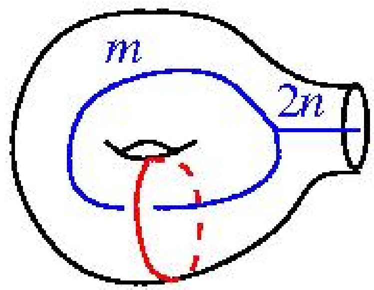
$$(1,0)v_{2n,m} = v_{2n,m+1} + \frac{[m-n][m+n+1]}{[m][m+1]}v_{2n,m-1}$$

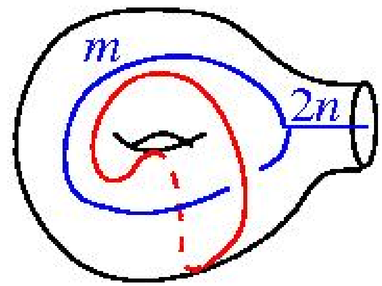
$$(0,1)v_{2n,m} = (-t^{2m+2} - t^{-2m-2})v_{2n,m}$$

$$(1,1)v_{2n,m} = (-t^{-2m-3})v_{2n,m+1} + (-t^{2m+1})\frac{[m-n][m+n+1]}{[m][m+1]}v_{2n,m-1}$$

for $n \le m \le r-2-n$, *with the convention that* $v_{2n,n-1} = v_{2n,r-1-n} = 0$.



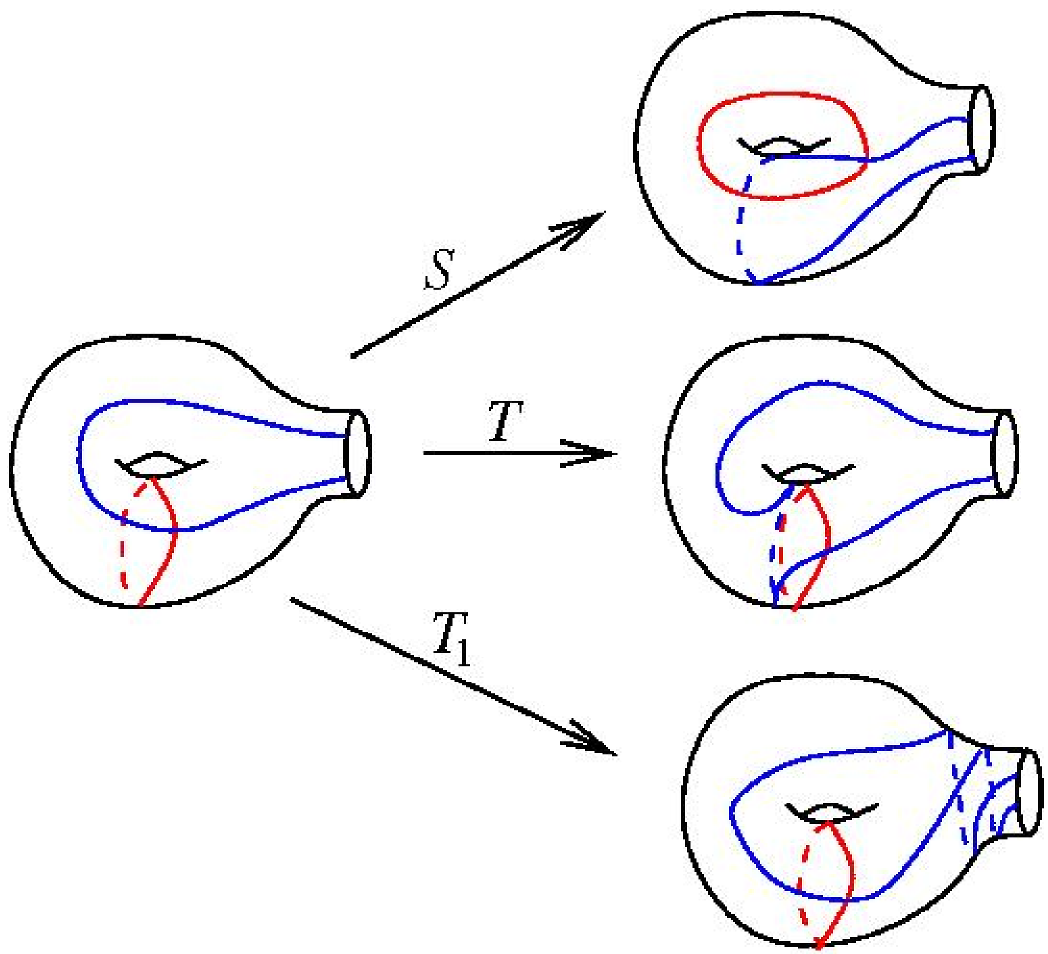




There is a projective representation of the mapping class group of the punctured torus on the space $V_{r,n}$, the Reshetikhin-Turaev representation. This interpolates the operators of the action of $K_t(\Sigma_{1,1}\times[0,1])$ on $K_{t,r}(S^1\times\mathbb{D}^2, 2n)$

$$h((p,q)) = \rho(h)^{-1}(p,q)\rho(h).$$

The mapping class group of the punctured torus is generated by S, T, T_1



Note that T_1 is a multiple of the identity.

The matrices of S and T can be computed from the action by using the equalities

$$
\begin{aligned}
(1,0)Sv_{2n,n+j} &= S(0,1)v_{2n,n+j} \\
(0,1)Sv_{2n,n+j} &= S(1,0)v_{2n,n+j} \\
(1,0)Tv_{2n,n+j} &= T(1,1)v_{2n,n+j} \\
(0,1)Tv_{2n,n+j} &= T(0,1)v_{2n,n+j}
\end{aligned}
$$

These yield the recursive relations

$$
\begin{aligned}
a_{j-1,k} &= (-t^{2n+2k+2} - t^{-2n-2k-2})a_{j,k} - \frac{[j+1][2n+j+2]}{[n+j+1][n+j+2]}a_{j+1,k} \\
a_{j,k-1} &= (-t^{2n+2j+2} - t^{-2n-2j-2})a_{j,k} - \frac{[k+1][2n+k+2]}{[n+k+1][n+k+2]}a_{j+1,k} \\
b_{j,j} &= -t^{2n+2j+1}b_{j-1,j-1}
\end{aligned}
$$

We obtain

$$a_{r-2n-2-j,r-2n-2-k} = P_j(\lambda_{r-n}, (x_l)_{l\geq 1}) \cdot P_k(\lambda_{r-n-j}, (x_l)_{l\geq 1}),$$

where

$$x_l = \frac{[r-n-1-l][n+r-l]}{[r-l-1][r-l]}, \quad l \geq 1,$$
$$\lambda_m = -t^{2m-2} - t^{-2m+2}, \quad m \geq 0$$

and the sequence $P_n(\lambda, (x_l)_{l\geq 1})$ *is defined recursively by*

$$P_{n+1}(\lambda, (x_l)_{l\geq 1}) = \lambda P_n(\lambda, (x_l)_{l\geq 1}) - x_n P_{n-1}(\lambda, (x_l)_{l\geq 1}),$$
$$P_0(\lambda, (x_l)_{l\geq 1}) = 1, \quad P_1(\lambda, (x_l)_{l\geq 1}) = \lambda$$

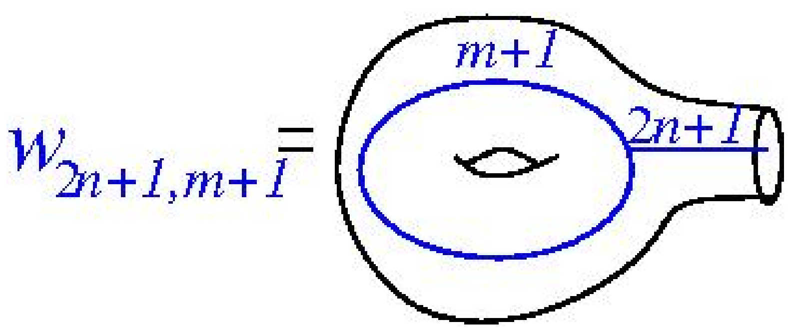
and

$$b_{j,j} = (-1)^{n+j} t^{(n+j)^2-1}$$

The fact that the representation of the Kauffman bracket skein algebra of the punctured torus is a quantization of the moduli space of $su(2)$-connections means that the operators should be self-adjoint. This means that we should normalize the given (orthogonal) basis to an orthonormal basis

$$w_{2n+1,m+1} = \left(\prod_{j=n+1}^{m} \frac{[j-n][j+n+1]}{[j][j+1]} \right)^{-1/2} v_{2n,m}$$

Also the quantization of interest is the quantum group quantization, in which the formulas differ slightly.



The stands are colored by the $m+1$- respectively $2n+1$-dimensional irreducible representations of the quantum group of $SU(2)$.

PROPOSITION. The quantization at $\hbar = \frac{1}{2r}$ of the moduli space of flat $su(2)$-connections on the punctured torus with the trace of the holonomy on the boundary equal to $2\cos\frac{4\pi i n}{r}$, for some $n \in \{0, 1, \ldots, r-2\}$, has the Hilbert space $\mathcal{H}_{r,n}$ with orthonormal basis $w_{2n+1,m}$, $n+1 \leq m \leq r-1-n$, and with the algebra of quantum observables acting by

$$Op(W_{(1,0)})w_{2n+1,m} = \sqrt{\frac{[m-n][m+n+1]}{[m][m+1]}}w_{2n+1,m+1} + \sqrt{\frac{[m-1-n][m+n]}{[m-1][m]}}w_{2n+1,m-1}$$

$$Op(W_{0,1})w_{2n+1,m} = (t^{2m} + t^{-2m})w_{2n+1,m}$$

$$Op(W_{(1,1)})w_{2n+1,m} = t^{-2m-1}\sqrt{\frac{[m-n][m+n+1]}{[m][m+1]}}w_{2n+1,m+1} + t^{2m-1}\sqrt{\frac{[m-1-n][m+n]}{[m-1][m]}}w_{2n+1,m-1}.$$