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CONCENTRATION OF AREA IN HALF-PLANES

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ABSTRACT. For the standard class S of normalized univalent functions f analytic in the unit disk $\mathbb{U}$, we consider a problem on the minimal area of the image $f(\mathbb{U})$ concentrated in any given half-plane. This question is related to a well-known problem posed by A. W. Goodman in 1949 that regards minimizing area covered by analytic univalent functions under certain geometric constraints. An interesting aspect of this problem is the unexpected behavior of the candidates for extremal functions constructed via geometric considerations.

1. Introduction

For a function $f \in S$,

$$f(z) = z + a_2(f)z^2 + \dots,$$

analytic and univalent in the unit disk $\mathbb{U} = \{z : |z| < 1\}$, the Dirichlet integral

$$D(f) = \int_{\mathbb{U}} |f'|^2 \, d\sigma = \pi \sum_{n=1}^{\infty} n|a_n(f)|^2 \tag{1.1}$$

measures the area of the image $f(\mathbb{U})$. From (1.1), it is immediate that

$$D(f) \geq \pi \tag{1.2}$$

with equality only for the identity mapping. (1.2) gives the best lower bound for the area of the whole image $f(\mathbb{U})$. In this note we are interested in a similar sharp lower bound for the area of $f(\mathbb{U})$ concentrated in a half-plane $\{w : \Re e^{-i\alpha} w > d\}$ for any given $0 \leq \alpha < 2\pi$ and $d \in \mathbb{R}$. In a certain sense this problem is a half-plane version of a well-known omitted area problem posed by A. W. Goodman in 1949, which has a long history, as noted in [2]. Goodman's problem concerned the minimization of the area of $f(\mathbb{U})$ concentrated in the disk $\mathbb{U}_r = \{w : |w| < r\}$ for any given $r > 0$.

Since the class S is rotationally invariant, i.e. $e^{-i\alpha} f(e^{i\alpha}) \in S$ if $f \in S$, we may assume that $\alpha = 0$ and thus consider the area in the half-plane $\mathbb{H}_d^+ = \{w : \Re w > d\}$.

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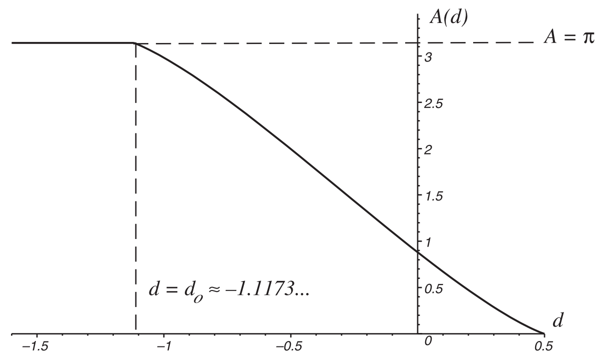

FIGURE 1. Graph of $A(d)$

Let $\mathbb{H}_d^- = \{w : \Re w < d\}$. Since $f(z) = z(1+z)^{-1}$ is in S and maps $\mathbb{U}$ onto $\mathbb{H}_{1/2}^-$, it follows that the minimal area is zero and the problem is trivial for $d \geq 1/2$. For the non-trivial range $d < 1/2$, the solution to the problem is given by

Theorem 1.1. *For $f \in S$, let $A_f(d) = \text{Area}(f(\mathbb{U}) \cap \mathbb{H}_d^+)$. Then*

$$A_f(d) \geq \begin{cases} \pi\beta^2\left(1 + 2d\beta^{-1}(3\beta - 4\beta^{1/2} + 1)\right) & \text{if} \quad d_0 \leq d < 1/2, \\ \pi & \text{if} \quad d \leq d_0, \end{cases} \tag{1.3}$$

where $\beta = \beta(d) > 1/4$ is the smallest root of the equation

$$2\beta\left((3\beta^{1/2} - 1)(1 - \beta^{1/2})\log\frac{1 - \beta^{1/2}}{\beta^{1/2}} + \frac{1}{2}(5 - 6\beta^{1/2})\right) = d, \tag{1.4}$$

which has a unique solution for $-1 < d < 1/2$ and two solutions for $-1.1174 < d \leq -1$, and $d_0 = -1.1173...$ is a solution of the equation

$$\beta^2\left(1 + 2d\beta^{-1}(3\beta - 4\beta^{1/2} + 1)\right) = 1 \tag{1.5}$$

with $\beta = \beta(d)$, unique in the interval $-1.1174 < d < 1/2$.

For $d_0 < d < 1/2$ there is a unique extremal function

$$f_d(z) = 4i\beta\sin\theta_0 \int_{ie^{-i\theta_0/2}}^{\tau} \frac{\tau(\tau + i)^2}{(\tau - i)^2(\tau^2 + 2i\cos(\theta_0/2) - 1)^2}\,d\tau, \tag{1.6}$$

where

$$\tau = ie^{i\theta_0/2}\sqrt{(z - e^{-i\theta_0})/(z - e^{i\theta_0})} \tag{1.7}$$

with the principle branch of the radical and

$$\theta_0 = \theta_0(d) = 4\cos^{-1}(4\beta)^{-1/4}. \tag{1.8}$$

For $d < d_0$ the unique extremal function is the identity mapping $f(z) = z$. For $d = d_0$, there are two extremal functions: the identity mapping and f_d defined by (1.6)–(1.8).

Let $A(d) = \inf\{A_f(d) : f \in S\}$. The graph of $A(d)$ is shown in Figure 1. Figure 2 displays the extremal domains $f_d(\mathbb{U})$ for some typical values of d. Looking at Figure 1, one may wonder why the function $A(d)$ is not differentiable at $d = d_0$ as such is rare in extremal problems concerning conformal mappings. The reason for this unexpected result is that the left-hand side of (1.4) is not monotone (see Lemma

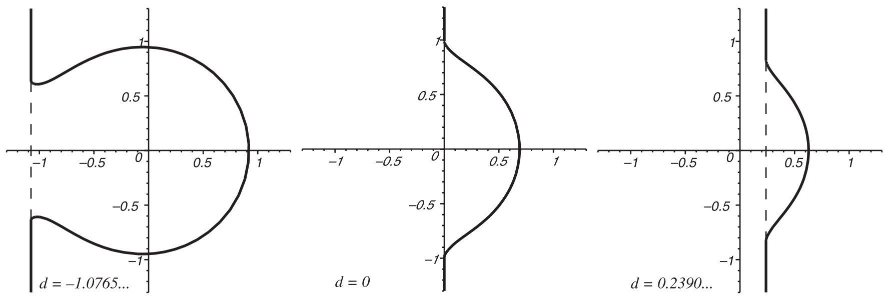

FIGURE 2. Extremal domains for some typical values of d

4.1) and for $d < -1$, (1.4) has two solutions. Taking into account the identity mapping, this gives three functions in S providing critical points for the considered minimal area problem. The graph of $A(d)$ reflects the fact that for different ranges of d, the global minimum is attained by different local minima.

The proof of Theorem 1.1 is given in Section 3. In Section 2, we apply symmetrization and local variations developed in [2] to prove some important qualitative properties of the extremal functions and extremal domains. This allows us to identify in Lemma 2.4 the closed form for the extremal functions. In Section 4, we prove two monotonicity results needed to justify the uniqueness assertions of Theorem 1.1.

2. Qualitative properties of the extremals

Since the area functional $A_f(d)$ is lower semi-continuous, the existence of an extremal function, at least one for each d, easily follows from the compactness of the class S. Thus the proof of our first lemma is standard (see [1, 2]) and left to the reader.

Lemma 2.1. *For every $d < 1/2$, there exists $f \in S$ such that $A_f(d) = A(d)$. In addition, $A(d)$ is continuous for $d \leq 1/2$.*

The following lemma describes the most important geometric properties of extremal domains. We remind the reader that a domain $D \subset \mathbb{C}$ is called Steiner symmetric (possesses Steiner symmetry) w.r.t. $\mathbb{R}$ if for every $x_0 \in \mathbb{R}$ the intersection $D \cap \{z = x_0 + it : -\infty < t < \infty\}$ is either empty or consists of a single interval symmetric w.r.t. $\mathbb{R}$. Similarly, a domain D is called circularly (Pólya) symmetric w.r.t. $\mathbb{R}_- = (-\infty, 0]$ if for every $r \geq 0$ the intersection $D \cap \{z : |z| = r\}$ is either empty, coincides with $\{z : |z| = r\}$, or consists of a single arc symmetric w.r.t. $\mathbb{R}_-$; see [3].

Lemma 2.2. *If $f \in S$ is extremal for $A(d), d < 1/2$, then $D_f = f(\mathbb{U})$ possesses Steiner symmetry w.r.t. $\mathbb{R}$ and circular symmetry w.r.t. $\mathbb{R}_-$. If D_f is bounded, then $D_f = \mathbb{U}$ and $f(z) = z$.*

If D_f is unbounded, then $\mathbb{H}_d^- \subset D_f$ and the free boundary $L_{fr} = \partial D_f \cap \mathbb{H}_d^+$ is a Jordan, locally rectifiable arc joining the points $d \pm ic_f$ for some $0 < c_f \leq \infty$ depending on f.

Further, if $c_f < \infty$, then the non-free part L_{nf} of the boundary consists of two rays $L_{nf}^{\pm} = \{w = d \pm it : t \geq c_f\}$ and L_{fr} is of finite length and satisfies the

Lavrent′ev condition

$$\text{length}\,(J(w_1,w_2)) \le C\,|w_1-w_2| \quad \textit{for} \quad w_1,w_2\in \bar{L}_{fr}, \tag{2.1}$$

where C is a constant independent of w_1, w_2 and $J(w_1, w_2)$ is the shortest arc of $\bar{L}_{fr}$ between w_1 and w_2.

Proof. The arguments establishing the Steiner and circular symmetries are standard (see [1, 2]) and based on the following well-known results (see [3, 4]). Steiner symmetrization w.r.t. $\mathbb{R}$ preserves the area in vertical strips and strictly increases the conformal radius unless the domain already possesses the symmetry. Similarly, circular symmetrization w.r.t. $\mathbb{R}_-$ diminishes the area in the half-plane $\mathbb{H}_d^+$ and strictly increases the conformal radius unless the domain already possesses the symmetry.

Let $\mathrm{R}(D, z_0)$ denote the conformal radius of the domain D at the point z_0 (see [3, 4]). If $D_f\cap\mathbb{H}_d^- \neq \emptyset$, then the Steiner symmetry of D_f implies that $D^* = D_f\cap\mathbb{H}_d^-$ is a simply connected domain such that $\mathrm{R}(D^*, 0) > \mathrm{R}(D_f, 0)$ and $\mathrm{Area}(D^*\cap\mathbb{H}_d^+) = A_f(d)$. These two relations lead, by a standard subordination argument, to a contradiction of the extremality of f. Thus, either $D_f \cap \mathbb{H}_d^- = \emptyset$ or $\mathbb{H}_d^- \subseteq D_f$.

If $D_f \cap \mathbb{H}_d^- = \emptyset$, then clearly $d < 0$ and $D_f \subset \mathbb{U}_{|d|}$ since D_f is circularly symmetric w.r.t. $\mathbb{R}_-$. Since $\mathrm{R}(D_f, 0) = 1$, the latter inclusion shows that $d \le -1$. Thus $A_f(d) = \mathrm{Area}(D_f) \ge \pi$, which implies that the identity mapping must be the unique extremal for the case $D_f \cap \mathbb{H}_d^- = \emptyset$.

Consider, then, $\mathbb{H}_d^- \subseteq D_f$. Let $a = \inf\{|w| : w \in L_{fr}\}$. To show that $L_{fr}^+ = L_{fr} \cap \{w : \Im w \ge 0\}$ is Jordan, we note that the real-valued function $\tau(w) = |w| + a - \Re w$, which is clearly continuous, is one-to-one on L_{fr}^+. Indeed, let w_1 and w_2 be two distinct points of L_{fr}^+. If $|w_1| = |w_2|$, then $\Re w_1 \neq \Re w_2$ and therefore $\tau(w_1) \neq \tau(w_2)$. If, for instance, $|w_1| < |w_2|$, then it follows from the Steiner and circular symmetries that $\Re w_2 \le \Re w_1$ and so again $\tau(w_1) \neq \tau(w_2)$. Since τ is continuous and one-to-one, it follows that L_{fr}^+ is Jordan; clearly the same is true for L_{fr}.

To show that L_{fr}^+ is locally rectifiable, we split it into two parts $L^{++} = L_{fr}^+\cap\overline{H_0^+}$ and $L^{+-} = L_{fr}^+\cap H_0^-$ (which may be empty). Since $\Re w$ and $\Im w$ both are monotone when w runs along L^{++}, the local rectifiability of L^{++} easily follows as well as the Lavrent′ev condition (2.1) with constant $C = 2$.

To show that L^{+-} is locally rectifiable, we fix points $w_0 \in L^{+-}$ and $w_T \in L^{++}$ such that $\Re w_T = 0$, then consider a polygonal line L_N with vertices $w_0, w_1, \ldots, w_N = w_T$ on $\overline{L^{+-}}$ such that all distances $|w_{k+1} - w_k|$ are small enough. Since D_f possesses Steiner and circular symmetry, it follows that $\Re w_k \le \Re w_{k+1}$ and $|w_k| \ge |w_{k+1}|$. To estimate the length of L_N, we replace each linear segment $[w_k, w_{k+1}]$ by the union of the vertical segment $[w_k, \Re w_k + ih_k]$ where $h_k = (|w_{k+1}|^2 - (\Re w_k)^2)^{1/2}$ together with the circular arc γ_k centered at the origin with end points at $\Re w_k + ih_k$ and w_{k+1}. Such vertical segments and circular arcs always exist if the distances $|w_{k+1} - w_k|$ are small enough. Then

$$\text{length}\, L_N \le \sum_{k=0}^{N-1} (\Im w_k - h_k) + \sum_{k=0}^{N-1} \text{length}\, \gamma_k. \tag{2.2}$$

It is not difficult to show that each sum in (2.2) decreases if we replace L_N with another polygonal line $L_{N'}$ by adding new vertices. This combined with (2.2) shows that the length of L_N is bounded by a constant independent of N. Therefore, L^{+-} is locally rectifiable.

If $c_f < \infty$, then the arguments above imply the desired assertion on L_{nf}. The same arguments applied to the shortest arc $J(w_k, w_{k+1})$ lead to the inequalities

$$\text{length}\,(J(w_k, w_{k+1})) \leq \Im w_k - h_k + \text{length}\,\gamma_k \leq C\,|w_k - w_{k+1}|$$

with some constant C independent of $w_k, w_{k+1} \in \bar{L}^{+-}$. The latter inequality implies that L^{+-} and therefore $\bar{L}_{fr} = \bar{L}^{++} \cup \bar{L}^{+-}$ satisfies the Lavrent′ev condition (2.1). □

Let $l_{fr} = \{e^{i\theta} : |\theta| \leq \theta_0\}$ be the "free arc"; that is, l_{fr} is the preimage of L_{fr} under the mapping f. Similarly, let $l^{\pm}_{nf} = f^{-1}(L^{\pm}_{nf})$, $e^{\pm i\theta_0} = f^{-1}(d \pm ic_f)$.

Lemma 2.3. *For a fixed $d < 1/2$, let $f \in S$ be an unbounded extremal for $A(d)$. Then: (i) $|f'(z)| = \beta$ with some $0 < \beta < 1$ for all $z \in l_{fr}$; (ii) $|f'(e^{i\theta})|$ strictly increases from β to ∞ as θ runs from θ_0 to π.*

Proof. First we show that $|f'(z)|$ is constant a.e. on l_{fr}. Since L_{fr} is Jordan locally rectifiable, it follows that the non-zero finite limit

$$f'(\zeta) = \lim_{z\to\zeta, z\in\bar{\mathbb{U}}} \frac{f(z) - f(\zeta)}{z - \zeta} \neq 0, \infty \tag{2.3}$$

exists a.e. on l_{fr}; see [5, Theorem 6.8, Exercise 6.4.5]. Assume that

$$0 < \beta_1 = |f'(e^{i\theta_1})| < |f'(e^{i\theta_2})| = \beta_2 < \infty \tag{2.4}$$

for $e^{i\theta_1}, e^{i\theta_2} \in l_{fr}$. Note that (2.3), (2.4) allow us to apply the two-point variational formulas of [2, Lemma 10]. Namely, for fixed positive k_1, k_2 such that $0 < k_1 < 1 < k_2$ and $k_1\beta_1^{-1} > k_2\beta_2^{-1}$ and fixed $\varphi > 0$ small enough, we consider the two-point variation $\tilde{D}$ of D centered at $w_1 = f(e^{i\theta_1})$ and $w_2 = f(e^{i\theta_2})$ with inclinations φ and radii $\varepsilon_1 = k_1\varepsilon$, $\varepsilon_2 = k_2\varepsilon$ respectively; see [2, Section 3]. Computing the change in the area by [2, formula (3.32)], we find

$$\text{Area}\,\tilde{D} - \text{Area}\,D = \frac{2\pi\varphi - \sin 2\pi\varphi}{2\sin^2 \pi\varphi}\varepsilon^2(k_1^2 - k_2^2) + o(\varepsilon^2) < 0 \tag{2.5}$$

for all $\varepsilon > 0$ small enough. Similarly, applying [2, formula (3.31)], we get

$$\log\frac{R(\tilde{D}, 0)}{R(D, 0)} = \left[\frac{\varphi(2+\varphi)}{6(1+\varphi)^2}\frac{k_1^2}{\beta_1^2} - \frac{\varphi(2-\varphi)}{6(1-\varphi)^2}\frac{k_2^2}{\beta_2^2}\right]\varepsilon^2 + o(\varepsilon^2) > 0 \tag{2.6}$$

for all $\varepsilon > 0$ small enough and φ chosen such that the expression in the brackets is positive.

Inequalities (2.5) and (2.6) lead to a contradiction to the extremality of f for $A(d)$, via a standard subordination argument. Thus $|f'(e^{i\theta})| = \beta$ a.e. on l_{fr} with some $\beta > 0$. This implies, in particular, that length $L_{fr} < \infty$ and therefore $c_f < \infty$, and $l_{nf} \neq \emptyset$.

Since D_f is Steiner symmetric w.r.t. $\mathbb{R}$, the strict monotonicity of $|f'|$ along l_{nf} follows from [2, Lemma 4]. To prove that $|f'(e^{i\theta})| > \beta$ for all $e^{i\theta} \in l_{nf}$, we assume that $\beta = |f'(e^{i\theta_1})| > |f'(e^{i\theta_2})| = \beta_2$ with $e^{\theta_1} \in l_{fr}$ and some $e^{\theta_2} \in l_{nf}$. Then applying the two-point variation as above, we get inequalities (2.5), (2.6), contradicting the extremality of f for $A(d)$, again via a subordination argument. Hence, $|f'(e^{i\theta})| \geq \beta$ for all $e^{i\theta} \in l_{nf}$ which, when combined with the strict monotonicity property of $|f'|$, leads to the strict inequality $|f'(e^{i\theta})| > \beta$ for $e^{i\theta} \in l_{nf}$.

To prove that $|f'| = \beta$ everywhere on l_{fr}, we consider the function $g = \varphi \circ f$ with $\varphi(w) = (w-(d-s))/(w-(d+s))$, where $s > \inf\{|w| : w \in L_{fr}\}$. Lemma 2.2 implies that $D_g = g(\mathbb{U})$ is Jordan rectifiable. Moreover, since L_{fr} satisfies the Lavrent′ev condition, it follows that D_g is a Lavrent′ev domain and hence a Smirnov domain; see [5, Sections 7.3, 7.4]. Thus, $\log |g'|$ can be represented by the Poisson integral

$$\log |\varphi'(w) f'(z)| = \log |g'(z)| = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t) \log |g'(e^{it})| \, dt \tag{2.7}$$

with boundary values defined a.e. on $\mathbb{T}$; see [5, p.155]. (2.7) easily implies that $|g'(e^{i\theta})| = \beta |\varphi'(f(e^{i\theta}))|$ and therefore $|f'(e^{i\theta})| = \beta$ for all $e^{i\theta} \in l_{fr}$. In addition, (2.7) implies that $\log f'$ is bounded on $\overline{\mathbb{U}}$ outside any neighbourhood of the point $z = -1$.

To show that f' is continuous at $e^{\pm i\theta_0}$, we note that by the reflection principle, f can be continued analytically through l_{nf} and f' can be continued analytically through l_{fr}. This implies that f can be considered as a function analytic in a slit disk $\{z : |z - e^{i\theta_0}| < \varepsilon\} \setminus [e^{i\theta_0}, (1+\varepsilon)e^{i\theta_0}]$ with $\varepsilon > 0$ small enough.

Using the Julia-Wolff lemma ([5, Proposition 4.13]), boundedness of $\log f'$, and well-known properties of the angular derivatives ([5, Propositions 4.7, 4.9]), one can prove that f' has a finite limit $f'(e^{i\theta_0})$, $|f'(e^{i\theta_0})| = \beta$, along any path in $\overline{\mathbb{U}}$ ending at $e^{i\theta_0}$. The details of this proof are similar to the arguments in [2, Lemma 13].

Since $|f'|$ takes its minimal values on $\mathbb{T}$, it follows that $|f'(z)| > \beta$ for all $z \in \mathbb{U}$. In particular, $\beta < |f'(0)| = 1$. The proof is complete. □

Summing up the results of this section we can prove the following lemma, which allows us to find a closed form for the unbounded extremal functions.

Lemma 2.4. *Let $f \in S$ be an unbounded extremal for $A(d)$, $d < 1/2$. Then $\varphi(z) = z f'(z)$ maps $\mathbb{U}$ univalently onto a "fork domain":*

$$F(\beta, \psi_0) = \mathbb{C} \setminus (\{w = \beta e^{i\theta} : |\theta| \le \psi_0\} \cup \{w = t : t \ge \beta\})$$

with $\psi_0 = \psi_0(\beta) = \pi - \cos^{-1}(8\beta - 8\beta^{1/2} + 1)$ and some $\beta = \beta(d) \in (1/4, 1)$.

Proof. (a) First we show that f' is univalent in $\mathbb{U}$. By Lemma 2.3, $|f'(e^{i\theta})|$ increases from β to ∞ as θ runs from θ_0 to π. Since $\arg f'(e^{i\theta}) = -\theta$ strictly decreases from $-\theta_0$ to $-\pi$ as θ runs from θ_0 to π, it follows that f' maps l_{nf}^+ one-to-one onto an analytic Jordan arc δ_+ lying in $\{w : |w| > \beta, -\pi < \arg w < -\theta_0\}$.

Since $|f'| > \beta$ in $\mathbb{U}$ and $|f'| = \beta$ on l_{fr}, it follows that $f''(e^{i\theta}) \neq 0$ for $e^{i\theta} \in l_{fr}$. Thus f' is locally univalent on l_{fr} and therefore $\arg f'(e^{i\theta})$ is monotone on l_{fr}. Let $\vec{n}(\theta)$ be the outer unit normal to L_{fr} at $f(e^{i\theta})$. Then $0 \le \arg \vec{n}(\theta) \le \pi$ for $0 \le \theta \le \theta_0$ since D_f is Steiner symmetric. Since $\arg \vec{n}(\theta) = \theta + \arg f'(e^{i\theta})$, we have $-\theta_0 \le \arg f'(e^{i\theta}) \le \pi$ for $0 \le \theta \le \theta_0$. The latter shows that the total variation of $\arg f'(e^{i\theta})$ on l_{fr}^+ is $< 2\pi$, which implies that f' maps l_{fr}^+ one-to-one onto the arc $\gamma_+ = \{\beta e^{i\psi} : -\theta_0 \le \psi \le 0\}$. Since f' is symmetric w.r.t. $\mathbb{R}$ and $f'(0) = 1$, the argument principle implies that f' maps $\mathbb{U}$ one-to-one onto a domain $G \ni 1$ bounded by $L = \bar{\delta}_+ \cup \delta_- \cup \bar{\gamma}_+ \cup \gamma_-$, where $\delta_- = \{w : \bar{w} \in \delta_+\}$, $\gamma_- = \{w : \bar{w} \in \gamma_+\}$.

Since $|f'(e^{i\theta})|$ is monotone on l_{nf}^+, it follows that G is circularly symmetric w.r.t. $\mathbb{R}_+ = \{w = t : t \ge 0\}$. Hence by [2, Lemma 5], $|f''(e^{i\theta})|$ strictly increases as θ runs from 0 to θ_0.

(b) Considering boundary values of φ, we have $\arg \varphi(e^{i\theta}) = 0$ for $0 < |\theta - \pi| \le \pi - \theta_0$ since $\Re f(e^{i\theta})$ is constant for such θ. Since $|\varphi(e^{i\theta})| = |f'(e^{i\theta})|$ strictly increases in $\theta_0 < \theta < \pi$, φ maps l_{nf}^+ continuously and one-to-one onto the ray $\{w = t : t \geq \beta\}$.

For $0 \leq \theta \leq \theta_0$, $|\varphi(e^{i\theta})| = \beta$ and

$$\frac{\partial}{\partial\theta}\arg\varphi(e^{i\theta}) = \frac{\partial}{\partial\theta}\Im\log(e^{i\theta}f'(e^{i\theta})) = 1 + \frac{e^{i\theta}f''(e^{i\theta})}{f'(e^{i\theta})} = 1 - \beta^{-1}|f''(e^{i\theta})|$$

since $e^{i\theta}f''(e^{i\theta})/f'(e^{i\theta})$ is real non-positive for $0 \leq \theta \leq \theta_0$. Since $|f''(e^{i\theta})|$ strictly increases in $0 < \theta < \theta_0$, it follows that $\frac{\partial}{\partial\theta}\arg\varphi(e^{i\theta})$ changes its sign at most once in $0 < \theta < \theta_0$. Since $\arg\varphi(1) = \arg\varphi(e^{i\theta_0}) = 0$ and the total variation of $\arg\varphi(e^{i\theta})$ on l_{fr} is $< 2\pi$, it follows that $\frac{\partial}{\partial\theta}\arg\varphi(e^{i\theta})$ changes its sign from '$-$' to '$+$' exactly once on $0 < \theta < \theta_0$, say at the point $\theta = \theta_1$. Let $\varphi(e^{i\theta_1}) = \beta e^{-i\psi_0}$, $0 < \psi_0 < \pi$.

The previous arguments show that φ maps l_{fr}^+ one-to-one in the sense of boundary correspondence onto the circular slit along the arc $\{w = \beta e^{i\psi} : -\psi_0 \leq \psi \leq 0\}$. By the reflection principle and the argument principle, φ maps $\mathbb{U}$ conformally and one-to-one onto the fork domain $F(\beta, \psi_0)$. Since $\varphi'(0) = f'(0) = 1$, the Koebe 1/4–theorem shows that $\beta > 1/4$. The same normalization $\varphi'(0) = 1$ leads, after a lengthy computation, to the relation $\psi_0 = \pi - \cos^{-1}(8\beta - 8\beta^{1/2} + 1)$, which has already appeared a few times in the literature; see [1]. □

3. Proof of Theorem 1.1

Proof. Assume that f is an unbounded extremal for $A(d)$ with $d < 1/2$. By Lemma 2.4, $\varphi = zf'$ maps $\mathbb{U}$ conformally onto a fork domain $F(\beta, \psi_0)$. The function φ can be represented as a composition $\varphi = g \circ \tau$ with

$$g(\tau) = \beta\frac{(\tau + i)^2(\tau - ie^{-i\theta_0/2})(\tau - ie^{i\theta_0/2})}{(\tau - i)^2(\tau + ie^{-i\theta_0/2})(\tau + ie^{i\theta_0/2})} \tag{3.1}$$

and $\tau = \tau(z)$ defined by (1.7). Indeed, the function $\tau = \tau(z)$ maps $\mathbb{U}$ onto the first quadrant $Q_1 = \{\tau : \Re\tau > 0, \Im\tau > 0\}$ and, considering boundary values and using the argument principle, one can easily check that $w = g(\tau)$ maps Q_1 onto a fork domain. Since $\tau(0) = ie^{-i\theta_0/2}$, the normalization $\varphi'(0) = g'(\tau(0))\tau'(0) = 1$ gives (1.8). Since $f(z) = \int_0^z z^{-1}\varphi(z)\,dz$, changing the variable of integration $z = z(\tau)$, we obtain (1.6).

To compute the area $A_f(d)$ of $D_f(d) = D_f \cap \mathbb{H}_d^+$, we apply the standard line integral formula for area to the function $f_1(z) = f(z) - d$:

$$\begin{aligned} A_f(d) =& \frac{1}{2}\Im\int_{\partial D_f(d)}\bar{w}\,dw = \frac{1}{2}\Im\int_{L_{fr}}\bar{w}\,dw = \frac{1}{2}\Re\int_{-\theta_0}^{\theta_0}\overline{f_1(e^{i\theta})}e^{i\theta}f_1'(e^{i\theta})\,d\theta \\ =& \frac{\beta^2}{2}\Re\int_{-\pi}^{\pi}\frac{f_1(e^{i\theta})e^{i\theta}}{e^{i\theta}f_1'(e^{i\theta})}\,d\theta = \frac{\beta^2}{2}\Im\int_{|z|=1}\frac{f_1(z)}{z^2f_1'(z)}\,dz \\ =& \pi\beta^2\Re\operatorname{Res}\left[\frac{f_1}{z^2f_1'}, 0\right] = \pi\beta^2\Re\left[(f_1/f_1')'\,|_{z=0}\right] = \pi\beta^2(1 + d\,f''(0)). \end{aligned} \tag{3.2}$$

Differentiating (1.6) with $\tau = \tau(z)$ defined by (1.7) yields

$$f''(0) = 2\beta^{-1}(3\beta - 4\beta^{1/2} + 1). \tag{3.3}$$

Combining (3.2) and (3.3) we obtain inequality (1.3) under the assumption that f is unbounded.

It turns out that the minimal area and closed form for the extremal function can be nicely expressed in terms of the parameter β or via (1.8) in terms of θ_0. To find the relation between β and d, we note that $\Re f(e^{i\theta_0}) = d$; thus,

$$d = \Re f(e^{i\theta_0}) = 4\beta \sin\theta_0 \Im \int_0^{ie^{-i\theta_0/2}} \frac{\tau(\tau+i)^2\,d\tau}{(\tau-i)^2(\tau+ie^{i\theta_0/2})^2(\tau+ie^{-i\theta_0/2})^2}. \tag{3.4}$$

Expanding the integrand in (3.4) into partial fractions and then integrating, we come to an equation equivalent to (1.4):

$$d = p(\beta^{1/2}), \tag{3.5}$$

where $\beta = \left(4\cos^4(\theta_0/4)\right)^{-1}$ and

$$p(x) = x^2[2(1-x)(3x-1)\log((1-x)/x) + 5 - 6x]. \tag{3.6}$$

Integration leading to (3.5), (3.6) is rather lengthy and was performed by hand, then checked with "Mathematica" and "Maple".

(3.2), (3.3), and (3.5) can be used to express the minimal area $A_f(d)$ in terms of β: $A_f(d) = \pi q(\beta^{1/2})$, where

$$q(x) = x^4(1 + 2x^{-2}p(x)(3x^2 - 4x + 1)). \tag{3.7}$$

It turns out that functions (3.6) and (3.7) are not monotone, which makes the problem harder and more interesting.

By Lemma 4.1, for every d, $-1 < d \le 1/2$ and for $d = \hat{d}$, where $\hat{d} = p(\hat{\beta}^{1/2}) = -1.1464\ldots$ and $\hat{\beta} = \hat{x}^2 = .9385\ldots$ are determined in Section 4, equation (3.5) has a unique solution $\beta = \beta(d)$. For $\hat{d} < d < -1$, it has two solutions $\beta_1 = \beta_1(d)$, $1/2 < \beta_1 < \hat{\beta}$ and $\beta_2 = \beta_2(d)$, $\hat{\beta} < \beta_2 < 1$. For $d < \hat{d}$, (3.5) has no solutions and therefore there are no unbounded extremal functions for such d.

According to Lemma 4.2, if the parameter β corresponding to the extremal function f is $> \beta_0$, where $\beta_0 = .8976\ldots$ is defined by equation $q({\beta_0}^{1/2}) = 1$, then $A_f(d) = \pi q(\beta^{1/2}) > \pi$. Since by Lemma 4.1, $d = p(\beta^{1/2}) < -1$ for all $\beta \ge \beta_*$ with $\beta_* = .8370\ldots$ defined by $p(\beta_*^{1/2}) = -1$ and since $\beta_* < \beta_0$, the identity mapping provides a smaller area than f does, contradicting conjectured extremality of f. Therefore, if f is an unbounded extremal for $A(d)$, then the corresponding value of β is $\le \beta_0$. Since $\beta_0 < \hat{\beta}$ and $p(\beta^{1/2})$ is monotone for $1/4 < \beta \le \hat{\beta}$, it follows that for each d, $d_0 \le d < 1/2$, where $d_0 = p(\beta_0^{1/2}) = -1.1173\ldots$, equation (3.5) has a unique solution $\beta = \beta(d) \le \beta_0$. This implies the uniqueness assertion of Theorem 1.1 and finishes its proof. □

4. Monotonicity lemmas

Lemma 4.1. *There exists $\hat{x} = .9688\ldots$ such that $p(x)$ strictly decreases from $1/2$ to $\hat{d} = -1.1464\ldots$ when x runs from $1/2$ to $\hat{x}$ and strictly increases from $\hat{d}$ to -1 when x runs from $\hat{x}$ to 1.*

Proof. Differentiating, we find $p'(x) = 4xu(x)$, where

$$u(x) = 3 - 6x - (6x^2 - 6x + 1)\log((1-x)/x). \tag{4.1}$$

To show that the equation $u(x) = 0$ has a unique solution on $1/2 < x < 1$, we note that

$$u'''(x) = \frac{-2}{x^3(x-1)^3};$$

so $u'''(x) > 0$ on $(1/2, 1)$ and hence $u''(x)$ is increasing on this interval. Now, $u''(1/2) = 0$ and so $u(x)$ is concave up on $(1/2, 1)$. Since $u(1/2) = 0, u(3/5) < 0$, and $u \to +\infty$ as $x \to 1^-$, it follows that u has exactly one zero $\hat{x}$ on $(1/2, 1)$. Solving with "Maple", we get $\hat{x} = .9688\ldots$ The lemma is proved. □

Lemma 4.2. *The function* $q(x)$ *strictly increases from* 0 *to* $\hat{q} = 1.0089\ldots$ *when* x *runs from* $1/2$ *to* $\hat{x}$ *and strictly decreases from* $\hat{q}$ *to* 1 *when* x *runs from* $\hat{x}$ *to* 1.

Proof. Differentiating, we find $q'(x) = 16x^3(x-1)(3x-1)u(x)$, where $u(x)$ is defined by (4.1), and the desired result follows from the proof of Lemma 4.1. □

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