Obtaining the Witten-Reshetikhin-Turaev Invariants from Quantum Mechanical Considerations

Razvan Gelca
Texas Tech University

Alejandro Uribe
University of Michigan

- Understand the Witten-Reshetikhin-Turaev invariants

- Find a more geometric approach to the Jones polynomial

Starting point:

The quantization of the moduli space of flat connections on a surface

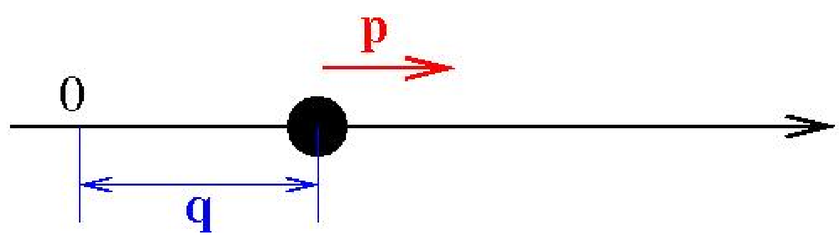
THE PROTOTYPE:

The Weyl quantization of a 1-dimensional particle.

- *the Hilbert space*
- *the Schrödinger representation*
- *the metaplectic representation*

The 1-dimensional particle.

Phase space $\mathbb{R}^2$ *has coordinates* $\mathbf{q}$*: position*, $\mathbf{p}$*: momentum.*
The classical observables are functions $f(\mathbf{p}, \mathbf{q})$.



Quantization: Planck's constant h *(*$\hbar = h/2\pi$*)*

$$\text{phase space} \mapsto L^2(\mathbb{R}, d\mathbf{q}),$$
$$\mathbf{q} \mapsto Q = M_{\mathbf{q}},$$
$$\mathbf{p} \mapsto P = -i\hbar \frac{d}{d\mathbf{q}}.$$

where $M_{\mathbf{q}}$ *is multiplication by* $\mathbf{q}$.

Heisenberg's canonical commutation relations:

$$PQ - QP = -i\hbar Id$$

Weyl quantization

$$e^{2\pi i(p\mathbf{p}+q\mathbf{q})} \mapsto e^{2\pi i(pP+qQ)}$$

In general, for a function $f(\mathbf{p},\mathbf{q})$*,*

$$f \mapsto \mathit{Op}(f),$$

where

$$\hat{f}(\xi,\eta) = \iint f(\mathbf{p},\mathbf{q}) \exp(-2\pi i\mathbf{p}\xi - 2\pi i\mathbf{q}\eta)dxdy$$

and then defining

$$\mathit{Op}(f) = \iint \hat{f}(\xi,\eta) \exp 2\pi i(\xi P + \eta Q) d\xi d\eta.$$

The Schrödinger representation

$$e(pP + qQ + tId) = e^{2\pi i(pP+qQ+tId)}$$

The cannonical commutation relations in exponential form

$$\mathsf{e}(P)\mathsf{e}(Q) = e^{\pi i h}\mathsf{e}(Q)\mathsf{e}(P).$$

Heisenberg group $\mathbf{H}(\mathbb{R}) = \mathbb{R} \times \mathbb{R} \times U(1)$,

$$(p, q, e^{2\pi i t})(p', q', e^{2\pi i t'}) = \left(p + p', q + q', e^{2\pi i(t+t'+\frac{h}{2}(pq'-qp'))}\right)$$

The Schrödinger representation of $\mathbf{H}(\mathbb{R})$ *on* $L^2(\mathbb{R}, d\mathbf{q})$ *is given by*

$$e(pP + qQ + tId)\psi(\mathbf{q}) = e^{2\pi i q\mathbf{q}+\pi i h p q+2\pi i t}\psi(\mathbf{q} + p).$$

The metaplectic representation

THEOREM (Stone-von Neumann) The Schrödinger representation is the unique irreducible unitary representation of $\mathbf{H}(\mathbb{R})$ that maps e(tId) to multiplication by $e^{2\pi it}$ for all $t \in \mathbb{R}$.

COROLLARY Linear (symplectic) changes of coordinates can be quantized.

One obtains a projective representation ρ of

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}$$

on $L^2(\mathbb{R}, d\mathbf{q})$. This "is" the metaplectic representation.

$\rho(h)$ is a Fourier transform.

It is the standard Fourier transform when the symplectic map h is the 90° rotation.

The exact Egorov identity

As above, ρ is the metaplectic representation, which tells us how (symplectic) changes of coordinates are quantized.

The fundamental symmetry of the Weyl quantization is

$$\mathit{Op}(f \circ h^{-1}) = \rho(h)\mathit{Op}(f)\rho(h)^{-1},$$

for every observable $f \in C^\infty(\mathbb{R}^2)$ and $h \in SL(2,\mathbb{R})$.

Other quantization models satisfy it only up to an error in Planck's constant:

$$\mathit{Op}(f \circ h^{-1}) = \rho(h)\mathit{Op}(f)\rho(h)^{-1} + O(\hbar).$$

For Weyl quantization it is satisfied exactly. This is the exact Egorov identity.

The Segal-Bargmann model

The Segal-Bargmann space, $\mathcal{H}L^2(\mathbb{C}^n, d\mu_\hbar)$, is the space of holomorphic functions on $\mathbb{C}^n$ that are L^2-integrable with respect to the measure

$$d\mu_\hbar = (\pi\hbar)^{-n/2} e^{-\|\operatorname{Im}\mathbf{z}\|^2/\hbar} d\mathbf{x} d\mathbf{y}.$$

Based on Fock's observation

$$\frac{\partial}{\partial z}(z\phi(z)) - z\frac{\partial}{\partial z}\phi(z) = \phi(z).$$

set

$$Q + iP = a = M_z + 2\hbar\frac{\partial}{\partial z}$$
$$Q - iP = a^\dagger = M_z.$$

Then

$$e(pP + qQ + tId)\psi(\mathbf{q}) = e^{\pi i h \mathbf{q}(\mathbf{p}+i\mathbf{q}) + 2\pi i \mathbf{q} z + 2\pi i t}\phi\left(z + h(\mathbf{p} + i\mathbf{q})\right).$$

The abstract version of the Schrödinger representation

$\mathbf{L}$: *Lagrangian subspace of* $\mathbb{R}^2 = \mathbb{R}\mathbf{p} + \mathbb{R}\mathbf{q}$, $\chi_{\mathbf{L}}(e(l + tI)) = e^{2\pi it}, l \in \mathbf{L}$.

- *The Hilbert space* $\mathcal{H}(\mathbf{L})$ *consists of functions* $\phi(u)$ *on* $\mathbf{H}(\mathbb{R})$ *satisfying*

$$\phi(uu') = \chi_{\mathbf{L}}(u')^{-1}\phi(u) \text{ for all } u' \in e(\mathbf{L} + \mathbb{R}I)$$

and such that $u \rightarrow |\phi(u)|$ *is a square integrable function on the left equivalence classes modulo* $e(\mathbf{L} + \mathbb{R}I)$.

- *The Schrödinger representation of the Heisenberg group is given by*

$$u_0\phi(u) = \phi(u_0^{-1}u).$$

- *The metaplectic representation is defined as*

$$\rho(h) : \mathcal{H}(\mathbf{L}) \rightarrow \mathcal{H}(h(\mathbf{L}))$$

$$(\rho(h)\phi)(u) = \int_{e(h(\mathbf{L}))/e(\mathbf{L}\cap h(\mathbf{L}))} \phi(uu_2)\chi_{h(\mathbf{L})}(u_2)du_2$$

QM in Witten-Reshetikhin-Turaev theory

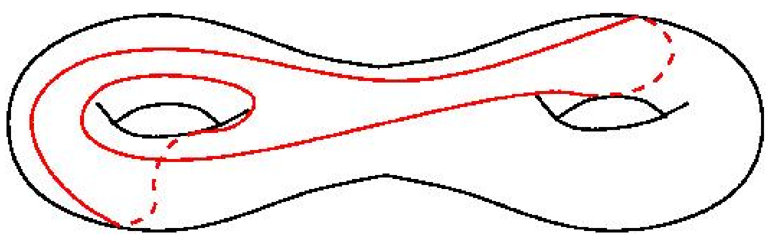
- G: compact simple Lie group, $\mathfrak{G}$: the Lie algebra of G,
- Σ_g: genus g closed surface,
- Planck's constant $h = \frac{1}{2r}$, r an integer.

The moduli space of flat $\mathfrak{G}$-connections on Σ_g:

$$\mathcal{M}_g^G = \{A \mid A : \mathfrak{G} - \textit{connection on } \Sigma_g\}/\textit{gauge transformations}$$
$$= \{\rho : \pi_1(\Sigma_g) \longrightarrow G\}/\textit{conjugation}$$

The observables are the Wilson lines

$$W_{\gamma,n}(A) = tr_{V^n} hol_\gamma(A)$$



The quantum group quantization

Associate to the gauge group its quantum group

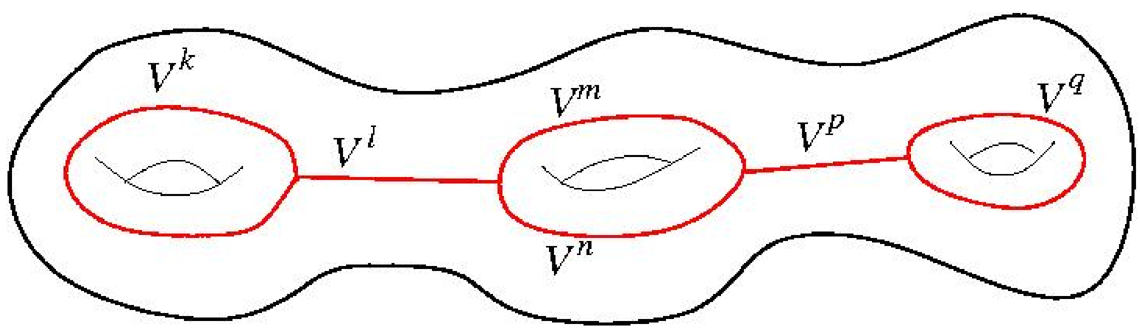
$$G \mapsto U_h(G).$$

V^j, $j \in J$, finite family of irreducible representations of $U_h(G)$ that generate a ring of representations.

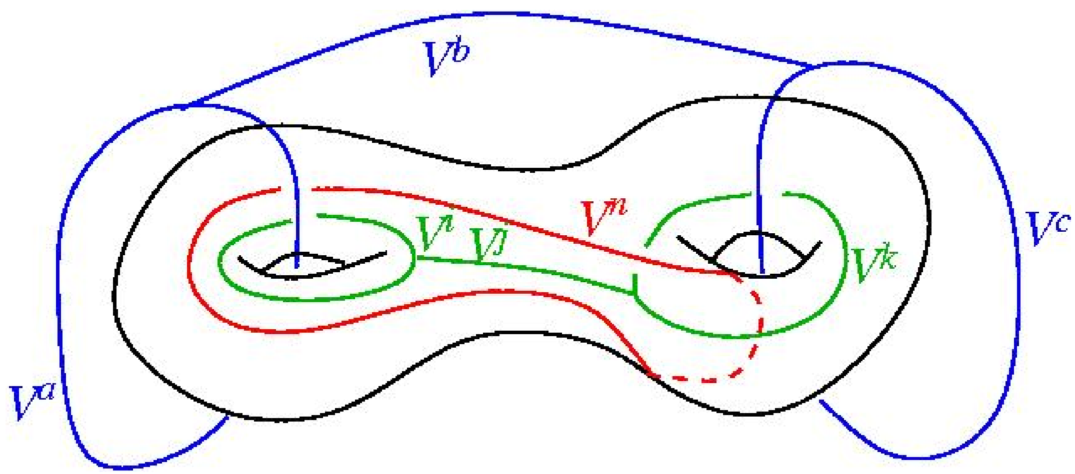
Quantization:

$$\mathcal{M}_g^G \mapsto \text{space of non-abelian theta functions.}$$

Non-abelian theta series of Σ_g can be parametrized by admissible colorings of the core of a genus g handlebody by irreducible representations of $U_h(G)$.



The quantization of a Wilson line, $W_{\gamma,n}(A) = tr_{V^n} hol_{\gamma}(A)$ is an operator $Op(W_{\gamma,n})$ whose matrix has "entries" that are Reshetikhin-Turaev invariants of diagrams of the form:



The quantization comes with a projective representation ρ of the mapping class group of Σ_g, defined by the condition

$$op(W_{h(\gamma),n}) = \rho(h) Op(W_{\gamma,n}) \rho(h)^{-1},$$

which is an exact Egorov identity.

Our paradigm: *There are the following analogies:*

At the level of the vector space

a. $L^2(\mathbb{R})$

b. non-abelian theta functions

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At the level of quantum observables

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b. The algebra generated by quantized Wilson lines $Op(W_{\gamma,n})$

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At the level of quantized changes of coordinates

a. *The metaplectic representation (by Fourier transforms)*

b. *The Reshetikhin-Turaev representation of the mapping class group*

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At the level of quantized changes of coordinates

a. The metaplectic representation (by Fourier transforms)

b. The Reshetikhin-Turaev representation of the mapping class group

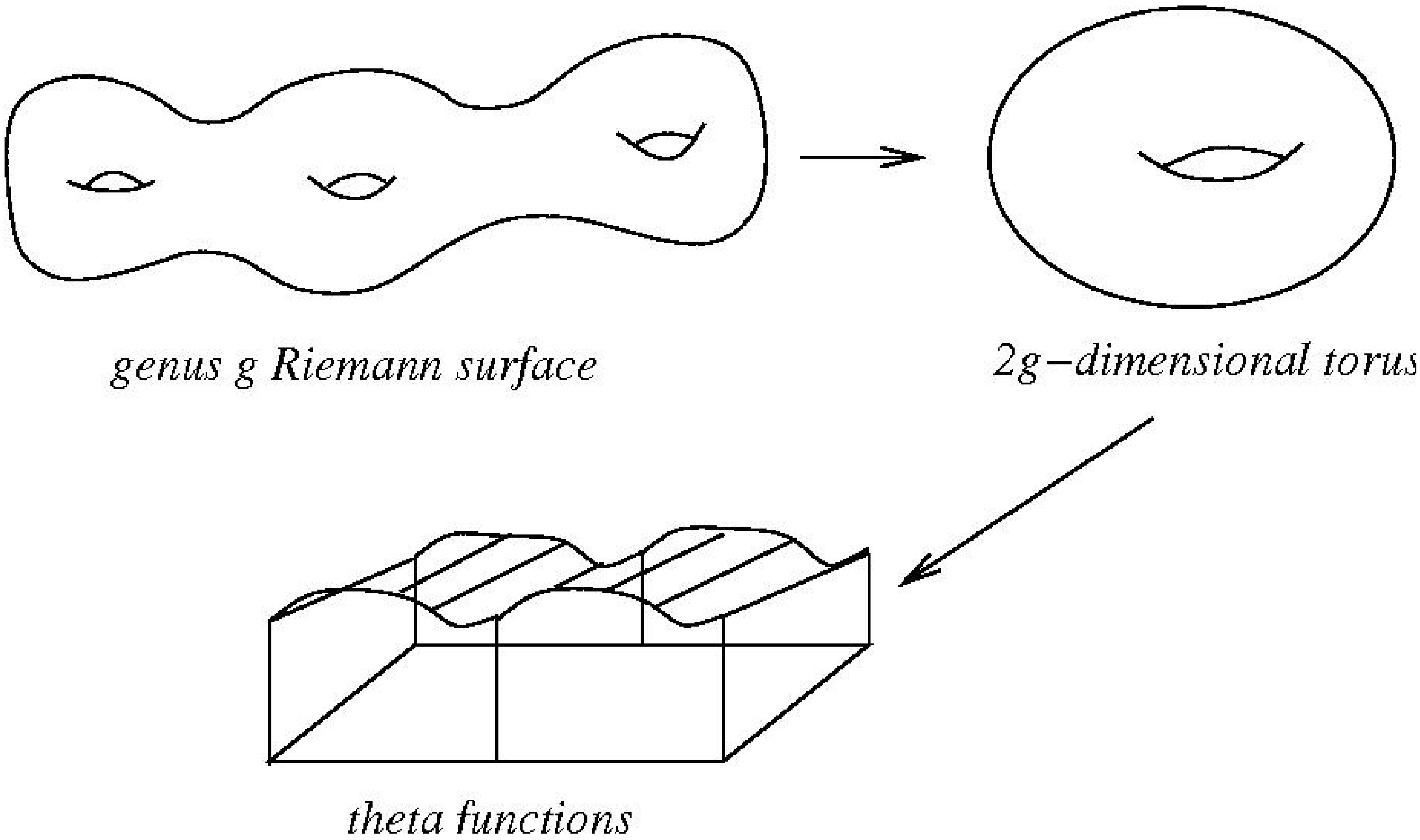
At the level of the relation between observables and changes of coordinates

a. The exact Egorov identity

b. Handle slides

Example 1. Classical theta functions

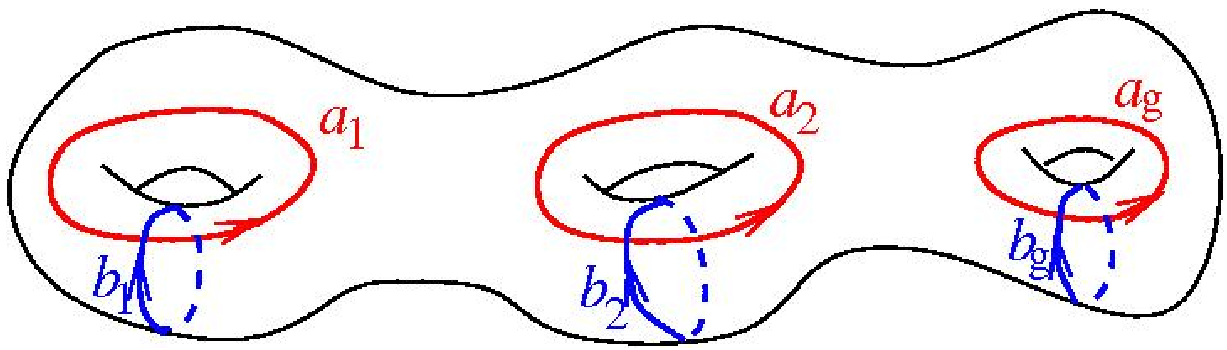
J.E. Andersen (2005), G.-Uribe (2009)



The Jacobian variety

Σ_g: closed Riemann surface of genus g, $a_1, a_2, \ldots, a_g$, $b_1, b_2, \ldots, b_g$ a canonical basis for $H_1(\Sigma_g, \mathbb{R})$, $\zeta_1, \zeta_2, \ldots, \zeta_g$: holomorphic 1-forms defined by $\int_{a_k} \zeta_j = \delta_{jk}$, $j, k = 1, 2, \ldots, g$.

$$(\Pi)_{jk} = \int_{b_k} \zeta_j, \quad j, k = 1, \ldots, g,$$



The columns of the $2g \times g$ matrix (I_g, Π) span a $2g$-dimensional lattice $L(\Sigma_g)$ in $\mathbb{C}^g = \mathbb{R}^{2g}$. The complex torus

$$\mathcal{J}(\Sigma_g) = \mathbb{C}^g / L(\Sigma_g) = H_1(\Sigma_g, \mathbb{R}) / H_1(\Sigma_g, \mathbb{Z})$$

is the Jacobian variety of Σ_g.

Theta functions and their symmetries

- $\mathcal{J}(\Sigma_g)$ *has an associated holomorphic line bundle, whose sections can be identified with holomorphic functions on* $\mathbb{C}^g$ *satisfying certain periodicity conditions. These are the classical theta functions (Jacobi).*
- *The mapping class group of* Σ_g *acts on theta functions (Hermite-Jacobi action).*
- *There is an action of a finite Heisenberg group (Weil) on theta functions which induces the Hermite-Jacobi action via a Stone-von Neumann theorem.*

These can be obtained by applying Weyl quantization to $\mathcal{J}(\Sigma_g)$.

The Weyl quantization of the Jacobian variety

Planck's constant $h = \frac{1}{N}$ *(Weil's integrality condition),* $N = 2r$, r *integer.*
Replace:

- $\mathcal{J}(\Sigma_g) \mapsto$ *space of classical theta functions* Θ_N *(geometric quantization)*
- $f \in C^\infty(\mathcal{J}(\Sigma_g)) \mapsto$ *linear operator Op*(f) *(Weyl quantization)*

An orthonormal basis for Θ_N *consists of the theta series*

$$\theta_\mu(z) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i N[\frac{1}{2}(\frac{\mu}{N}+n)^T \Pi (\frac{\mu}{N}+n) + (\frac{\mu}{N}+n)^T z]}, \quad \mu \in \mathbb{Z}_N^g.$$

$$\mathit{Op}\left(e^{2\pi i(px+qy)}\right)\theta_\mu(z) = e^{-\frac{\pi i}{N}p\cdot q - \frac{2\pi i}{N}\mu\cdot q}\theta_{\mu+p}(z), \quad p, q \in \mathbb{Z}^g$$

The Schrödinger representation of the finite Heisenberg group

$$Op\left(e^{2\pi i(px+qy+k/2)}\right) = e(pP + qQ + kI), p, q \in \mathbb{Z}^g, k \in \mathbb{Z}$$

Mod out by the kernel of the representation to obtain a finite Heisenberg group, $\mathbf{H}(\mathbb{Z}_N^g)$, which is a $\mathbb{Z}_{2N}$-extension of $\mathbb{Z}_N^{2g} = H_1(\Sigma_g, \mathbb{Z}_N)$.

The Schrödinger representation

$$e(pP + qQ + kI)\theta_\mu(z) = e^{-\frac{\pi i}{N}p\cdot q - \frac{2\pi i}{N}\mu\cdot q + \frac{2\pi i}{N}k}\theta_{\mu+p}(z).$$

Stone - von Neumann theorem. The Schödinger representation is the unique irreducible unitary representation of the finite Heisenberg group with the property that $e(kI)$ acts as multiplication by $e^{\frac{\pi i}{N}k}$.

The action of the modular group

An element h of the mapping class group $Mod(\Sigma_g)$ induces a linear symplectomorphism $\tilde{h}$ on $\mathcal{J}(\Sigma_g)$

$$\tilde{h} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

h acts on quantum observables

$$h \cdot \mathit{Op}\left(e^{2\pi i(p^T x + q^T y)}\right) = \mathit{Op}\left(e^{2\pi i[(Ap+Bq)^T x + (Cp+Dq)^T y]}\right)$$

By Stone-von Neumann there is a unique automorphism $\rho(h)$ of the space of theta functions satisfying the exact Egorov identity

$$h \cdot \mathit{Op}\left(e^{2\pi i(px+qy)}\right) = \rho(h)\mathit{Op}\left(e^{2\pi i(px+qy)}\right)\rho(h)^{-1}.$$

$h \rightarrow \rho(h)$: projective representation of the mapping class group (the Hermite-Jacobi action). It is given by discrete Fourier transforms.

The abstract version of the Schrödinger representation

Recall that the finite Heisenberg group is defined by the multiplication:

- $e(pP + qQ)e(p'P + q'Q) = e^{\frac{\pi i}{N}(pq' - qp')}e((p + p')P + (q + q')Q)$

• $\mathbf{L} \subset H_1(\Sigma_g, \mathbb{Z}_N)$ *arising from a Lagrangian subspace of* $H_1(\Sigma_g, \mathbb{R})$ *wrt. to the intersection form. Define*

$$\chi : e(\mathbf{L} + \mathbb{Z}_{2N}I) \to U(1), \quad \chi(l + kI) = e^{\frac{ik\pi}{N}}.$$

The Hilbert space of the quantization, $\mathcal{H}(\mathbf{L})$*, is the quotient of* $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$ *by*

$$u \sim \chi(u')^{-1}uu', \quad u' \in e(\mathbf{L} + \mathbb{Z}_{2N}I).$$

• *The Schrödinger representation is the left action of* $\mathbf{H}(\mathbb{Z}_N^g)$ *on the quotient* $\mathcal{H}(\mathbf{L})$ *of the group algebra of the Heisenberg group:*

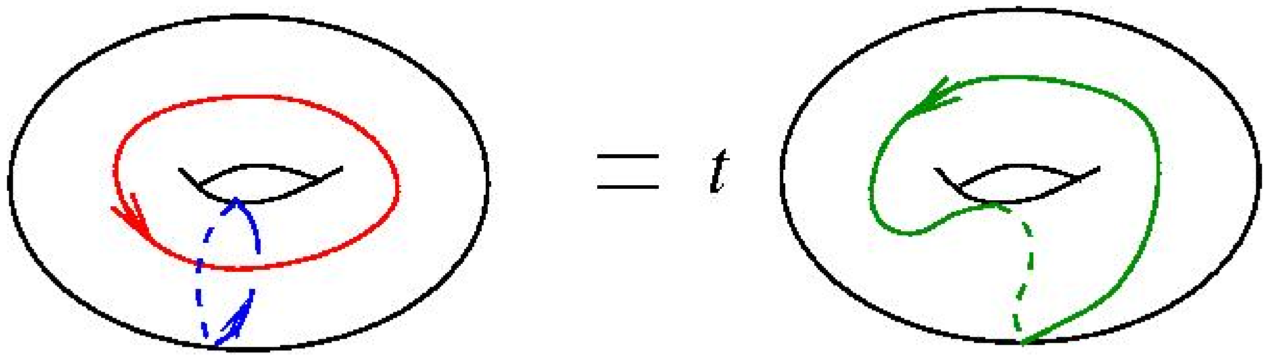
$$u_0 \cdot \hat{u} = \widehat{u_0 u}$$

The topological version of the Schrödinger representation

The multiplication rule for quantized exponentials on the torus:

$$e(pP + qQ)e(p'P + q'Q) = t^{pq' - qp'} e((p + p')P + (q + q')Q)$$

where $t = e^{\frac{i\pi}{N}}$. The determinant is the algebraic intersection number of the curve (p, q) of slope p/q with the curve (p', q') of slope p'/q'.

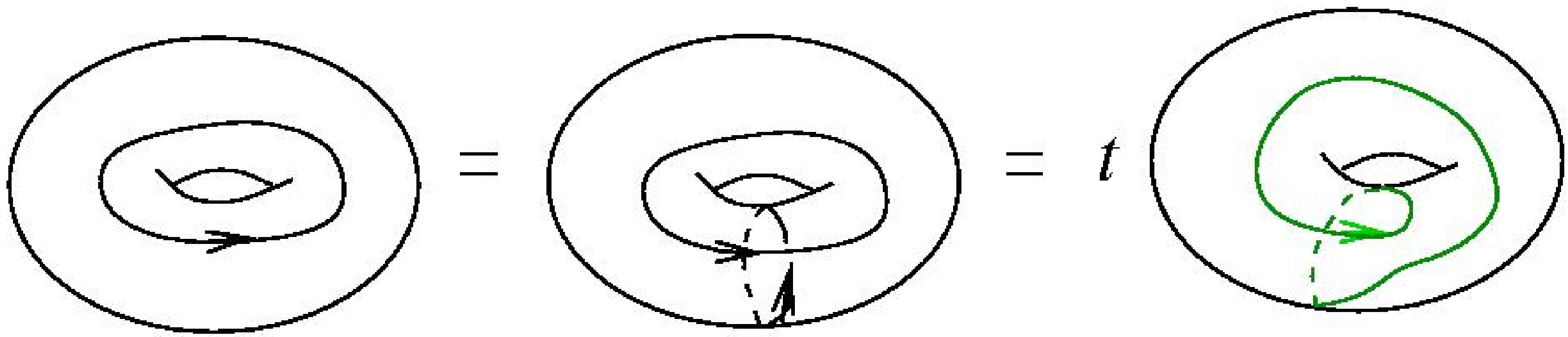


$$(1, 0) \cdot (0, 1) = t(1, 1)$$

The equivalence relation on $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$

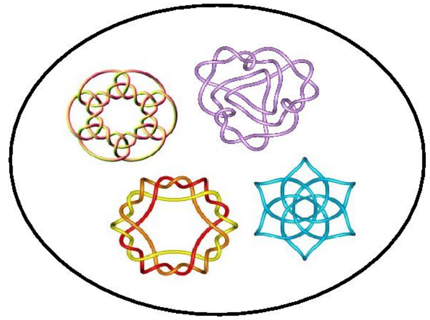
The case of the torus, with $\mathbf{L}$ *spanned by the meridian.*

$$u \sim \chi_{\mathbf{L}}(u')^{-1}uu', \quad u' \in e(\mathbf{L}+\mathbb{R}I)$$

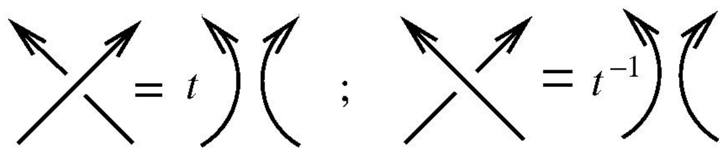


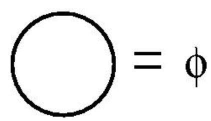
$$(1,0) \sim (1,0) \bullet (0,1) = t\,(1,1)$$

Linking number skein modules



Definition (J. Przytycki) The *linking number skein module* $\mathcal{L}_t(M)$ *of the 3-manifold* M *is the quotient of the free* $\mathbb{C}[t, t^{-1}]$*-module with basis the framed oriented links in* M *by the skein relations*

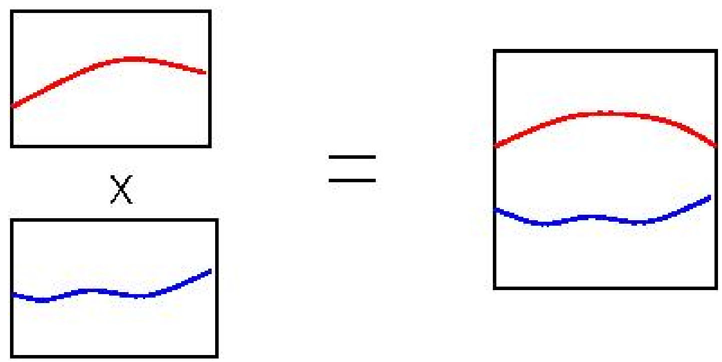




Definition The reduced linking number skein module $\widetilde{\mathcal{L}}_t(M)$ *is obtained by setting* $t = e^{\frac{i\pi}{N}}$ *and deleting any* N *parallel copies of a curve.*

Skein algebras

If $M = \Sigma_g \times [0,1]$, *then by gluing two copies of* M *along* Σ_g *we obtain a multiplication on* $\mathcal{L}_t(M)$ *and* $\widetilde{\mathcal{L}}_t(M)$.



$\mathcal{L}_t(\Sigma_g \times [0,1])$: *linking number skein algebra of* Σ_g

If M *has boundary, then* $\mathcal{L}_t(M)$ *is a* $\mathcal{L}_t(\partial M \times [0,1])$*-module. The module structure is defined by gluing* $\partial M \times [0,1]$ *to* M *along* ∂M, *and identifying the result with* ∂M.

The topological version of the quantization

We have topological descriptions of the group algebra of the finite Heisenberg group and the Schrödinger representation:

THEOREM. The group algebra of the finite Heisenberg group $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$ is isomorphic to the reduced linking number skein algebra $\widetilde{\mathcal{L}}_t(\Sigma_g \times [0,1])$.

THEOREM. The Schrödinger representation of $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$ on theta functions coincides with the action of $\widetilde{\mathcal{L}}_t(\Sigma_g \times [0,1])$ on $\widetilde{\mathcal{L}}_t(H_g)$, where H_g is the genus g handlebody.

The discrete Fourier transform

THEOREM. Let h be an element of $Mod(\Sigma_g)$. Then for a skein $\sigma \in \widetilde{\mathcal{L}}_t(H_g)$, $\rho(h)(\sigma)$ is obtained by lifting σ in all possible nonequivalent ways to the boundary Σ_g, mapping those skeins by h taking the average and viewing the result as a skein in H_g.

The discrete Fourier transform

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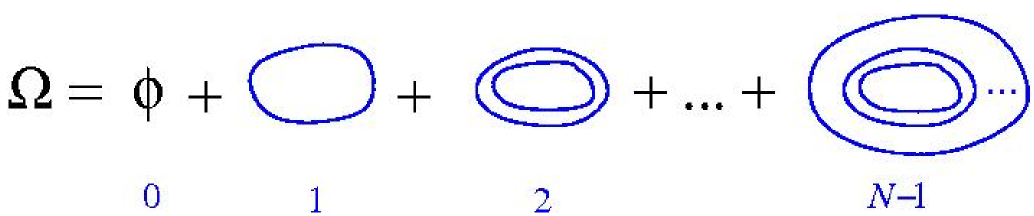
THEOREM. Let h be an element of $Mod(\Sigma_g)$ defined by surgery on a framed link L_h in $\Sigma_g \times [0,1]$. The discrete Fourier transform

$$\rho(h) : \widetilde{\mathcal{L}}_t(H_g) \to \widetilde{\mathcal{L}}_t(H_g)$$

is given by

$$\rho(h)\beta = \Omega_{U(1)}(L_h)\beta$$

where $\Omega_{U(1)}(L_h)$ is obtained from L_h by replacing each link component by

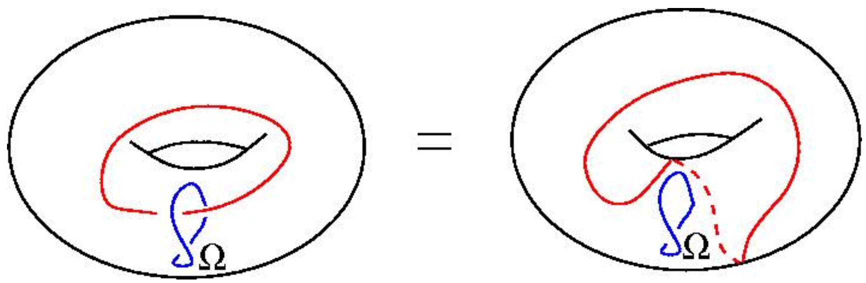


The exact Egorov identity as a handle-slide

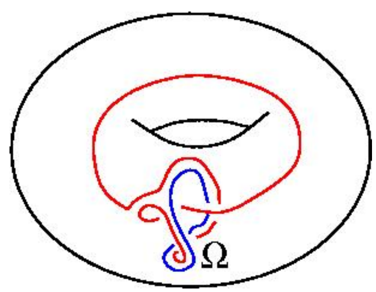
The exact Egorov identity

$$\mathit{Op}\left(f \circ h^{-1}\right) = \rho(h)\,\mathit{Op}\,(f)\,\rho(h)^{-1}$$

can be rewritten for skeins as $\rho(h)\sigma = h(\sigma)\rho(h)$



The diagram on the right is the same as

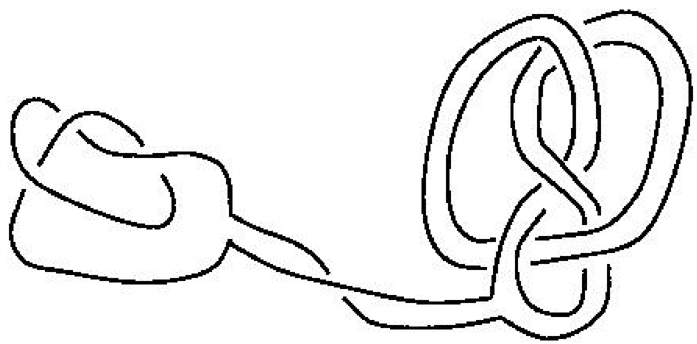


which is the slide of the red curve along the blue curve.

$\Rightarrow$ *Invariants of 3-manifolds!*

The WRT invariant

Stone-von Neumann $\Rightarrow$ handle slides



We obtain the well known Witten-Reshetikhin-Turaev invariant for abelian Chern-Simons theory:

THEOREM Let M be a 3-manifold obtained by surgery on the framed link L in S^3. Then

$$Z(M) = e^{-\frac{\pi i}{4}\mathsf{sign}(L)}\Omega(L)$$

is a topological invariant of M.

The quantum group for $G = U(1)$

This fits the Reshetikhin-Turaev picture. The quantum group is

$$\mathbb{C}[\mathbb{Z}_{2N}] = \mathbb{C}[K]/K^{2N} = 1$$
$$\Delta(K) = K \otimes K, \quad S(K) = K^{2N-1}, \quad \epsilon(K) = 1.$$

The irreducible representation of interest are V^k, $k = 0, 1, \ldots, N-1$, where $V^k \cong \mathbb{R}$ and K acts by

$$K \cdot v = t^k v.$$

Here $t = e^{\frac{\pi i}{N}}$.

Example 2. Non-abelian theta functions for $G = SU(2)$.

G.-Uribe (2003), G.-Uribe (2009)

In the Reshetikhin-Turaev picture, the quantum group $U_h(sl(2,\mathbb{C}))$ is an algebra generated by X, Y, K satisfying

$$KX = t^2XK,\ KY = t^{-2}YK,\ XY - YX = \frac{K^2 - K^{-2}}{t^2 - t^{-2}},$$
$$X^r = Y^r = 0,\ K^{4r} = 1$$

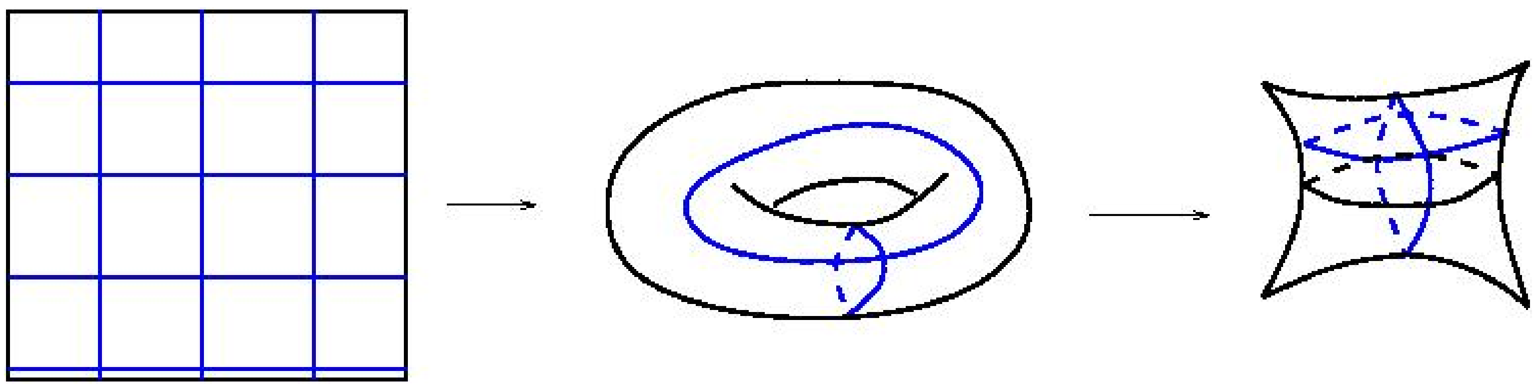
Irreducible representations: V^1, V^2, ..., V^{r-1};

V^k has basis e_j, $j = -k_0, \ldots,, k_0$, where $k_0 = \frac{k-1}{2}$, and the quantum group acts on it by

$$Xe_j = [k_0 + j + 1]e_{j+1}, \quad Ye_j = [k_0 - j + 1]e_{j-1}, \quad Ke_j = t^{2j}e_j.$$

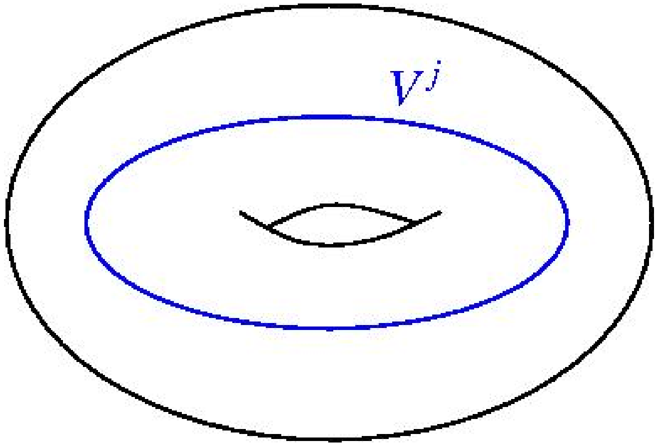
The pillow case

The moduli space of flat $su(2)$*-connections on the torus*

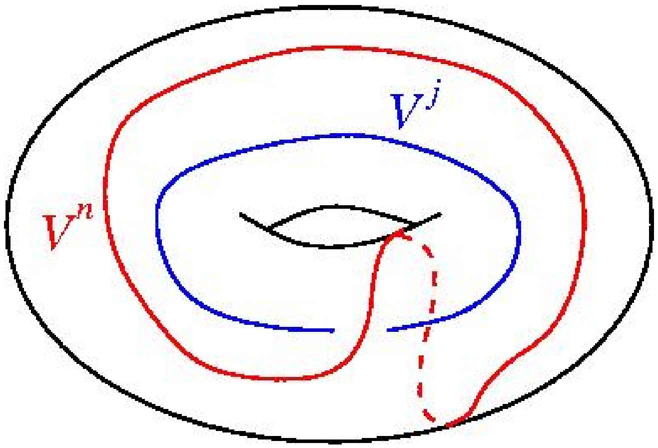


The quantum group quantization for the torus

The basis of the Hilbert space



The operators



Weyl quantization again

THEOREM The quantum group quantization and the Weyl quantization of the moduli space of flat $su(2)$-connections on the torus are unitary equivalent.

As a corollary we obtain that the Weyl quantization of the pillow case can be obtained as the left action of the skein algebra of the Jones polynomial of the torus on the skein module of the solid torus.

THEOREM The representation of the skein algebra of the torus defined by the Weyl quantization of the moduli space of flat $su(2)$-connections on the torus is the unique irreducible representation of this algebra that maps simple closed curves to self-adjoint operators and t to multiplication by $e^{\frac{\pi i}{2r}}$.

We thus have an exact Egorov identity, hence an element $\Omega_{SU(2)}$ and handle slides.

The formulas for the Weyl quantization of the pillow case

Complex structure $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$

$$\theta_j^\tau(z)=\sum_n e^{2\pi i N\left[\frac{\tau}{2}\left(\frac{j}{N}+n\right)^2+z\left(\frac{j}{N}+n\right)\right]}, \quad j=0,1,2,\ldots,N-1.$$

$$\zeta_j^\tau(z)=(\theta_j^\tau(z)-\theta_{-j}^\tau(z)), \quad j=1,2,\ldots,r-1,$$

$$\textit{Op}(2\cos 2\pi(px+qy))\zeta_j^\tau(z)=e^{-\frac{\pi i}{2r}pq}\left(e^{\frac{\pi i}{r}qj}\zeta_{j-p}^\tau(z)+e^{-\frac{\pi i}{r}qj}\zeta_{j+p}^\tau(z)\right)$$