Complex Analysis – Homework 5

1. Show that any biholomorphic map from the upper half-plane onto the open unit disk has the form

$$f(z) = e^{i\theta}\frac{z-a}{z-\bar{a}}.$$

2. Let f be the Riemann map of a simply connected domain D onto the open unit disk satisfying $f(z_0) = 0$, $f'(z_0) = A > 0$. Show that if g is an analytic function on D satisfying $g(z_0) = 0$ and $|g| < 1$, then $|g'(z_0)| \leq A$, with equality if an only if $f = e^{i\theta}g$, $\theta \in \mathbb{R}$.

3. Let $f(z)$ be analytic for $0 < |z| < 1$, and define $f_n(z) = f(z/n)$ for $0 < |z| < 1$, $n \geq 1$. Show that $\{f_n\}$ is a normal family in the space of meromorphic functions on the punctured disk if and only if the singularity of f at 0 is a pole or removable.

4. Find a biholomorphic map from $\mathbb{C}\backslash(-\infty, 0]$ to $D = \{z \,|\, |z| < 1\}$.

5. Find a biholomorphic map from the domain

$$\{z = re^{i\theta} \,|\, r \in \mathbb{R}, \quad 0 < \theta < 2\pi/3\}$$

to the upper half plane $\text{Im } z > 0$.

6. Show that $w(z) = \frac{1}{2}(z+\frac{1}{z})$ maps the interior of the unit disk conformally onto a domain D on the Riemann sphere. What is D? What is the inverse map?