Introduction to Topology – Homework 5

1. Given the topological spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps from X to Y. Show that $[X, [0, 1]]$ has one element.

2. Show that the fundamental groups of homeomorphic spaces are isomorphic.

3. Show that

$$\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y).$$

(The group $G \times H$ is defined by the multiplication law $(g, h)(g', h') = (gg', hh')$. What is the fundamental group of the 2-dimensional torus?

4. Let G be a topological group with operation $\cdot$ and identity element 1. Let $\Omega(G, 1)$ be the set of loops in G based at 1. For $f, g \in \Omega(G, 1)$, define

$$(f \otimes g)(t) = f(t) \cdot g(t).$$

(a) Show that this operation makes $\Omega(G, 1)$ into a group.
(b) Show that this operation induces a group structure on $\pi_1(G, 1)$.
(c) Show that this group structure on $\pi_1(G, 1)$ is the same as the standard group structure which defines the fundamental group.
(d) Using this fact, show that $\pi_1(G, 1)$ is abelian.

5. Let $SO(3)$ be the group of rotations of $\mathbb{R}^3$ about an axis passing through the origin. This is the same as the group of matrices with determinant 1 of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

satisfying $a_{1i}a_{1j} + a_{2i}a_{2j} + a_{3i}a_{3j} = \delta_{ij}$ (where δ_{ij} is the Kronecker symbol, equal to 1 if $i = j$ and 0 otherwise). Find $\pi_1(SO(3))$.