NON-ABELIAN THETA FUNCTIONS A LA ANDRÉ WEIL

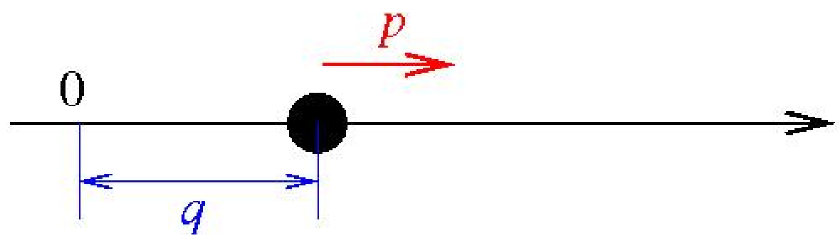
Răzvan Gelca — *Texas Tech University*
Alejandro Uribe — *University of Michigan*

WE WILL SHOW THAT THE CONSTRUCTS OF THE THEORY OF CLASSICAL THETA FUNCTIONS IN THE REPRESENTATION THEORETIC POINT OF VIEW OF ANDRÉ WEIL HAVE ANALOGUES FOR THE NON-ABELIAN THETA FUNCTIONS OF THE WITTEN-RESHETIKHIN-TURAEV THEORY

THE PROTOTYPE:

- *the Schrödinger representation*
- *the metaplectic representation*

They arise when quantizing a free 1-dimensional particle ($\hbar = 1$).



Phase space has coordinates q: position, p: momentum.

Quantization: phase space $\mapsto L^2(\mathbb{R})$,

$$q \mapsto Q = \text{ multiplication by } q,$$
$$p \mapsto P = \frac{1}{i}\frac{d}{dq}.$$

Add to this the operator $E = Id$.

Heisenberg uncertainty principle:

$$QP - PQ = i\hbar E$$

Weyl quantization:

$$\hat{f}(\xi,\eta) = \iint f(x,y)\exp(-2\pi i x\xi - 2\pi i y\eta)dxdy$$

and then defining

$$\mathit{Op}(f) = \iint \hat{f}(\xi,\eta)\exp 2\pi i(\xi Q + \eta P)d\xi d\eta.$$

Notation:

$$\exp(xP + yQ + tE) = e^{2\pi i(xP+yQ+tE)}.$$

EXAMPLE:

$$\exp(x_0 P)\phi(x) = \phi(x + x_0),$$
$$\exp(y_0 Q)\phi(x) = e^{2\pi i x y_0}\phi(x).$$

The elements $\exp(xP + yQ + tE)$, $x, y, t \in \mathbb{R}$ *form the Heisenberg group with real entries* $\mathbf{H}(\mathbb{R})$*:*

$$\begin{aligned} &\exp(xP + yQ + tE)\exp(x'P + y'Q + t'E) \\ &\quad = \exp[(x + x')P + (y + y')Q + (t + t' + \frac{1}{2}(xy' - yx'))E] \end{aligned}$$

The action of $\mathbf{H}(\mathbb{R})$ *on* $L^2(\mathbb{R})$ *is called the Schrödinger representation.*

THEOREM (Stone-von Neumann) The Schrödinger representation is the unique irreducible unitary representation of $\mathbf{H}(\mathbb{R})$ *that maps* $\exp(tE)$ *to multiplication by* $e^{2\pi it}$ *for all* $t \in \mathbb{R}$.

COROLLARY Linear (symplectic) changes of coordinates can be quantized.

Recall that the linear maps that preserve the classical mechanics are

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1 \right\}.$$

If $h \in SL(2,\mathbb{R})$, *and* $h(x,y) = (x', y')$ *then*

$$\exp(xP + yQ + tE) \circ \phi = \exp(x'P + y'Q + tE)\phi$$

is another representation, which by the Stone-von Neumann theorem is unitary equivalent to the Schrödinger representation.

Hence there is a unitary map $\rho(h) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *such that the exact Egorov identity is satisfied:*

$$\exp(x'P + y'Q + tE) = \rho(h)\exp(xP + yQ + tE)\rho(h)^{-1}.$$

The map $h \to \rho(h)$ *is a projective representation of* $SL(2,\mathbb{R})$ *on* $L^2(\mathbb{R})$. *One can make this into a true representation by passing to a double cover* $M(2,\mathbb{R})$ *of* $SL(2,\mathbb{R})$. *This is the metaplectic representation.*

EXAMPLE: If

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

then

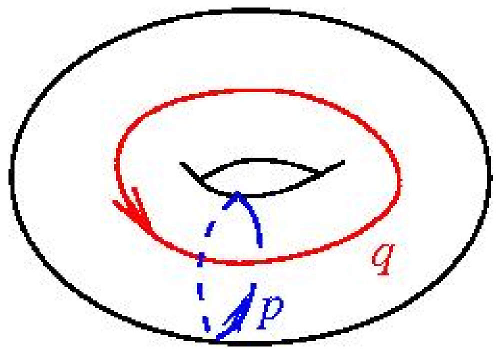
$$\rho(S)\phi(x) = \int_{\mathbb{R}} \phi(y) e^{-2\pi i x y} dy$$
$$\rho(T)\phi(x) = e^{2\pi i x^2 a}\phi(x).$$

The metaplectic representation can be interpreted as a general Fourier transform. It is a Fourier-Mukai transform.

In general, a Heisenberg group is a $U(1)$ (or cyclic) extension of a locally compact abelian group. It has an associated Schrödinger representation and Fourier-Mukai transform. The two are related by the exact Egorov identity.

A. Weil's representation theoretic point of view:

Weyl quantization of a particle with periodic position and momentum ($\hbar = 1/N$, N an even integer) .



Quantization: phase space $\mapsto$ space of theta functions which has an orthonormal basis consisting of the theta series

$$\theta_j(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i N\left[\frac{i}{2}\left(\frac{j}{N}+n\right)^2 + z\left(\frac{j}{N}+n\right)\right]}, \quad j = 0, 1, 2, \ldots, N-1.$$

The quantizations of the exponential functions on the torus generate a finite Heisenberg group $\mathbf{H}(\mathbb{Z}_N)$ which is a $\mathbb{Z}_{2N}$-extension of $\mathbb{Z}_N \times \mathbb{Z}_N$:

$$\exp(pP + qQ + kE)\theta_j = e^{-\frac{\pi i}{N}pq - \frac{2\pi i}{N}jq + \frac{\pi i}{N}k}\theta_{j+p}, \quad p, q, k \in \mathbb{Z}.$$

THEOREM (Stone-von Neumann) The Schrödinger representation of $\mathbf{H}(\mathbb{Z}_N)$ *is the unique irreducible unitary representation of this group with the property that* $\exp(kE)$ *acts as* $e^{\frac{\pi i}{N}k} Id$ *for all* $k \in \mathbb{Z}$.

The linear maps on the torus that preserve classical mechanics are

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\},$$

i.e. the mapping class group of the torus.

COROLLARY There is a projective representation ρ *of the mapping class group of the torus on the space of theta functions that satisfies the exact Egorov identity*

$$\exp(p'P + q'Q + kE) = \rho(h)\exp(pP + qQ + kE)\rho(h)^{-1}$$

where $(p', q') = h(p, q)$. *Moreover, for every* h, $\rho(h)$ *is unique up to multiplication by a constant.*

This action of the mapping class group is called the Hermite-Jacobi action.

Non-abelian theta functions for the group $SU(2)$

Arise from the quantization of the moduli space of flat $su(2)$-connections on a genus g surface Σ ($\hbar = \frac{1}{2N}$, $N = 2r$, r an integer).

$$\mathcal{M}_g^{SU(2)} = \{A \,|\, A:\ su(2) - \textit{connection}\}/\mathcal{G}$$
$$= \{\rho : \pi_1(\Sigma) \longrightarrow SU(2)\}/\textit{conjugation}$$

Quantization: $\mathcal{M}_g^{SU(2)} \mapsto$ *space of non-abelian theta functions.*

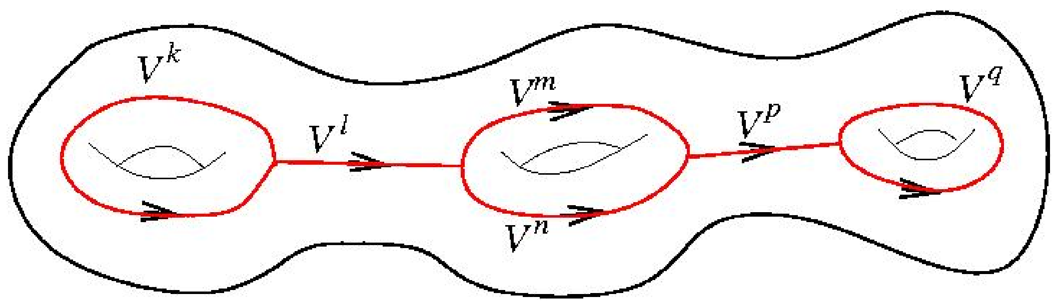
Non-abelian theta series can be parametrized by coloring the core of a genus g handlebody by irreducible representations of $U_\hbar(SL(2,\mathbb{C}))$, the quantum group of $SU(2)$.

$U_\hbar(SL(2,\mathbb{C}))$ *has irreducible representations* $V^1, V^2, \ldots, V^{r-1}$*. They satisfy a Clebsch-Gordan theorem*

$$V^m \otimes V^n = \oplus_p V^p$$

where $|m-n|+1 \leq p \leq \min(m+n-1, 2r-2-m-n)$.

The *theta series* *are the colorings of the core by* $V^1, V^2, \ldots, V^{r-1}$ *such that at each vertex the conditions from the Clebsch-Gordan theorem are satisfied.*

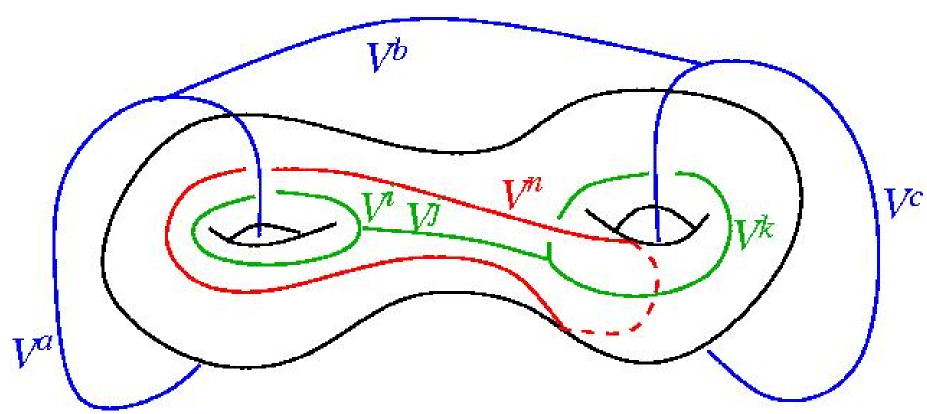


The analogues of the exponentials are the *Wilson lines*

$$W_{\gamma,n}(A) = tr_{V^n} hol_{\gamma}(A)$$

where γ *is a simple closed curve on the surface.*

The *operator* *associated to* $W_{\gamma,n}$*, denoted by* $op(W_{\gamma,n})$*, has a matrix whose "entries" are the Reshetikhin-Turaev invariants of diagrams of the form:*



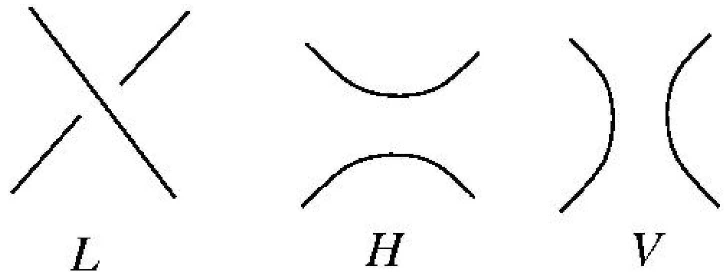
The Reshetikhin-Turaev theory yields a projective representation ρ of the mapping class group of the surface on the space of non-abelian theta functions.

This projective representation satisfies the following relation with the quantizations of Wilson lines

$$op(W_{h(\gamma),n}) = \rho(h)\,\mathit{Op}(W_{\gamma,n})\rho(h)^{-1}$$

which is an exact Egorov identity.

Skein modules! *There is an easy way to see these using skein modules.*



The skein relations are those of the Reshetikhin-Turaev version of the Jones polynomial:

$$L = tH + t^{-1}V \text{ or } L = \epsilon(tH - t^{-1}V)$$

depending on whether the two crossing strands come from different components or not, where ϵ is the sign of the crossing. Set also the trivial knot equal to $t^2 + t^{-2}$.

Let $RT_t(M)$ be the Reshetikhin-Turaev skein module of the 3-manifold M, obtained by factoring the free module with basis the isotopy classes of framed links in M by these relations.

As a module, $RT_t(M)$ is isomorphic to the Kauffman bracket of the manifold, but...

If $M = \Sigma \times [0,1]$ then $RT_t(\Sigma \times [0,1])$ is an algebra:

$$\Sigma \times [0,1] \cup \Sigma \times [0,1] \approx \Sigma \times [0,1].$$

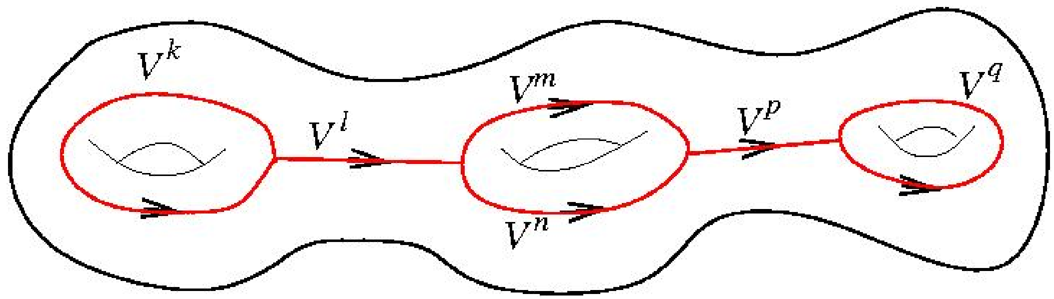
This algebra is not isomorphic to the Kauffman bracket skein algebra.

If M has a boundary, then $RT_t(M)$ is a $RT_t(\partial M \times [0,1])$-module:

$$\partial M \times [0,1] \cup M \approx M.$$

The reduced RT skein module of M, denoted by $\widetilde{RT}_t(M)$ is obtained by factoring $RT_t(M)$ by $t = e^{\frac{i\pi}{2r}}$ and $f_{r-1} = 0$ where f_{r-1} is the $r-1$st Jones-Wenzl idempotent.

The space of non-abelian theta functions of a genus g surface Σ_g is $\widetilde{RT}_t(H_g)$ where H_g is the genus g handlebody.



The skein theoretic versions of non-abelian theta series are obtained by replacing

$$V^k \to f_{k-1}.$$

Algebra generated by quantized Wilson lines is isomorphic to $\widetilde{RT}_t(\Sigma_g \times [0,1])$ with the isomorphism given by

$$Op(W_{\gamma,2}) \mapsto \gamma.$$

The action of operators on non-abelian theta functions coincides with the action of $\widetilde{RT}_t(\Sigma_g \times [0,1])$ on $\widetilde{RT}_t(H_g)$.

Our paradigm: *There are the following analogies*

At the level of the vector space

a. $L^2(\mathbb{R})$

b. classical theta functions

c. non-abelian theta functions

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b. The group algebra of the finite Heisenberg group $\mathbf{H}(\mathbb{Z}_N)$

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At the level of quantized changes of coordinates

a. The metaplectic representation (i.e. the Fourier transform)

b. The Hermite-Jacobi action (i.e. the discrete Fourier transform)

c. The Reshetikhin-Turaev representation

For the exact Egorov identity

a. The exact Egorov identity for the metaplectic representation

$$\exp(x'P + y'Q + tE) = \rho(h)\exp(xP + yQ + tE)\rho(h)^{-1}.$$

where $h(x, y) = (x', y')$.

b. The exact Egorov identity for the Hermite-Jacobi action

$$\exp(p'P + q'Q + kE) = \rho(h)\exp(pP + qQ + kE)\rho(h)^{-1}$$

where $(p', q') = h(p, q)$.

c. The exact Egorov identity for the Reshetikhin-Turaev representation

$$op(W_{h(\gamma),n} = \rho(h)\,\mathit{Op}(W_{\gamma,n})\rho(h)^{-1}$$

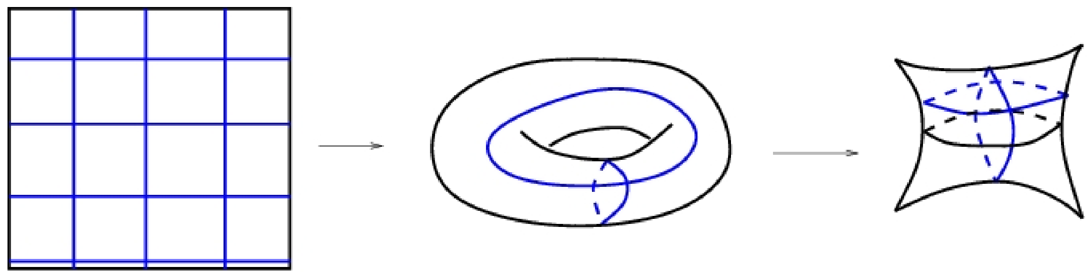
Applications of this point of view:

1. The quantum group quantization of Wilson lines determines the Reshetikhin-Turaev representation.

2. Andersen, Freedman-Wang: Proof of the asymptotic faithfulness of the Reshetikhin-Turaev representation.

3. The case of the torus:

The moduli space is the pillow case:



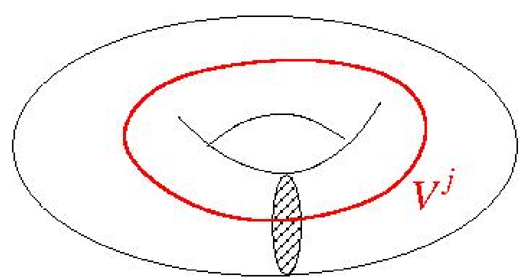
A basis of the space of non-abelian theta functions is

$$\zeta_j(z) = (\theta_j(z) - \theta_{-j}(z)), \quad j = 1, 2, \ldots, r-1,$$

where θ_j are the theta series

$$\theta_j(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i 2r\left[\frac{i}{2}\left(\frac{j}{2r}+n\right)^2 + z\left(\frac{j}{2r}+n\right)\right]}, \quad j = 0, 1, 2, \ldots, 2r-1.$$

The ζ_j's can be represented graphically as



Fact: Just in genus one, the algebra of quantized Wilson lines, which is the reduced Reshetikhin-Turaev skein algebra, is isomorphic to the reduced Kauffman bracket skein algebra of the torus at $t = e^{\frac{i\pi}{2r}}$.

THEOREM (Frohman-G.) The Kauffman bracket skein algebra has the multiplication rule

$$(p_1, q_1)_T (p_2, q_2)_T = t^{p_1 q_2 - p_2 q_1}(p_1 + p_2, q_1 + q_2)_T \\ + t^{-(p_1 q_2 - p_2 q_1)}(p_1 - p_2, q_1 - q_2)_T$$

where (p, q) is the curve of slope p/q on the torus if $gcd(p, q) = 1$ and $(p, q) = T_n((p/n, q/n))$ if $n = gcd(p, q) > 1$, T_n being the Chebyshev polynomial of first kind.

A Stone-von Neumann theorem can be proved in this case:

THEOREM (G.-Uribe) The representation of the reduced Reshetikhin-Turaev skein algebra of the torus defined by the Weyl quantization of the moduli space of flat $SU(2)$-connections on the torus is the unique irreducible representation of this algebra that maps simple closed curves to self-adjoint operators and t to multiplication by $e^{\frac{\pi i}{2r}}$. Moreover, quantized Wilson lines span the algebra of all linear operators on the Hilbert space of the quantization.

This implies the existence of the Reshetikhin-Turaev representation

$$h \mapsto \rho(h)$$

of the mapping class group of the torus without apriori knowing it.

Because the skein algebra contains all linear operators, $\rho(h)$ can be represented as multiplication by a skein. Here is the computation...

Let $h = T$, the twist. Recall that $[n]$ denotes the quantized integer $\sin \frac{n\pi}{r} / \sin r$. The exact Egorov identity, in skein form, reads

$$h(\gamma) = \rho(h)\gamma\rho(h)^{-1}$$

where γ is a simple closed curve.

From this identity we deduce that

$$\rho(h) = \sum_{j=1}^{r-1} \alpha_j (0,1)^j.$$

Rewrite this as

$$\rho(h) = \sum_{j=1}^{r-1} c_j S_{j-1}((0,1))$$

where S_n is the nth Chebyshev polynomial of second type.

Then

$$(1,1)\rho(T)\zeta_k = \rho(T)(1,0)\zeta_k.$$

$$\sum_j c_j \frac{[jk]}{k} t^{-1}(t^{-2k}\zeta_{k+1} + t^{2k}\zeta_{k-1}$$

$$= \sum_j c_j \left(\frac{[j(k+1)]}{[k+1]}\zeta_{k+1} + \frac{[j(k-1)]}{[k-1]}\zeta_{k-1} \right).$$

Setting the coefficients of ζ_{k+1} *on both sides equal yields the system*

$$\sum_{j=1}^{r-1} c_j[j(k+1)] = \frac{[k+1]}{[k]} t^{-2k-1} \sum_{j=1}^{r-1} c_j[jk].$$

Solving we obtain

$$\rho(h) = \sum_{j=1}^{r-1} [j] t^{j^2} S_{j-1}((0,1)).$$

We conclude that $\rho(h)$ is the skein obtained by coloring the surgery curve of T by

$$\Omega = \sum_j [j] V^j$$

Using the fact that each element of the mapping class group is a composition of twists we obtain:

THEOREM *Let h be an element of the mapping class group of the torus defined by surgery on the framed link L_h in $T^2 \times [0,1]$. Then*

$$\rho(h) : \widetilde{RT}_t(S^1 \times D^2) \to \widetilde{RT}_t(S^1 \times D^2)$$

is given by

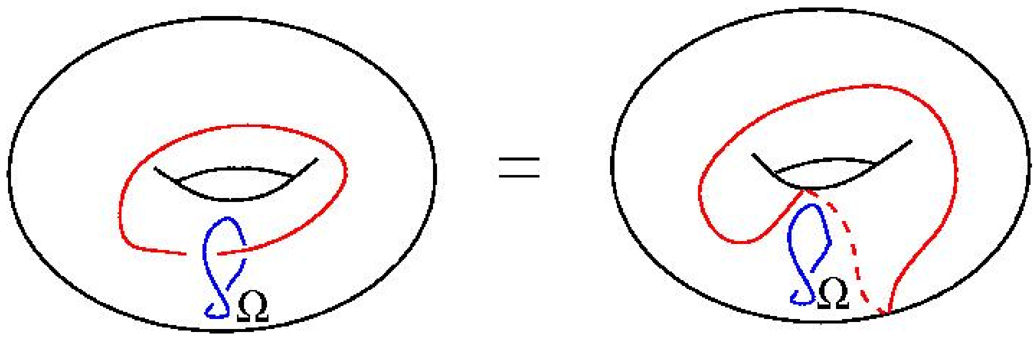
$$\rho(h)\beta = \Omega(L_h)\beta$$

where $\Omega(L_h)$ is the skein obtained by coloring all components of L by Ω.

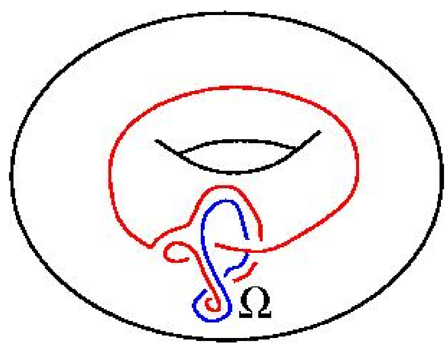
The exact Egorov identity

$$h(\gamma) = \rho(h)\gamma\rho(h)^{-1}$$

gives



The skein on the right is



Exact Egorov identity $\Longrightarrow$ *handle slides in the cylinder over the torus.*

In conclusion, by looking at non-abelian theta functions from André Weil's point of view we can introduce in a natural way the element Ω*, which is the building block of the Reshetikhin-Turaev topological quantum field theory, and we arrive at slides along link components colored by* Ω*, which is the main principle behind constructing quantum 3-manifold invariants.*