HOMEWORK 2

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Solution: *Let* $H : X \times [0, 1] \to X$ *be the deformation retraction. Define* $r : X \to A$, $r(x) = H(x, 1)$. *Then* $r \circ i = 1_A$. *On the other hand* $i \circ r$ *is homotopic to* 1_X, *the homotopy being* H *itself. The conclusion follows from Theorem 4.2.3.*

Problem 2. *The Klein bottle is obtained as a quotient space of* $[0,1] \times [0,1]$ *by the equivalence relations* $(s,0) \equiv (s,1)$ *and* $(0,t) \equiv (1,1-t)$*, for all* $s,t \in [0,1]$*. Compute the fundamental group of the Klein bottle.*

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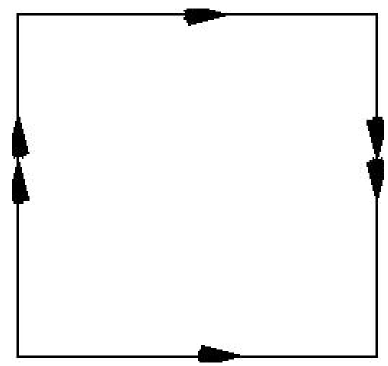

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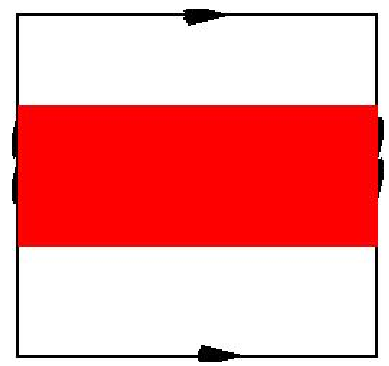

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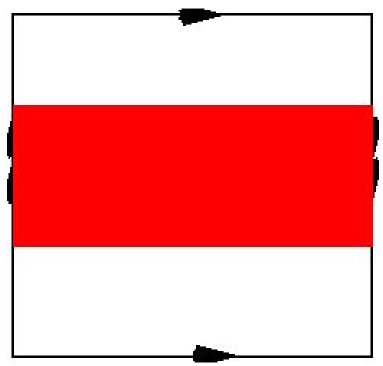

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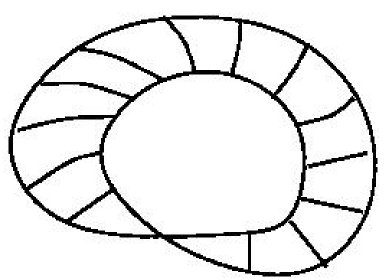

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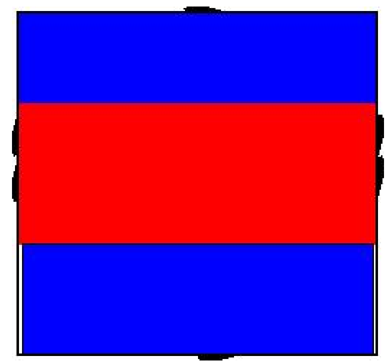

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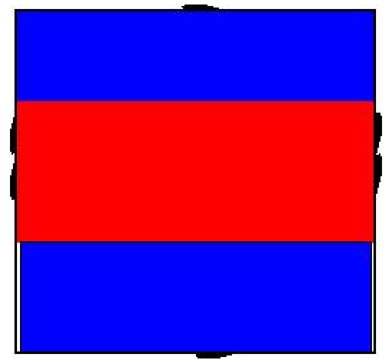

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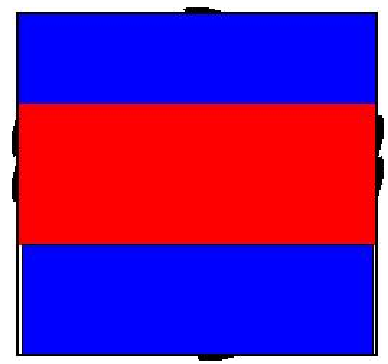

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Now we can apply the Seifert-van Kampen theorem.

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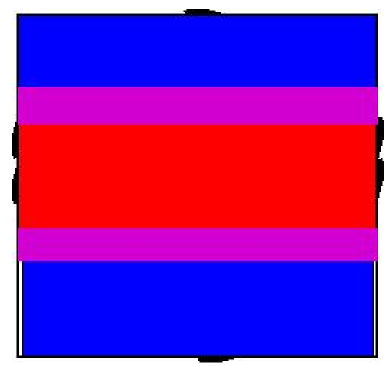

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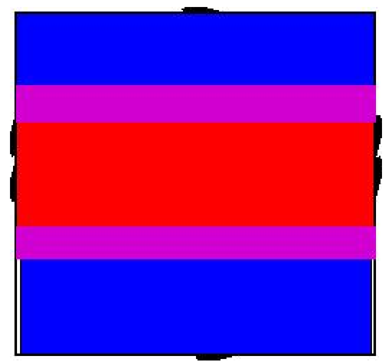

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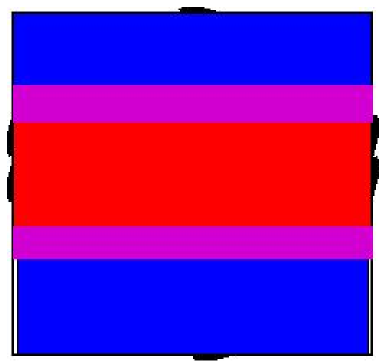

What is the pink region topologically?

It is a cylinder! If you don't believe me go home, make a Möbius band out of paper (use some Scotch tape), then cut out a regular neighborhood of its boundary. You will see that the regular neighborhood twists twice, thus is homeomorphic to a cylinder.

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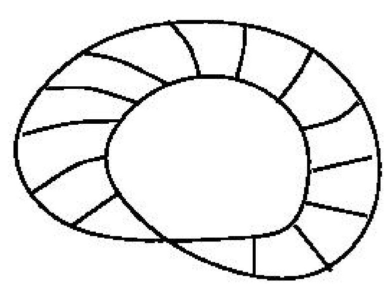

 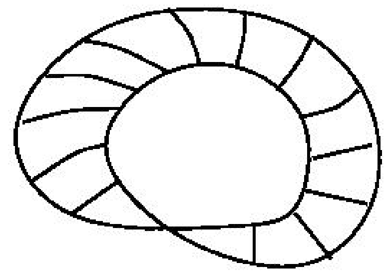

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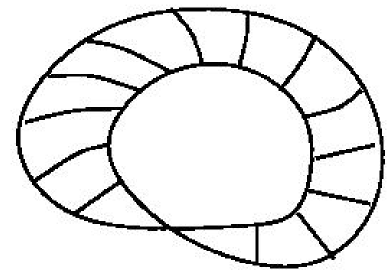

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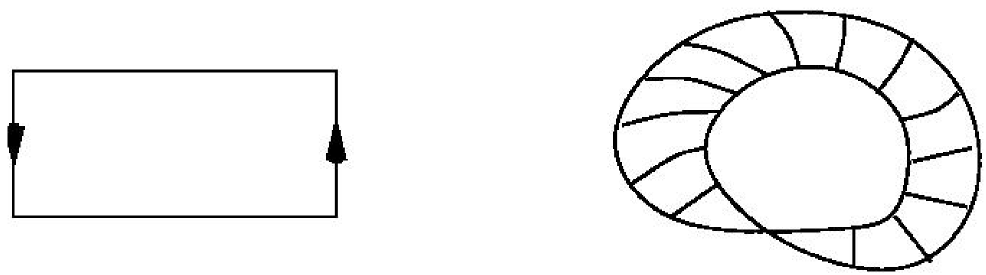
Thus N, which consists of the elements of the form $i_{1*}(h)(i_{2*}(h)^{-1})$, is generated by the element x^2y^{-2}, where x and y are the generators of the first, respectively second $\mathbb{Z}$.



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Consequently N *introduces the only relation* $x^2 = y^2$ *and so*

$$\pi_1(K) = \left\langle x, y \,|\, x^2 = y^2 \right\rangle$$

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In fact we have the following general result, which is a consequence of the Seifert-van Kampen theorem.

Proposition: *Let* X *and* Y *be path connected spaces and* $x_0 \in X$*,* $y_0 \in Y$ *be points that have simply connected neighborhoods. Define* $X \bigvee Y$ *as the quotient of* $X \sqcup Y$ *by the equivalence relation* $x_0 \sim y_0$*. Then*

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In particular

$$\pi_1(S^1 \bigvee S^2) = \mathbb{Z}$$

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Consequently

$$\pi_1(X) = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$$

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