

TOPOLOGICAL HILBERT NULLSTELLENSATZ FOR BERGMAN SPACES

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We prove Bergman space analogues of a conjecture of Douglas and Paulsen related to the classification of invariant subspaces for multiplication operators in several variables.

1. INTRODUCTION

In the present paper we intend to discuss problems related to the classification of invariant subspaces for multiplication operators on spaces of functions of several variables. Unlike the one-dimensional situation, the several variables case proves to be very hard, and little progress has been made until now. In [4] and [6] the attention was focused on invariant subspaces that arise as closures of ideals from some dense ring, the main result proved in this direction being the rigidity theorem (see [4]). In order to understand this kind of invariant subspaces, one needs first to understand their corresponding ideals. In particular one needs to classify the ideals that are closed in the relative topology induced on the ring by the space of functions.

In [3] it was conjectured that an ideal of polynomials is closed in the relative topology induced by the Hardy space of the unit polydisk if and only if every irreducible component of the variety of the ideal intersects the polydisk. In [5] the conjecture was proved in dimension 2. The idea of the proof is to relate the conjecture to a topological version of the Hilbert Nullstellensatz for the ring of polynomials. In this paper we intend to prove similar results for Bergman space topologies. One should note that our methods provide new proofs for the one-dimensional case as well. Some of the techniques come from commutative algebra, and our basic source is [1].

Let us start by recalling some facts from [5]. Let \mathcal{R} be a commutative Noetherian ring with unit and let τ be a topology on \mathcal{R} for which the addition is continuous and multiplication is separately continuous in each variable. Examples of such rings are the ring of polynomials with the topology induced by the Hardy space or by some Bergman space, the ring of germs of analytic functions in a neighborhood of the closure of some domain satisfying certain properties [7] with the topology induced by the Bergman space, etc.

In [5] it has been shown that the radical of a closed ideal is closed, and that the prime ideals associated to a closed ideal are also closed. Define \mathcal{C} to be the set of all maximal

ideals \mathcal{M} with the property that \mathcal{M}^n is closed for all $n \in \mathbf{N}$. In our examples \mathcal{C} will consist of the maximal ideals that correspond to points in the domain on which the spaces of functions are defined. The pair (\mathcal{R}, τ) is said to satisfy the topological Hilbert Nullstellensatz if every ideal $I \subset \mathcal{R}$ is either dense or there exists $\mathcal{M} \in \mathcal{C}$ with $I \subset \mathcal{M}$. The Nullstellensatz is easy to prove in the one-dimensional case, but becomes hard when passing to several variables. In what follows we show that it holds for various topologies in the two variable case. We will need the following elementary result, whose proof can be found in [5].

Proposition 1.1. *Let $p(z) = a(z - z_1)(z - z_2) \cdots (z - z_m)$ be a polynomial with the property that $|z_i| \geq 1$, $i = 1, 2, \dots, m$. Then for all $1/2 < r < 1$ and z with $|z| \leq 1$ we have $|p(z)/p(rz)| \leq 2^m$.*

2. THE RING OF POLYNOMIALS

Let $\Omega \in \mathbf{C}^n$ be a bounded complete Reinhardt domain (i.e. a domain with the property that for all $x \in \Omega$ and $|z| \leq 1$, zx is in Ω), and let $L_a^2(\Omega)$ be the Bergman space of Ω . For a polynomial $p \in \mathbf{C}[z_1, z_2, \dots, z_n]$ we denote by $d(p)$ the sum of the degrees of p in each variable.

Lemma 2.1. *Let $p \in \mathbf{C}[z_1, z_2, \dots, z_n]$ be a polynomial having no zeros in Ω . Then for every $1/2 < r < 1$ and $(z_1, z_2, \dots, z_n) \in \Omega$, $|p(z_1, z_2, \dots, z_n)/p(rz_1, rz_2, \dots, rz_n)| \leq 2^{d(p)}$.*

Proof: The proof will be done by induction on the number of variables. By Proposition 1.1 the property is true for $n = 1$.

Fix $(z_1, z_2, \dots, z_n) \in \Omega$. Let $D = \{w, (w, z_2, \dots, z_n) \in \Omega\}$. Then D is a disk and $p(w, z_2, \dots, z_n)$ has no zeros in D . Thus $|p(z_1, z_2, \dots, z_n)/p(rz_1, z_2, \dots, z_n)| \leq 2^{d_1}$, where we denote by d_i the degree of p in the variable z_i . Applying the induction hypothesis to $p(z_1, \cdot, \dots, \cdot)$ we get $|p(rz_1, z_2, \dots, z_n)/p(rz_1, rz_2, \dots, rz_n)| \leq 2^{d_2 + \dots + d_n}$, thus $|p(z_1, z_2, \dots, z_n)/p(rz_1, rz_2, \dots, rz_n)| \leq 2^{d(p)}$. \square

Proposition 2.2. *Let $p \in \mathbf{C}[z_1, z_2, \dots, z_n]$ be a polynomial having no zeros in Ω . Then $pL_a^2(\Omega)$ is dense in $L_a^2(\Omega)$.*

Proof: Let $f_r(z_1, z_2, \dots, z_n) = p(z_1, z_2, \dots, z_n)/p(rz_1, rz_2, \dots, rz_n)$. Then for $1/2 < r < 1$, $f_r \in pL_a^2(\Omega)$. By Lemma 2.1 the family f_r is bounded, and since for $r \rightarrow 1$, f_r converges uniformly on compact subsets of Ω to 1, it follows that it converges in $L_a^2(\Omega)$ to 1. The conclusion follows by using the fact that the ring of polynomials is dense in $L_a^2(\Omega)$. \square

Corollary 2.3. *The topological Hilbert Nullstellensatz holds for principal ideals in the ring of polynomials with the topology induced by $L_a^2(\Omega)$.*

Proof: If \mathcal{M} is a maximal ideal that corresponds to a point in Ω then \mathcal{M}^n is closed for all n , thus $\mathcal{M} \in \mathcal{C}$. This shows that if the variety of an ideal intersects Ω then the ideal is contained in some maximal ideal from \mathcal{C} . If the variety of a principal ideal does not intersect Ω , then the ideal is dense and the conclusion follows. \square

We now exhibit some examples of topologies for which the topological Hilbert Nullstellensatz will hold for the ring of polynomials in two variables. Following [2] we define

$$\Omega_{p,q} := \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^p + |z_2|^q < 1\}.$$

where $0 < p, q < \infty$. Note that $\Omega_{2,2}$ is the unit ball in \mathbf{C}^2 .

Theorem 2.4 *The ring $\mathbf{C}[z_1, z_2]$ with the topology induced by $L^2(\Omega_{p,q})$ satisfies the topological Hilbert Nullstellensatz.*

Proof: By Proposition 4.1 in [5] $\mathcal{C} = \Omega$ and the maximal ideals not contained in \mathcal{C} are dense, thus the Nullstellensatz holds for maximal ideals. The previous corollary shows that it also holds for principal ideals. Since the ring $\mathbf{C}[z_1, z_2]$ has dimension 2, a prime ideal is either maximal or principal, hence the Nullstellensatz holds for prime ideals.

Now let I be an arbitrary ideal. Then I is either contained in an element from \mathcal{C} , or its variety does not intersect Ω . In the latter case all prime ideals associated to I are dense. Let P_1, P_2, \dots, P_m be these ideals. By Noetherianess (see Proposition 7.14 in [1]) there is a k such that $(P_1 P_2 \cdots P_m)^k \subset I$, and since $(P_1 P_2 \cdots P_m)^k$ is dense it follows that I is dense. This proves the theorem. \square

By using Theorem 1.3 in [5] we get the following analogue of the Douglas-Paulsen conjecture.

Corollary 2.5. *Let $\mathbf{C}[z_1, z_2]$ be endowed with the topology induced by $L_a^2(\Omega_{p,q})$. Then an ideal is closed if and only if each of the irreducible components of its zero set intersects $\Omega_{p,q}$.*

3. THE RING OF ANALYTIC FUNCTIONS IN A NEIGHBORHOOD OF THE UNIT BALL

Let us denote by \mathbf{B}^2 the open unit ball in \mathbf{C}^2 and by $\mathcal{O}(\overline{\mathbf{B}^2})$ the ring of germs of analytic functions defined in a neighborhood of $\overline{\mathbf{B}^2}$. We want to prove the topological Hilbert Nullstellensatz for $\mathcal{O}(\overline{\mathbf{B}^2})$ with the topology induced by the Bergman space. For $k \geq 1$, denote by S^k the unit sphere in \mathbf{R}^{k+1} and by \mathbf{D} the unit disk in the plane.

Lemma 3.1. *Let $f \in \mathcal{O}(\overline{\mathbf{B}^2})$, f not identically equal to zero. Then there exists a unitary transformation $\rho : \mathbf{C}^2 \rightarrow \mathbf{C}^2$ such that $f \circ \rho(0, e^{i\alpha}) \neq 0$ for all $\alpha \in [0, 2\pi)$.*

Proof: For $S^3 = \partial\mathbf{B}^2$ consider the Hopf fibration

$$S^1 \rightarrow S^3 \xrightarrow{\pi} S^2$$

where we recall that the projection π is given by the equivalence relation $(a, b) \sim (\lambda a, \lambda b)$ for $|\lambda| = 1$. If we denote by $V(f)$ the zero set of f , then $V(f) \cap S^3$ has dimension at most one, hence $\pi(V(f) \cap S^3)$ has dimension at most one. It follows that there exists $x \in S^2 \setminus \pi(V(f) \cap S^3)$. Let $(a, b) \in \mathbf{C}^2$, $|a|^2 + |b|^2 = 1$ with the property that $\pi(a, b) = x$. Then $f(ae^{i\alpha}, be^{i\alpha}) \neq 0$ for all $\alpha \in [0, 2\pi)$. If we choose $\rho = \begin{pmatrix} \bar{b} & a \\ -\bar{a} & b \end{pmatrix}$ then $f \circ \rho$ satisfies the desired property. \square

Proposition 3.2. *Let $f \in \mathcal{O}(\overline{\mathbf{B}^2})$ be an analytic function having no zeros in \mathbf{B}^2 . Then the space $fL_a^2(\mathbf{B}^2)$ is dense in $L_a^2(\mathbf{B}^2)$.*

Proof: By Lemma 3.1 we may assume that if $(w_1, w_2) \in \overline{\mathbf{B}^2}$ and $w_1 = 0$ then $f(w_1, w_2) \neq 0$. Let us show that there exists $C > 0$ such that for every $1/2 < r < 1$ and $(z_1, z_2) \in \overline{\mathbf{B}^2}$, $|f(z_1, z_2)/f(rz_1, z_2)| < C$.

Fix $(w_1, w_2) \in \overline{\mathbf{B}^2}$, and assume that $w_1 \neq 0$. Note that $f(z, w_2)$ is not identically zero as a function of z_1 , otherwise $(0, w_2)$ would be a zero for f lying inside of \mathbf{B}^2 . Thus there exists $a > 0$ such that $w_1 \in a\mathbf{D}$ and $f(\cdot, w_2)$ is defined in a neighborhood of $a\overline{\mathbf{D}}$ and has no zeros on $aS^1 = \{z \mid |z| = a\}$. It follows that $f(\cdot, w_2)$ has a finite number of zeros in $a\mathbf{D}$, say m , counting multiplicities. By Rouché's Theorem there is a compact neighborhood K of w_2 such that $a\overline{\mathbf{D}} \times K$ is contained in the domain of f , and for every $z_2 \in K$, $f(\cdot, z_2)$ has exactly m zeros in $a\mathbf{D}$, counting multiplicities, and no zero on aS^1 .

Thus on $a\overline{\mathbf{D}} \times K$ we can write $f(z_1, z_2) = p_{z_2}(z_1)g_{z_2}(z_1)$ where for each z_2 , $p_{z_2}(z_1)$ is a polynomial of degree m and $g_{z_2}(z_1)$ is an analytic function having no zeros in $a\overline{\mathbf{D}}$. Another application of Rouché's Theorem and the maximum modulus principle shows that g_{z_2} depends continuously on z_2 .

It follows that for $1/2 \leq r \leq 1$ the family $\{g_{z_2}(rz_1)\}_r$ is bounded away from zero, hence

$$C_1 = \sup_{1/2 \leq r \leq 1} \sup_{a\overline{\mathbf{D}} \times K} |g_{z_2}(z_1)/g_{z_2}(rz_1)| < \infty.$$

By Proposition 1.1, for $1/2 \leq r \leq 1$ and $(z_1, z_2) \in a\overline{\mathbf{D}} \times K \cap \overline{\mathbf{B}^2}$

$$|p_{z_2}(z_1)/p_{z_2}(rz_1)| \leq 2^m.$$

Thus there exists a neighborhood U of (w_1, w_2) and a constant $C_2 > 0$ such that for $1/2 < r < 1$ and $(z_1, z_2) \in U \cap \overline{\mathbf{B}^2}$

$$|f(z_1, z_2)/f(rz_1, z_2)| < C_2.$$

If $w_1 = 0$ then $f(z_1, z_2) \neq 0$ in a neighborhood of (w_1, w_2) , thus a similar inequality holds there. From the compactness of $\overline{\mathbf{B}^2}$ it follows that there exists a constant $C > 0$ such that for $1/2 < r < 1$ and $(z_1, z_2) \in \overline{\mathbf{B}^2}$, $|f(z_1, z_2)/f(rz_1, z_2)| < C$.

As in the proof of Proposition 2.2, the family $h_r(z_1, z_2) = f(z_1, z_2)/f(rz_1, z_2)$ is in $fL_a^2(\mathbf{B}^2)$ and tends to 1 as $r \rightarrow 1$, so the conclusion follows. \square

Theorem 3.3. *The ring $\mathcal{O}(\overline{\mathbf{B}^2})$ with the topology induced by $L_a^2(\mathbf{B}^2)$ satisfies the topological Hilbert Nullstellensatz.*

Proof: The ring $\mathcal{O}(\overline{\mathbf{B}^2})$ is Noetherian [7], and has dimension 2. Indeed, if there existed distinct prime ideals $P_0 \subset P_1 \subset P_2 \subset P_3$, by localizing at a maximal ideal $\mathcal{M} \supset P_3$ we would get a chain of four distinct prime ideals in the local ring $\mathcal{O}_{\mathcal{M}}$, which would contradict the fact that the latter ring has dimension 2. So the proof of Theorem 2.4 applies mutatis mutandis to give the desired conclusion. \square

Corollary 3.4. *Let $\mathcal{O}(\overline{\mathbf{B}^2})$ be endowed with the topology induced by the Bergman space. Then an ideal is closed if and only if each irreducible component of its zero set intersects the unit ball.*

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