HOMEWORK 1

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If $m = 1$, we obtain the trivial cover of the space by itself via a homeomorphism.

If $m = 0$, we obtain the universal covering $\mathbb{R}$ with covering map $p(x) = e^{2\pi ix}$.

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Recall that the fundamental group of S^1 *is*

$$\pi_1(S^1, 1) = \{[z^k] \mid k \in \mathbb{Z}\}$$

with

$$[z^k] * [z^n] = [z^{k+n}].$$

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This implies that $p_* : \mathbb{Z} \to \mathbb{Z}$, *is* $k \to mk$. *So* $p_*(\mathbb{Z}) = m\mathbb{Z}$. *Hence the conclusion.*

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Lemma. *The subgroups of* $\mathbb{Z} \times \mathbb{Z}$ *are of the form*

1. *the trivial subgroup,*
2. *free abelian groups with one generator* (p, q)*,*
3. *free abelian groups with two generators* (p, q) *and* (r, s) *such that* $ps - qr \neq 0$*.*

Proof of the lemma: *Because we have a subgroup of a free abelian group, it is free abelian. Assume that it must have more than two generators.*

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By taking linear combintations with integer coefficients we can obtain an element of the form (p, q) *with* p *the greatest common divisor of the generators. Add this element to the set of generators. The other generators can then be transformed into elements of the form* $(0, r)$ *by subtracting multiples of* (p, q).

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Then (p, q) *and* $(0, s)$ *generate the subgroup, a contradiction. The conclusion follows.*

For each subgroup of $\mathbb{Z} \times \mathbb{Z}$ we will construct a covering space. If the subgroup is trivial we obtain the universal covering space $\mathbb{R}^2$ with covering map $p : \mathbb{R}^2 \to S^1 \times S^1$, $p(x, y) = (e^{2\pi i x}, e^{2\pi i y})$.

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If H *is the subgroup generated by the element* (p, q) *the covering space is* $S^1 \times \mathbb{R}$, *with covering map* $p(e^{2\pi ix}, y) = (e^{2\pi ipx}, e^{2\pi i(qx+y)})$.

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Indeed, note that $p_*([z]) = ([z^p], [z^q])$, hence $p_*(1) = (p, q) \in \mathbb{Z} \times \mathbb{Z}$.

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Indeed, note that $p_*([z]) = ([z^p], [z^q])$, hence $p_*(1) = (p, q) \in \mathbb{Z} \times \mathbb{Z}$.

This implies that $p_*(\mathbb{Z}) = (p, q)\mathbb{Z} = H$ which solves the case of one generator.

If the subgroup has two generators (p, q) *and* (r, s) *with* $ps - rq \neq 0$*, the corresponding covering space is* $S^1 \times S^1$*, with covering map*

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So $p_*(1, 0) = (p, q)$ *and* $p_*(0, 1) = (r, s)$, *which shows that* $p_*(\mathbb{Z} \times \mathbb{Z})$ *is the subgroup generated by* (p, q) *and* (r, s).

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Mission accomplished.

Problem 3. *Let* G *be a topological group and* $p : (\tilde{G}, \tilde{e}) \longrightarrow (G, e)$ *a covering map, where* e *is the identity element. Show that there is a unique multiplication on* $\tilde{G}$ *with* $\tilde{e}$ *the identity element, such that* p *is a group homomorphism.*

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Solution: *Recall that the covering* $p : (\tilde{G}, \tilde{e}) \longrightarrow (G, e)$ *is obtained by a standard procedure in which the elements of* $\tilde{G}$ *are equivalence classes of paths in* G.

More precisely, one considers paths in G *starting at* e *modulo the equivalence relation*

$$\alpha \sim \beta \text{ if and only if } \alpha(1) = \beta(1) \text{ and } [\alpha * \bar{\beta}] \in H$$

where $H = p(\pi_1(\tilde{G}, \tilde{e}))$. *We denote by* $\hat{\alpha}$ *the equivalence class of* α.

Define the multiplication on $\tilde{G}$ *by*

$$\hat{\alpha} \cdot \hat{\beta} = \hat{\gamma}$$

where the path γ *is defined by*

$$\gamma(t) = \alpha(t) \cdot \beta(t).$$

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Associativity follows from the pointwise associativity of G*, the identity element is the constant path at* $\tilde{e}$*, and* $\hat{\alpha}^{-1}$ *is the equivalence class of the path* $\alpha(t)^{-1}$*,* $t \in [0, 1]$*. Done.*

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Take $e_0 \in p^{-1}(b_0)$ and $e_0' \in p^{-1}(b_0')$ and consider some path α in E from e_0 to e_0'. For every $e \in p^{-1}(b_0)$, the path $p(\alpha)$ has a unique lift that starts at e.

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Moreover, every point in $p^{-1}(b_0')$ is the endpoint of a unique such lift. Indeed, run $p(\alpha)$ backwards with b_0' its initial point and lift it to a path that starts at e'. Then this path ends at some e in $p^{-1}(b_0)$, and reversing again we obtain a lift from e to e'.

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It follows that the group of deck transformations acts transitively on the endpoints, and we are done.

Problem 5. *Find the universal covering space of the figure eight. Compute the fundamental group of figure eight.*

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Solution: *We need to learn a little bit more, so let us get back to the theory.*