Complex Analysis – Homework 4

1. Let γ be the counterclocwise rectangular path whose vertices are $n+1/2+ni, -n-1/2+ni, -n-1/2-ni, n+1/2-ni$. Evaluate
$$\int_\gamma \frac{\pi \cot \pi z}{(z+a)^2} dz,$$
for a not an integer. Show that the limit of this integral as $n \to \infty$ is zero, and use this fact to prove that
$$\frac{\pi^2}{\sin^2 \pi a} = \sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2}.$$

2. Evaluate
$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1}.$$

3. Evaluate
$$\int_0^\infty \frac{\cos x - 1}{x^2} dx,$$
$$\int_0^\infty \frac{x^a}{1+x} dx, \quad a<0,$$
$$\int_0^\infty \frac{x^a}{(1+x)^2} dx, \quad 0<a<1.$$

4. Prove that
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin a\pi} \text{ if } 0<a<1$$
$$\int_0^{2\pi} \log \sin^2 2\theta d\theta = 4\int_0^\pi \log \sin \theta d\theta = -4\pi \log 2.$$
Note in the second example the relationship to the Lobachevski function
$$\Lambda(\theta) = -\int_0^\theta \log 2|\sin \theta| d\theta$$
or the Clausen function
$$\mathrm{Cl}_2(\theta) = -\int_0^\theta \log|2\sin\frac{\theta}{2}| d\theta.$$

5. Evaluate
$$\int_0^1 \frac{x^4}{\sqrt{x(1-x)}} dx$$
$$\int_0^\infty \frac{\log x}{(x+a)(x+b)} dx, \quad a,b>0.$$