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Abstract. Polynomials to be used in interpolation of digital signals are called interpolating polynomials. They may require modification to assure convergence of their reciprocals on the unit circle.

This paper concerns discrete time windowing, which consists of scaled truncation of a series such as

$$P_N(z) \overset{\Delta}{=} 1 + \sum_{m=1}^{\infty} (z^m + z^{-m}) \operatorname{sinc} \frac{m\pi}{N}, \qquad \operatorname{sinc} x \overset{\Delta}{=} \frac{\sin x}{x},$$

where $N > 1$, to obtain an expression of the form

$$P^*_{N,L}(z) \overset{\Delta}{=} z^{L-1}\left(1 + \sum_{m=1}^{L-1} (z^m + z^{-m}) c_m \operatorname{sinc} \frac{m\pi}{N}\right).$$

We delete the asterisk to write $P_{N,L}$ when each $c_m = 1$.

The zeros of $P_{N,L}$ are shown to have unit modulus for $L \leq N$. Examples are given to show that little can be said of the zeros of $P_{N,L}$ for $L > N$. Conditions are found to define real sequences of the form, $\{c_m : 1 \leq m < \infty\}$, so that $P^*_{N,L}$ has no zero of unit modulus. Several standard discrete time windows are shown to define real sequences which are special cases of the conditions developed.

Introduction. Polynomials to be used in interpolation of digital signals are called interpolating polynomials. These polynomials may require modification to assure convergence of their reciprocals on the unit circle. Such modification is a principal concern of this paper.

A real function, g, defined for all values of the real independent variable time, t, is called a signal. A digital signal, γ, is a real sequence, $\{\gamma_m : -\infty < m < \infty\}$, consisting of equally spaced values or samples, $\gamma_m = g(m\Delta t)$, from the signal, g, with a time increment or sample interval, Δt. Thus, the independent variable for digital signals such as γ is sample time, $m\Delta t$, or simply sample number, m.

The signal, g, is studied in terms of its classical Fourier transform, G, as a function of real frequency, ω. The digital analog of the Fourier transform consists of the study of a sequence such as γ in terms of its Z-transform, which is defined to be the power series, Γ, having γ_m as the coefficient of z^m. Frequency's digital analog comes from evaluation of Z-transforms such as Γ on the unit circle with the negative of the θ in $z = e^{i\theta}$ referred to as frequency. If the coefficients in Γ are used without any actual evaluation of $\Gamma(z)$ or g is used without computation of G, such use is said to be in the time domain. But if $\Gamma(z)$ is used with evaluation for some z of unit modulus or G is used, such use is said to be in the frequency domain.

Signals are based on even functions in a number of applications and in this paper. This restricts digital signals to self-inversive cases meaning that $\Gamma(z) = \Gamma(z^{-1})$ for $z \neq 0$. Equivalently, γ is a symmetric sequence meaning that $\gamma_m = \gamma_{-m}$ for all m.

A second signal, f, with Fourier transform, F, poses as a filter of the signal, g, if the convolution integral, $g * f$, of g and f is considered. Of course, the Fourier transform of $g * f$ is the product of the Fourier transforms, G of g and F of f. The

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discrete analogy consists of the product of Z-transforms, Γ and Φ, where the latter refers to the power series with the sample, $\Phi_m = f(m\Delta t)$, taken from the filter, f, as the coefficient of z^m.

Reduction of certain frequencies is a fundamental aim in application of a filter, f, to a function, g. This can involve definition of f by the requirement that $F(\omega)$ be a constant, c, for $|\omega| < \omega_0$ but zero otherwise. If so, c can be chosen so that

$$f(t) = \operatorname{sinc} \omega_0 t, \tag{1.1}$$

where

$$\operatorname{sinc} x \overset{\Delta}{=} \frac{\sin x}{x}. \tag{1.2}$$

These equations illustrate definition of a real signal from specification of its Fourier transform. Similarly, digital signals are often defined by specification of Z-transforms.

The Fourier transform, F, of the f in (1.1) is referred to as a frequency window since it has compact support in frequency. Application of such a window to a signal, g, is known as frequency windowing. This paper concerns discrete time windowing. This consists of scaled truncation of an infinite sequence such as γ to obtain a finite sequence of the form $\{c_m\gamma_m : -L < m < L\}$ wherein the finite sequence, $\{c_m : -L < m < L\}$, is referred to as a time window.

Suppose a given digital signal, $\{b_k : -\infty < k < \infty\}$, is such that b_k is understood to correspond to the time, $kN\Delta t$, with the sample interval, $N\Delta t$, where N is a natural number such that $N > 1$. If this digital signal is to be compared with digital signals based on the smaller sample interval, Δt, the given digital signal must be interpolated to the smaller sample interval, Δt. For example, insertion of $N-1$ zeros between every b_k and b_{k+1} followed by multiplication of the Z-transform of the result by the interpolating series,

$$P_N(z) \overset{\Delta}{=} 1 + \sum_{m=1}^{\infty} (z^m + z^{-m}) \operatorname{sinc} \frac{m\pi}{N}, \tag{1.3a}$$

leads to

$$A(z) \overset{\Delta}{=} \sum_{n=-\infty}^{\infty} a_n z^n \overset{\Delta}{=} \left(\sum_{j=-\infty}^{\infty} b_j z^{jN} \right) P_N(z). \tag{1.3b}$$

Since the coefficient of z^{kN}, a_{kN}, in A comes from products of b_j and $\operatorname{sinc}(m\pi/N)$ such that $kN = jN \pm m$, it follows that $m \equiv 0 \pmod{N}$, $\operatorname{sinc}(m\pi/N) = 0$ for nonzero m, and $a_{kN} = b_k$. Thus, A is an interpolation of the given B.

A major purpose of this paper is to study possible alternatives to the interpolation used in (1.3a) in terms of truncation of the interpolating series in (1.3b). We consider the interpolating polynomial,

$$P_{N,L}(z) \overset{\Delta}{=} z^{L-1} \left(1 + \sum_{m=1}^{L-1} (z^m + z^{-m}) \operatorname{sinc} \frac{m\pi}{N} \right), \tag{1.4}$$

where $N > 1$.

Note that $P_{N,L}$ is a polynomial of degree $2L-2$ except that it has degree $2L-3$ and $P_{N,L}(0) = 0$ when $L \equiv 1 \pmod{N}$. In any case, its real coefficients imply conjugates of nonzero roots to be roots, and symmetry of coefficients implies reciprocals of nonzero roots to be roots. Since the conjugate and the reciprocal of a root of unit modulus are equal, and the conjugate of a real root is the root itself, nonzero roots can occur in pairs. In other cases, a nonzero root, its conjugate, and their reciprocals are all different and plot as the vertices of a trapezoid in the complex plane.

We show that the zeros of $P_{N,L}$ are all of unit modulus for $L \leqq N$. Since

$$P_{N,N+1}(z) = P_{N,N}(z), \tag{1.5}$$

$P_{N,N+1}$ has a zero at the origin in addition to the zeros of unit modulus of $P_{N,N}$. We use examples to show that little can be said of the zeros of $P_{N,L}$ for $L > (N+1)$.

Conditions are then developed to define real sequences of the form, $\{c_m : 1 \leqq m < \infty\}$, so that the polynomial,

$$P^*_{N,L}(z) \stackrel{\Delta}{=} z^{L-1}\left(1 + \sum_{m=1}^{L-1} (z^m + z^{-m}) c_m \operatorname{sinc}\frac{m\pi}{N}\right), \tag{1.6}$$

has no zero of unit modulus. A number of standard discrete time windows are shown to define real sequences which are special cases of the conditions developed.

2. Zeros of $P_{N,L}$ for $L \leqq N$. Our study of $P_{N,L}$, for $L \leqq N$, is based on the properties of $H_{N,L}$ as defined by

$$H_{N,L}(\theta) \stackrel{\Delta}{=} (2\pi/N) P_{N,L}(z)/z^{L-1}\Big|_{z=e^{i\theta}}. \tag{2.1}$$

LEMMA 2.1.

$$H_{N,L}(\theta) = \int_{\theta-\pi/N}^{\theta+\pi/N} \frac{\sin[(2L-1)t/2]}{\sin(t/2)}\, dt. \tag{2.2}$$

Proof. Use the identity,

$$\int_{\theta-\pi/N}^{\theta+\pi/N} e^{imt}\, dt = e^{im\theta} \frac{e^{im\pi/N} - e^{-im\pi/N}}{im} \tag{2.3}$$

$$= \frac{\pi}{N} e^{im\theta} \frac{2\sin(m\pi/N)}{m\pi/N} = \frac{2\pi}{N} e^{im\theta} \operatorname{sinc}\frac{m\pi}{N},$$

to eliminate the sinc in (1.4). Then, substitute (1.4) in (2.1) and compute

$$H_{N,L}(\theta) = \int_{\theta-\pi/N}^{\theta+\pi/N} \left(1 + \sum_{m=1}^{L-1} (e^{imt} + e^{-imt})\right) dt$$

$$= \int_{\theta-\pi/N}^{\theta+\pi/N} \left(1 + e^{it}\frac{1 - e^{it(L-1)}}{1 - e^{it}} + e^{-it}\frac{1 - e^{-it(L-1)}}{1 - e^{-it}}\right) dt$$

$$= \int_{\theta-\pi/N}^{\theta+\pi/N} \left(1 + \frac{e^{it}(1 - e^{it(L-1)}) - (1 - e^{-it(L-1)})}{1 - e^{it}}\right) dt$$

$$= \int_{\theta-\pi/N}^{\theta+\pi/N} \left(1 + \frac{e^{it} - e^{itL} - 1 + e^{-it(L-1)}}{1 - e^{it}}\right) dt$$

$$= \int_{\theta-\pi/N}^{\theta+\pi/N} \frac{e^{i(2L-1)t/2} - e^{-i(2L-1)t/2}}{e^{it/2} - e^{-it/2}}\, dt,$$

which implies (2.2). □

LEMMA 2.2. *Suppose* $0<\theta-\pi/N<\theta+\pi/N<2\pi$. *Then,*

$$\frac{dH_{N,L}(\theta)}{d\theta}=\left.\frac{\sin[(2L-1)t/2]}{\sin(t/2)}\right|_{\theta-\pi/N}^{\theta+\pi/N} \tag{2.4}$$

is zero if and only if

$$\sin(L\theta)\sin[(L-1)\pi/N]=\sin(L\pi/N)\sin[(L-1)\theta]. \tag{2.5a}$$

Proof. Differentiate (2.2) to verify (2.4). Then, set the derivative in (2.4) to zero and clear fractions to obtain

$$\begin{aligned}0&=\sin[(2L-1)(\theta+\pi/N)/2]\sin[(\theta-\pi/N)/2]\\&\quad-\sin[(2L-1)(\theta-\pi/N)/2]\sin[(\theta+\pi/N)/2]\\&\equiv\sin(L\theta)\sin[(L-1)\pi/N]\\&\quad-\sin(L\pi/N)\sin[(L-1)\theta],\end{aligned} \tag{2.5b}$$

involving a trigonometric identity which can most easily be verified by writing the sines in (2.5b) in terms of complex exponentials and combining terms on both sides to compare exponents. This completes the proof. □

LEMMA 2.3.

$$\begin{aligned}H_{N,L}(\theta)&=-(2/L)\sin(L\pi/N)\cos(L\theta)\\&\quad+2\int_0^{\pi/N}\frac{\sin(L\theta)\cos(Ls)\sin\theta-\cos(L\theta)\sin(Ls)\sin s}{\cos s-\cos\theta}\,ds\\&=2\int_0^{\pi/N}\frac{\cos(Ls)\cos[(L-1)\theta]-\cos(L\theta)\cos[(L-1)s]}{\cos s-\cos\theta}\,ds.\end{aligned} \tag{2.6}$$

Proof. Use the variable of integration, $s=t-\theta$, to write (2.2) in the form,

$$\begin{aligned}H_{N,L}(\theta)&=\int_{\theta-\pi/N}^{\theta+\pi/N}\frac{\sin(Lt)\cos(t/2)-\cos(Lt)\sin(t/2)}{\sin(t/2)}\,dt\\&=\int_{-\pi/N}^{\pi/N}\frac{\sin[L(\theta+s)]\cos[(\theta+s)/2]}{\sin[(\theta+s)/2]}-\cos[L(\theta+s)]\,ds,\end{aligned} \tag{2.7}$$

wherein the latter integrand can be written in the form,

$$\begin{aligned}\int_{-\pi/N}^{\pi/N}\cos[L(\theta+s)]\,ds&=\left.\frac{1}{L}\sin[L(\theta+s)]\right|_{-\pi/N}^{\pi/N}\\&=L^{-1}\{\sin[L(\theta+\pi/N)]-\sin[L(\theta-\pi/N)]\}\\&=(2/L)\sin(L\pi/N)\cos(L\theta).\end{aligned} \tag{2.8}$$

Change the integration interval from $(-\pi/N, \pi/N)$ to $(0, \pi/N)$ to write (2.7) in the form

$$(2.9)\quad H_{N,L}(\theta)+(2/L)\sin(L\pi/N)\cos(L\theta)$$
$$=\int_0^{\pi/N}\frac{\sin[L(\theta+s)]\cos[(\theta+s)/2]}{\sin[(\theta+s)/2]}+\frac{\sin[L(\theta-s)]\cos[(\theta-s)/2]}{\sin[(\theta-s)/2]}\,ds$$
$$=\int_0^{\pi/N}\left(\frac{\cos[(\theta+s)/2]}{\sin[(\theta+s)/2]}+\frac{\cos[(\theta-s)/2]}{\sin[(\theta-s)/2]}\right)\sin(L\theta)\cos(Ls)$$
$$+\left(\frac{\cos[(\theta+s)/2]}{\sin[(\theta+s)/2]}-\frac{\cos[(\theta-s)/2]}{\sin[(\theta-s)/2]}\right)\cos(L\theta)\sin(Ls)\,ds$$
$$=\int_0^{\pi/N}\frac{\sin(L\theta)\cos(Ls)\sin\theta-\cos(L\theta)\sin(Ls)\sin s}{\sin[(\theta+s)/2]\sin[(\theta-s)/2]}\,ds,$$

which implies the first equality in (2.6). Observe that

$$\sin(L\pi/N)\cos(L\theta)=L\int_0^{\pi/N}\frac{\cos(Ls)\cos(L\theta)[\cos s-\cos\theta]}{\cos s-\cos\theta}\,ds,$$

shows that $H_{N,L}(\theta)$ is given by an integral in which the numerator of the integrand has the form,

$$-\cos(Ls)\cos(L\theta)[\cos s-\cos\theta]+\sin(L\theta)\cos(Ls)\sin\theta-\cos(L\theta)\sin(Ls)\sin s$$
$$=\cos(Ls)[\cos(L\theta)\cos\theta+\sin(L\theta)\sin\theta]-\cos(L\theta)[\cos(Ls)\cos s+\sin(Ls)\sin s],$$

which implies the remaining equality in (2.6). □

THEOREM 2.1. *The zeros of $P_{N,L}$ have unit modulus for $L \leq N$.*

Proof. Set L to N in (2.5) to observe that

$$(2.10)\qquad \frac{dH_{N,N}(\theta)}{d\theta}=0 \quad\text{iff}\quad \sin(N\theta)=0 \quad\text{iff}\quad \theta=k\pi/N.$$

Although $\theta = k\pi/L$ is not a zero of the derivative of $H_{N,L}$, we use it to write the first equality in (2.6) in the form,

$$(2.11)\qquad H_{N,L}(k\pi/L)=(-1)^{k+1}(2/L)\sin(L\pi/N)$$
$$+2(-1)^{k+1}\int_0^{\pi/N}\frac{\sin(Ls)\sin s}{\cos s-\cos(k\pi/L)}\,ds.$$

Since the above integrand is positive on $(0, \pi/N)$ if $L \leq Nk$, the integral in (2.11) is positive. Thus, $H_{N,L}(k\pi/L)$ has $(L-1)$ changes of sign as k counts from 1 to L, $P_{N,L}(z)$ has $(L-1)$ zeros in the upper half of the unit circle, $(L-1)$ conjugate zeros in the lower half of the unit circle, and the proof is complete. □

3. Zeros of $P_{N,L}$ for $L > N$. Several examples are given to show that Theorem 2.1 cannot be extended to cover $L > N$. The most trivial example,

$$(3.1)\qquad P_{N,N+1}=zP_{N,N},$$

has a zero at the origin in addition to the zeros of unit modulus of $P_{N,N}$.

We discuss $P_{N,L}$, for $L>N+1$, in terms of $H_{N,L}$, given in (2.1), and $F_{N,L}$, defined by

$$F_{N,L}(x) \triangleq H_{N,L}(\theta)\Big|_{\cos\theta=x}. \tag{3.2}$$

Observe that each zero of $F_{N,L}$ in the interval, $(-1,1)$, implies two zeros of $H_{N,L}$, which implies two zeros of unit modulus of $P_{N,L}$.

The example,

$$\begin{aligned} H_{2,4}(\theta) &= \pi + 4\sum_{m=1}^{3} \frac{1}{m}\sin\frac{m\pi}{2}\cos m\theta \\ &= \pi + 4\cos\theta - \frac{4}{3}\cos 3\theta \\ &= \pi + 8\cos\theta - \frac{16}{3}\cos^3\theta, \end{aligned} \tag{3.3a}$$

defines

$$F_{2,4}(x) = \pi + 8x - \frac{16}{3}x^3, \qquad x=\cos\theta, \tag{3.3b}$$

which has three real roots, two in $(-1,0)$ and one in $(1,\infty)$, since it is positive at $x=-1$, $x=0$, and $x=1$, but it is negative for $x=-\frac{1}{2}$ and for large positive x. Its two roots in $(-1,0)$ force four roots of unit modulus on the polynomial,

$$\begin{aligned} P_{2,4}(z) &= \sum_{m=0}^{6} z^m \operatorname{sinc}\frac{(m-3)\pi}{2} \\ &= \frac{1}{3\pi}(-2+6z^2+3\pi z^3+6z^4-2z^6). \end{aligned} \tag{3.3c}$$

Since $P_{2,4}(0)<0$ and $P_{2,4}(1)>0$, the two remaining roots consist of one root in $(0,1)$ and its reciprocal in $(1,\infty)$.

The second example,

$$\begin{aligned} H_{2,6}(\theta) &= \pi + 4\sum_{m=1}^{5} \frac{1}{m}\sin\frac{m\pi}{2}\cos m\theta \\ &= \pi + 4\cos\theta - \frac{4}{3}\cos 3\theta + \frac{4}{5}\cos 5\theta \\ &= \pi + 4\left(3\cos\theta - \frac{16}{3}\cos^3\theta + \frac{16}{5}\cos^5\theta\right) \end{aligned} \tag{3.4a}$$

defines

$$F_{2,6}(x) = \pi + 4\left(3x - \frac{16}{3}x^3 + \frac{16}{5}x^5\right), \qquad x=\cos\theta, \tag{3.4b}$$

which has three real roots in $(-1,0)$ and two complex roots, since it has the same sign as x as $|x|\to\infty$, a positive maximum at $-\sqrt{3}/2$, a negative minimum at $-1/2$, a positive maximum at $1/2$, and a positive minimum at $\sqrt{3}/2$. The three roots of $F_{2,6}$ in

$(-1,0)$ force six roots of unit modulus on the polynomial,

$$P_{2,6}(z)=\sum_{m=0}^{10} z^m \operatorname{sinc}\frac{(m-5)\pi}{2} \tag{3.4c}$$
$$=\frac{2}{15\pi}\left(3-5z^2+15z^4+\frac{15\pi}{2}z^5+15z^6-5z^8+3z^{10}\right)$$
$$=\left(1+\frac{2}{15\pi}B(z)\right)z^5,\quad \text{where } B(z)=Q(z^{-1})+Q(z)$$

with

$$Q(z)=3z^5-5z^3+15z. \tag{3.5a}$$

Since

$$\frac{dQ}{dz}=\frac{15(z^6+1)}{z^2+1}\quad\text{and}\quad\frac{dB}{dz}=\frac{15(z^{12}-1)}{z^6(z^2+1)}, \tag{3.5b}$$

it follows that $xB(x)>0$ for nonzero real x,

$$\inf\{|B(x)|: x \text{ real}\}=|B(-1)|=26,$$
$$\sup\{1+2B(x)/(15\pi): x<0\}=1-52/(15\pi)<0,$$

and $P_{2,6}$ has no real roots. Thus, its remaining four roots must form the vertices of a trapezoid in the complex plane.

4. Zeros of $P^*_{N,L}$. We seek real sequences of the form, $\{c_m: 1\leq m<\infty\}$, such that polynomials defined by (1.6) have no roots in $\{z:|z|=1\}$. The search will be based on using the same real sequences in defining the polynomials,

$$Q_{N,L}(z)\triangleq 1+2\sum_{m=1}^{L-1} c_m z^m \operatorname{sinc}\frac{m\pi}{N}, \tag{4.1}$$

which will then be such that

$$\operatorname{Re}[Q_{N,L}(e^{i\theta})]\cos(L-1)\theta=\operatorname{Re}[P^*_{N,L}(e^{i\theta})]. \tag{4.2}$$

DEFINITION 4.1. R denotes the class of functions which are analytic and of positive real part on $\{z:|z|<1\}$.

First, take L to be infinite in (4.1), set $c_m=1$ for all m, and denote the result by

$$w(\zeta)\triangleq 1+2\sum_{m=1}^{\infty}\zeta^m \operatorname{sinc}\frac{m\pi}{N}, \tag{4.3}$$

which will be shown below to lie in R. A classical result will then be used to develop conditions on $\{c_m: 1\leq m<\infty\}$ to imply $Q_{N,L}\in R$. Another classical result will then be used to show that the same conditions imply that $P^*_{N,L}$, as given in (1.6), has no zeros of unit modulus. Alternative criteria for determining whether a given $\{c_m: 1\leq m<\infty\}$ has the desired properties will then be shown.

LEMMA 4.1. *The function, w, maps $\{\zeta:|\zeta|<1\}$ onto the vertical strip bounded by $\{w: \operatorname{Re} w=0\}\cup\{w:\operatorname{Re} w=N\}$. Thus, $w\in R$.*

Proof. Compute

$$w(\zeta)=1+\frac{N}{i\pi}\sum_{m=1}^{\infty}\frac{\zeta^m}{m}\left(e^{im\pi/N}-e^{-im\pi/N}\right) \tag{4.4a}$$

$$=1+\frac{N}{i\pi}\left(\sum_{m=1}^{\infty}\frac{\zeta^m e^{im\pi/N}}{m}-\sum_{m=1}^{\infty}\frac{\zeta^m e^{-im\pi/N}}{m}\right)$$

$$=1+\frac{N}{i\pi}\left(-\ln\left(1-\zeta e^{i\pi/N}\right)+\ln\left(1-\zeta e^{-i\pi/N}\right)\right)$$

$$=1+\frac{N}{i\pi}\ln\frac{1-\zeta e^{-i\pi/N}}{1-\zeta e^{i\pi/N}}.$$

Since w is analytic for $|\zeta|<1$ and $w(0)=1$, it suffices to consider $w(\zeta)$ for $|z|=1$. Compute

$$\operatorname{Re} w(e^{i\Phi})=1+\operatorname{Re}\left(\frac{N}{i\pi}\ln\frac{1-e^{i\Phi-i\pi/N}}{1-e^{i\Phi+i\pi/N}}\right) \tag{4.4b}$$

$$=1+\frac{N}{\pi}\operatorname{Im}\ln\frac{e^{i(\Phi-\pi/N)/2}\left(e^{-i(\Phi-\pi/N)/2}-e^{i(\Phi-\pi/N)/2}\right)}{e^{i(\Phi+\pi/N)/2}\left(e^{-i(\Phi+\pi/N)/2}-e^{i(\Phi+\pi/N)/2}\right)}$$

$$=1+\frac{N}{\pi}\operatorname{Im}\ln\left(e^{-i\pi/N}\frac{\sin[(\Phi-\pi/N)/2]}{\sin[(\Phi+\pi/N)/2]}\right)$$

$$=\begin{cases}1+\dfrac{N}{\pi}\left(-\dfrac{\pi}{N}+\pi\right)\\[2ex] 1+\dfrac{N}{\pi}\left(-\dfrac{\pi}{N}\right)\end{cases}=\begin{cases}N, & \Phi\in(\pi/N,\ 2\pi-\pi/N),\\ 0, & \Phi\notin(\pi/N,\ 2\pi-\pi/N)\end{cases}. \qquad \square$$

LEMMA 4.2. *Let L be an integer exceeding unity. Let r_L denote the unique positive root of*

$$2r^L+r-1=0. \tag{4.5}$$

Then, $0<r_L<1$ and $r_L<r_{L+1}$. Also, $r_L>1-(2/L)\ln L$, and $r_L\to 1$ as $L\to\infty$. Adopt the definitions,

$$f(\zeta)\stackrel{\Delta}{=}\sum_{n=0}^{\infty}a_n\zeta^n \quad \text{and} \quad s_L(\zeta)\stackrel{\Delta}{=}\sum_{n=0}^{L-1}a_n\zeta^n. \tag{4.6}$$

If $f\in R$, then $\operatorname{Re}[s_L(\zeta)]>0$ on $\{\zeta:|\zeta|<r_L\}$. Moreover, the example using $a_0=1$ and $a_n=2$ for $n>0$ shows that r_L cannot be increased in the conclusion.

Proof. [2, p. 523]. □

LEMMA 4.3. *Let $L>1$, $r\in(0,r_L)$ with r_L defined by* (4.5), *and f as denoted in* (4.6). *If $f\in R$, then*

$$\operatorname{Re}\left(\sum_{m=0}^{L-1}a_m r^m z^m\right)>0 \quad \text{on } \{z:|z|\le 1\}. \tag{4.7}$$

Proof. Set $\zeta=rz$ in partial sums of f as denoted in (4.6), and note that $|z|\le 1$ if and only if $|\zeta|\le r<r_L$. Then, Lemma 4.1 implies (4.7). □

LEMMA 4.4. *Let* $\{b_n : 0 \leqq n < \infty\} \in l_1$ *such that* $b_0 \neq 0$, *and let* f *be denoted as in* (4.6). *Then,*

$$\operatorname{Re}\left(\frac{1}{b_0}\sum_{n=0}^{\infty} b_n a_n\right) \geqq 0 \tag{4.8}$$

for all $f \in R$ *if and only if*

$$\operatorname{Re}\left(\frac{1}{b_0}\sum_{n=0}^{\infty} b_n z^n\right) \geqq \frac{1}{2} \quad on\ \{z : |z| = 1\}. \tag{4.9}$$

Proof. [2, pp. 517–518]. □

LEMMA 4.5. *Lemma* 4.4 *remains valid with* (4.8) *replaced by*

$$\operatorname{Re}\left(\frac{1}{b_0}\sum_{n=0}^{\infty} b_n a_n z^n\right) \geqq 0 \quad for\ |z| < 1. \tag{4.10}$$

Proof. Clearly, validity of (4.10) for all $f \in R$ implies validity of (4.8) for all $f \in R$. It remains to show that Lemma 4.4 implies validity of (4.10) for all $f \in R$. Fix ζ with $0 < |\zeta| < 1$ and let $f \in R$. Then,

$$g_\zeta(z) \stackrel{\Delta}{=} \sum_{n=0}^{\infty}\left(\frac{a_n \zeta^n}{|\zeta|^n}\right) z^n \tag{4.11}$$

satisfies $g_\zeta \in R$. Thus, (4.8) can be written in the form,

$$\operatorname{Re}\left(\frac{1}{b_0}\sum_{n=0}^{\infty} \frac{b_n a_n \zeta^n}{|\zeta|^n}\right) \geqq 0 \quad \text{for } |\zeta| \leqq 1, \tag{4.12}$$

wherein setting $|\zeta| = 1$ merely reduces (4.12) to (4.8). Let $z = \zeta/|\zeta|$ to write (4.12) in the form,

$$\operatorname{Re}\left(\frac{1}{b_0}\sum_{n=0}^{\infty} b_n a_n z^n\right) \geqq 0 \quad \text{for } |z| = 1. \tag{4.13}$$

Since $f \in R$ implies $\operatorname{Re}(a_0) \geqq 0$, (4.10) applies for $z = 0$, which combines with (4.13) to imply (4.10). □

THEOREM 4.1. *Suppose* $\{b_n : 0 \leqq n < \infty,\ b_0 = 1\}$ *is a real sequence satisfying* (4.9). *Let*

$$c_m \equiv r^m b_m, \quad where\ 0 \leqq r < r_L \tag{4.14}$$

with r_L *being the positive root of* (4.5). *Then,* $P^*_{N,L}$, *defined by* (1.6), *has no zeros in* $\{z : |z| = 1\}$.

Proof. Lemma 4.1 shows $w \in R$. Apply Lemma 4.5 to w to show that $f \in R$, where

$$f(\zeta) \stackrel{\Delta}{=} 1 + 2\sum_{m=1}^{\infty} b_m \zeta^m \operatorname{sinc}\frac{m\pi}{N}. \tag{4.15}$$

Apply Lemma 4.3 to show that

$$Q_{N,L}(z) \stackrel{\Delta}{=} 1 + 2\sum_{m=1}^{L-1} b_m r^m z^m \operatorname{sinc}\frac{m\pi}{N} \tag{4.16}$$

has positive real part on $\{z:|z|\leqq 1\}$. Since (4.14) shows (4.16) to be the same as (4.1), (4.2) implies the desired result. □

Classical tests to determine whether a given finite sequence, $\{c_m: 0<m<L\}$, can be used to define a window implying the results in Theorem 4.1 are given below.

THEOREM 4.2. *A sequence, $\{b_m: 0\leqq m<L,\ b_0=1\}$, initiates some infinite sequence, $\{b_m: 0\leqq m<\infty,\ b_0=1\}$, such that*

$$\operatorname{Re}\left(1+\sum_{m=1}^{\infty} b_m z^m\right)>\frac{1}{2} \quad \text{for } |z|<1 \tag{4.17}$$

if and only if

$$\begin{vmatrix} 1 & b_1 & \cdot & & & b_{k-1} & b_k \\ \bar{b}_1 & 1 & b_1 & \cdot & & & b_{k-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{b}_{k-1} & & & \cdot & \bar{b}_1 & 1 & b_1 \\ \bar{b}_k & \bar{b}_{k-1} & & & \cdot & \bar{b}_1 & 1 \end{vmatrix} \geqq 0 \tag{4.18}$$

for $0<k<L$ [6]. *Moreover,* (4.17) *is equivalent to the existence of a probability measure, Ψ, on $[0,2\pi]$ such that*

$$b_m=\frac{1}{2\pi}\int_0^{2\pi} e^{im\theta}\, d\Psi(\theta), \qquad 0\leqq m<\infty. \tag{4.19}$$

Proof. [5] for (4.18) and [7] for (4.19). □

THEOREM 4.3. *If the real sequence, $\{b_m: 0\leqq m<L,\ b_0=1\}$, is such that* (4.18) *is satisfied for $0<k<L$, let*

$$c_m=b_m\left(1-\frac{2\log L}{L}\right)^m \tag{4.20}$$

define the coefficients in (1.6). *Then $P^*_{N,L}$ has no zero of unit modulus.*

Proof. The inequality above (4.6) shows that (4.20) defines c_m satisfying the hypotheses of Theorem 4.1. □

5. Windows. The generalized Hamming window [4] is a standard parameterized time window which defines real sequences satisfying the conditions in Theorem 4.2. This window is defined by

$$\begin{aligned} &b_m=\alpha+(1-\alpha)\cos\frac{2\pi m}{K-1}=\alpha+(1-\alpha)\cos\frac{\pi m}{J} \\ &\text{for } K=2J+1 \quad \text{and} \quad -J=-\frac{K-1}{2}\leqq m\leqq\frac{K-1}{2}=J, \end{aligned} \tag{5.1a}$$

or

$$\begin{aligned} &b_m=\alpha+(1-\alpha)\cos\frac{2\pi(2m+1)}{2(K-1)}=\alpha+(1-\alpha)\cos\frac{\pi(2m+1)}{2J-1} \\ &\text{for } K=2J \quad \text{and} \quad -J=-\frac{K}{2}\leqq m\leqq\frac{K}{2}-1=J-1, \end{aligned} \tag{5.1b}$$

wherein α ordinarily lies in $[\frac{1}{2},1)$. The generalized Hamming window is known as the Hamming window if $\alpha=0.54$ and as the Hanning window if α is one-half.

The even case, with $K=2J$, is discarded here for lack of symmetry. Then, the generalized Hamming window becomes

$$b_m=\alpha+\frac{1-\alpha}{2}\left(e^{i\pi m/J}+e^{-i\pi m/J}\right)$$
$$=\alpha+(1-\alpha)\cos\frac{\pi m}{J} \quad \text{for } -J\leqq m\leqq J. \tag{5.2}$$

THEOREM 5.1. *Use* b_m *given by* (5.2) *with* $J=N-1>0$ *and* $0<\alpha<1$. *Then*, $\{b_m:0\leqq m<N-1,\ b_0=1\}$ *initiates the infinite sequence*, $\{b_m:0\leqq m<\infty,\ b_0=1\}$, *satisfying* (4.17) *if* b_m *is defined by* (5.2) *for all* m.

Proof. Compute

$$\sum_{m=0}^{\infty} b_m z^m=\frac{\alpha}{1-z}+\frac{1-\alpha}{2}\left(\frac{1}{1-ze^{i\pi/N}}+\frac{1}{1-ze^{-i\pi/N}}\right). \tag{5.3}$$

Since $(1-z)^{-1}$ maps the unit disc onto $\{w:\mathrm{Re}(w)>1/2\}$, and the bracket in (5.3) is a sum of compositions of $(1-z)^{-1}$ and rotations of the unit disc,

$$\mathrm{Re}\left(\sum_{m=0}^{\infty} b_m z^m\right)>\frac{\alpha}{2}+\frac{1-\alpha}{2}=\frac{1}{2}, \qquad |z|<1. \tag{5.4}$$

□

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