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# Weighted Inverse Hölder Inequalities

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In this paper, we study weighted inverse Hölder inequalities obtained by replacing Lebesgue measure $dt$ first by $t^\alpha\,dt$ and then by $e^{-xt}\,dt$. The first case generalizes an old result of Frank and Pick and, later, Bellman. The second provides a new inverse Hölder inequality for Laplace transforms. © 1990 Academic Press, Inc.

## 1. Introduction and Main Results

Many authors have studied inverse Hölder inequalities for concave functions of a single variable. One of the earliest results is due to Frank and Pick [10], who proved that

$$\int_0^1 u(t)\,v(t)\,dt \geqslant \frac{1}{2}\left(\int_0^1 u^2(t)\,dt\right)^{1/2}\left(\int_0^1 v^2(t)\,dt\right)^{1/2} \tag{1.1}$$

for all functions $u(t)$, $v(t)$ that are nonnegative and concave on the interval $[0, 1]$. Bellman [2], via a general minimization procedure, established the same result and, in collaboration with H. F. Weinberger, obtained the inverse Hölder inequality

$$\int_0^1 u(t)\,v(t)\,dt \geqslant C_{pq}\left(\int_0^1 u^p(t)\,dt\right)^{1/p}\left(\int_0^1 v(t)^q\,dt\right)^{1/q}, \tag{1.2}$$



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valid for the same class of functions, with

$$C_{pq} = \tfrac{1}{6}(p+1)^{1/p}(q+1)^{1/q}, \qquad 1 < p, q < \infty,\ 1/p + 1/q = 1.$$

For additional historical remarks and an elegant generalization of (1.2) to finite products $u_1 u_2 \cdots u_n$ of concave functions, we refer the reader to Borell [5]. For other related results see [1, 4, 6, 7, 8, 11, 14].

In this paper we consider generalizations of (1.1) and (1.2) in which Lebesgue measure $dt$ is replaced by $w(t)\,dt$. The choice $w(t) = t^\alpha$ in (1.1) leads to our first result.

THEOREM 1. *If $u$ and $v$ are nonnegative and concave functions on the interval $[0, 1]$ and $\alpha > -1$, then*

$$\int_0^1 u(t)\,v(t)\,t^\alpha\,dt \geqslant \sqrt{\frac{\alpha+1}{2(\alpha+2)}}\left(\int_0^1 u^2(t)\,t^\alpha\,dt\right)^{1/2}\left(\int_0^1 v^2(t)\,t^\alpha\,dt\right)^{1/2}. \qquad (1.3)$$

*Equality occurs for the choices $u(t) = t$, $v(t) = 1 - t$, $0 \leqslant t \leqslant 1$. Moreover, if $p \geqslant 1$,*

$$\int_0^1 u(t)\,t^\alpha\,dt \geqslant \frac{1}{(\alpha+1)(\alpha+2)\,B(p+1, \alpha+1)^{1/p}}\left(\int_0^1 u^p(t)\,t^\alpha\,dt\right)^{1/p}, \qquad (1.4)$$

*equality occurring for the choice $u(t) = 1 - t$, $0 \leqslant t \leqslant 1$.*

*Here $B$ denotes the beta function*

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt.$$

*In case $\alpha = 0$, (1.4) reduces to Favard's inequality* [9, 5],

$$\int_0^1 u(t)\,dt \geqslant \frac{(1+p)^{1/p}}{2}\left(\int_0^1 u(t)^p\,dt\right).$$

The weight $w(t) = e^{-xt}$ $(0 < x < \infty)$, with integration over $0 \leqslant t < \infty$, yields an inverse Hölder inequality for Laplace transforms.

THEOREM 2. *If $u, v$ are nonnegative and concave on $[0, \infty)$, $1 < p, q < \infty$, and $1/p + 1/q = 1$, then, for $0 < x < \infty$,*

$$\int_0^\infty u(t)\,v(t)\,e^{-xt}\,dt \geqslant C_{pq}\left(\int_0^\infty u^p(t)\,e^{-xt}\,dt\right)^{1/p}\left(\int_0^\infty v^q(t)\,e^{-xt}\,dt\right)^{1/q}, \qquad (1.5)$$

*where*

$$C_{pq} = \mathrm{Min}(\Gamma(p+1)^{-1/p}, \Gamma(q+1)^{-1/q}). \tag{1.6}$$

*Equality holds for $u(t)=1$ and $v(t)=t$ or vice versa.*

In (1.6), $\Gamma$ denotes the gamma function $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,dt$.

Both results follow a procedure suggested by Bellman in [2]. The idea is to represent $u$ and $v$ as integral transforms of the type

$$\int_a^b K(x,t)\, f(t)\, dt,$$

for some appropriately chosen symmetric kernel, and ultimately reduce the proof to a problem of determining the minimum value of the ratios of iterated kernels. These investigations are carried out in Sections 3 and 4.

## 2. Representrations of Concave Functions

The proof of Theorem 1 relies on Green's function

$$K(x,t) = \begin{cases} x(1-t), & 0 \leqslant x \leqslant t \leqslant 1 \\ t(1-x), & 0 \leqslant t \leqslant x \leqslant 1 \end{cases} \tag{2.1}$$

for the operator $L(u)=u''$, with the boundary conditions $u(0)=u(1)=0$. This is a symmetric kernel and as $f$ and $g$ range over the nonnegative functions in $L^1[0,1]$, the functions

$$u(t) = \int_0^1 K(t,x)\, f(x)\, dx, \qquad v(t) = \int_0^1 K(t,x)\, g(x)\, dx \tag{2.2}$$

constitute a dense subset of the nonnegative concave functions on $[0,1]$; hence it suffices to find

$$\mathrm{Min} \int_0^1 u(t)\, v(t)\, t^\alpha\, dt, \tag{2.3}$$

where $u$ and $v$, given by (2.2), satisfy the normalization conditions

$$\int_0^1 u^2(t)\, t^\alpha\, dt = 1, \qquad \int_0^1 v^2(t)\, t^\alpha\, dt = 1. \tag{2.4}$$

Next we substitute $u$ and $v$ from (2.2) into (2.3). Using the notation

$$K_\alpha^{(2)}(x,y) = \int_0^1 K(x,t)\, K(t,x)\, t^\alpha\, dt \tag{2.5}$$

and

$$G_\alpha(x, y) = \frac{K_\alpha^{(2)}(x, y)}{\sqrt{K_\alpha^{(2)}(x, x)}\sqrt{K_\alpha^{(2)}(y, y)}}, \tag{2.6}$$

this substitution leads to the inequality

$$\begin{aligned}\int_0^1 & u(t)\, v(t)\, t^\alpha\, dt \\ &\geqslant \operatorname*{Min}_{0 \leqslant x, y \leqslant 1} [G_\alpha(x, y)] \\ &\quad \times \int_0^1 \int_0^1 \sqrt{K_\alpha^{(2)}(x, x)}\sqrt{K_\alpha^{(2)}(y, y)}\, f(x)\, g(y)\, dx\, dy. \end{aligned} \tag{2.7}$$

Now it follows from (2.2), (2.4), and Schwarz's inequality that

$$\begin{aligned} 1 &= \int_0^1 u^2(t)\, t^\alpha\, dt = \int_0^1 u(t) \left[\int_0^1 K(t, x)\, f(x)\, dx\right] t^\alpha\, dt \\ &= \int_0^1 f(x) \left[\int_0^1 K(t, x)\, u(t)\, t^\alpha\, dt\right] dx \\ &\leqslant \int_0^1 f(x) \left[\int_0^1 K^2(t, x)\, t^\alpha\, dt\right]^{1/2} \left[\int_0^1 u^2(t)\, t^\alpha\, dt\right]^{1/2} dx \\ &= \int_0^1 f(x) \sqrt{K_\alpha^{(2)}(x, x)}\, dx. \end{aligned}$$

Similarly,

$$1 \leqslant \int_0^1 g(y) \sqrt{K_\alpha^{(2)}(y, y)}\, dy.$$

Hence (2.7) becomes

$$\int_0^1 u(t)\, v(t)\, t^\alpha\, dt \geqslant \operatorname*{Min}_{0 \leqslant x, y \leqslant 1} G_\alpha(x, y).$$

This is the function we wish to minimize. We will show that

$$\operatorname*{Min}_{0 \leqslant x, y \leqslant 1} G_\alpha(x, y) = \sqrt{\frac{\alpha + 1}{2(\alpha + 2)}}. \tag{2.8}$$

The proof of (1.4) follows similar lines. Our proof of Theorem 2 begins with the observation that the integrals in (1.5) exist for all positive $x$; hence it suffices to establish the inequality at $x=1$ and make a simple change of variable. Accordingly we can establish the required inequality by finding the infimum of

$$\int_0^\infty u(t)\, v(t)\, e^{-t}\, dt \tag{2.9}$$

over all nonnegative concave functions $u$ and $v$ that satisfy

$$\int_0^\infty u^p(t)\, e^{-t}\, dt = 1, \qquad \int_0^\infty v^q(t)\, e^{-t}\, dt = 1. \tag{2.10}$$

The infimum in (2.9) is unchanged over the smaller class of nonnegative concave functions that vanish at the origin, are polygonal, and eventually constant. All members of the latter class are found among the uniform limits of nonnegative concave functions of the form

$$u(t) = \int_0^\infty k(t, x)\, f(x)\, dx, \qquad v(t) = \int_0^\infty k(t, y)\, g(y)\, dy, \tag{2.11}$$

where $f$ and $g$ are nonnegative functions in $L^1[0, \infty)$ with compact support and $k$ is the symmetric kernel

$$k(t, x) = \begin{cases} x, & 0 \leqslant x \leqslant t < \infty \\ t, & 0 \leqslant t \leqslant x < \infty. \end{cases} \tag{2.12}$$

Substitutions of (2.11) into (2.9), followed by an application of Fubini's theorem, leads to

$$\int_0^\infty u(t)\, v(t)\, e^{-t}\, dt = \int_0^\infty \int_0^\infty k^{(2)}(x, y)\, f(x)\, g(y)\, dx\, dy, \tag{2.13}$$

where

$$k^{(2)}(x, y) = \int_0^\infty k(x, t)\, k(t, y)\, e^{-t}\, dt \tag{2.14}$$

for $0 \leqslant x, y < \infty$. Using the same procedure as before, except that now Hölder's inequality is required, we arrive at

$$\int_0^\infty u(t)\, v(t)\, e^{-t}\, dt \geqslant \inf_{0 \leqslant x, y < \infty} H(x, y), \tag{2.15}$$

where

$$H(x, y) = \frac{k^{(2)}(x, y)}{\left(\int_0^\infty k(x, t)^p\, e^{-t}\, dt\right)^{1/p} \left(\int_0^\infty k(y, t)^q\, e^{-t}\, dt\right)^{1/q}}. \tag{2.16}$$

The remainder of the proof consists of showing that

$$\inf_{0 \leqslant x,\, y < \infty} H(x, y) = \operatorname{Min}\{\Gamma(p+1)^{-1/p}, \Gamma(q+1)^{-1/q}\}. \tag{2.17}$$

The analysis is given in Section 4.

## 3. Proof of Theorem 1

If $0 \leqslant x \leqslant y \leqslant 1$ and $\alpha > -1$, we find from (2.1) and (2.5) that

$$\begin{aligned} K_\alpha^{(2)}(x, y) &= \int_0^1 K(x, t)\, K(t, x)\, t^\alpha\, dt \\ &= x(1-y) \int_0^1 t^{\alpha+2}\, dt + x(1-y) \int_x^y (1-t)\, t^{\alpha+1}\, dt \\ &\quad + xy \int_y^1 (1-t)^2\, t^\alpha\, dt. \end{aligned}$$

The change of variable $\tau = (t-y)/(1-y)$ in the last integral leads to

$$\int_y^1 (1-t)^2\, t^\alpha\, dt = (1-y)^3\, \phi_\alpha(y)$$

where

$$\phi_\alpha(y) = \int_0^1 (1-\tau)^2\, [y + (1-y)\tau]^\alpha\, d\tau. \tag{3.1}$$

Hence

$$\begin{aligned} K_\alpha^{(2)}(x, y) &= (1-x)(1-y)\, \frac{x^{\alpha+3}}{\alpha+3} \\ &\quad + x(1-y) \left[ \frac{y^{\alpha+2} - x^{\alpha+2}}{\alpha+2} - \frac{y^{\alpha+3} - x^{\alpha+3}}{\alpha+3} \right] \\ &\quad + xy(1-y)^3\, \phi_\alpha(y) \end{aligned}$$

and the quotient

$$G_\alpha(x, y) = \frac{K_\alpha^{(2)}(x, y)}{\sqrt{K_\alpha^{(2)}(x, x)}\, \sqrt{K_\alpha^{(2)}(y, y)}}$$

becomes

$$G_\alpha(x, y) = \frac{\begin{pmatrix}(1-x)(1-y)\,x^{\alpha+3}/(\alpha+3) + x(1-y) \\ [(y^{\alpha+2}-x^{\alpha+2})/(\alpha+2) - (y^{\alpha+3}-x^{\alpha+3})/(\alpha+3)] + xy(1-y)^3\,\phi_\alpha(y)\end{pmatrix}}{\begin{pmatrix}\sqrt{(1-x)^2\,x^{\alpha+3}/(\alpha+3) + x^2(1-x)^3\,\phi_\alpha(x)}\cdot \\ \sqrt{(1-y)^2\,y^{\alpha+3}/(\alpha+3) + y^2(1-y)^3\,\phi_\alpha(y)}\end{pmatrix}} \tag{3.2}$$

for $0 \leqslant x \leqslant y \leqslant 1$. Where necessary $G_\alpha(x, y)$ is defined by continuity. Since $G_\alpha$ is a symmetric function it is necessary only to find the minimum over $0 \leqslant x \leqslant y \leqslant 1$. As Bellman observed, the minimum of

$$G_0(x, y) = \frac{-x^2 - y^2 + 2y}{2(1-x)\,y} \qquad (0 \leqslant x \leqslant y \leqslant 1).$$

is $\frac{1}{2}$ at the point $x = 0$, $y = 1$ and at the symmetric point $x = 1$, $y = 0$. It turns out that $G_\alpha(x, y)$ also has its minimum at these two points. It is convenient to write

$$G_\alpha(x, y) = \frac{\begin{pmatrix}(1-x)\,x^{\alpha+2}/(\alpha+3) + (y^{\alpha+2}-x^{\alpha+2})/(\alpha+2) \\ -(y^{\alpha+3}-x^{\alpha+3})/(\alpha+3) + y(1-y)^2\,\phi_2(y)\end{pmatrix}}{\begin{pmatrix}y(1-x)\sqrt{x^{\alpha+1}/(\alpha+3) + (1-x)\,\phi_\alpha(x)} \\ \sqrt{y^{\alpha+1}/(\alpha+3) + (1-y)\,\phi_\alpha(y)}\end{pmatrix}}. \tag{3.3}$$

Clearly,

$$G_\alpha(0, 1) = \frac{1/(\alpha+2)(\alpha+3)}{\sqrt{\phi_\alpha(0)}\,\sqrt{1/(\alpha+3)}}$$

and, from (3.1), $\phi_\alpha(0) = 2/(\alpha+1)(\alpha+2)(\alpha+3)$. Therefore

$$G_\alpha(0, 1) = \sqrt{\frac{\alpha+1}{2(\alpha+2)}}. \tag{3.4}$$

We approach the problem of showing that (3.4) is the minimum of $G_\alpha$ on $0 \leqslant x \leqslant y \leqslant 1$ by analyzing $\partial G_\alpha(x, y)/\partial y$ for $0 \leqslant x < 1$, $x \leqslant y \leqslant 1$ and $\partial G_\alpha(x, 1)/\partial x$ for $0 < x < 1$.

LEMMA 3.1. $(\partial G_\alpha/\partial y)(x, y) \leqslant 0$ *for* $0 \leqslant x < 1$ *and* $x < y \leqslant 1$.

*Proof.* First, fix $x$, $0 \leqslant x < 1$; then define functions $P$ and $Q$ by

$$P(y) = \frac{K_\alpha^{(2)}(x, y)}{x(1-y)} = \frac{(1-x)\, x^{\alpha+2}}{\alpha+3} + \frac{y^{\alpha+2} - x^{\alpha+2}}{\alpha+2} - \frac{y^{\alpha+3} - x^{\alpha+3}}{\alpha+3} + y(1-y)^2 \phi_\alpha(y) \tag{3.5}$$

and

$$Q(y) = \frac{K_\alpha^{(2)}(y, y)}{(1-y)^2} = \frac{y^{\alpha+3}}{\alpha+3} + y^2(1-y)\, \phi_\alpha(y) \tag{3.6}$$

for $x \leqslant y \leqslant 1$. From (2.6),

$$G_\alpha(x, y) = \frac{x}{\sqrt{K_\alpha^{(2)}(x, x)}} \frac{P(y)}{\sqrt{Q(y)}}.$$

Note that the first factor is $1/\sqrt{\phi_\alpha(0)}$ at $x = 0$. Since the denominator is positive if $0 \leqslant x < 1$ and $x < y \leqslant 1$, we can determine the sign of $\partial G_\alpha(x, y)/\partial y$ from its numerator which, except for positive factors that depend only on $x$, is

$$N(y) = 2Q(y)\, P'(y) - P(y)\, Q'(y). \tag{3.7}$$

At first glance this expression appears quite intractible. Nonetheless, it can be factored usefully. We start with

$$P'(y) = y^{\alpha+1} - y^{\alpha+2} + (1 - 4y + 3y^2)\, \phi_\alpha(y) + y(1-y)^2\, \phi'_\alpha(y). \tag{3.8}$$

In order to eliminate $\phi'_\alpha$ we integrate by parts in (3.1) to obtain

$$\phi'_\alpha(y) = -\frac{y^\alpha}{1-y} + \frac{3}{1-y}\, \phi_\alpha(y). \tag{3.9a}$$

The result is

$$P'(y) = (1-y)\, \phi_\alpha(y). \tag{3.9b}$$

Similarly,

$$Q'(y) = 2y\phi_\alpha(y). \tag{3.10}$$

Substitution of these derivative relations into (3.7) gives

$$N(y) = 2\phi_\alpha(y)[(1-y)\, Q(y) - yP(y)]. \tag{3.11}$$

Finally, substituting from (3.5) and (3.6) for $P(y)$ and $Q(y)$, we arrive at

$$N(y) = \frac{2\phi_\alpha(y)\, y(x^{\alpha+2} - y^{\alpha+2})}{(\alpha+2)(\alpha+3)}. \tag{3.12}$$

It follows that $(\partial G_\alpha/\partial y)(x, y) \leqslant 0$ for $0 \leqslant x < 1$ and $x < y \leqslant 1$. ∎

LEMMA 3.2. *The function $G_\alpha(x, 1)$ is strictly increasing for $0 \leqslant x \leqslant 1$.*

*Proof.* Let $y = 1$ in (3.3). After some simplification,

$$G_\alpha(x, 1) = \frac{1}{\sqrt{2(\alpha+2)}} \frac{1 - x^{\alpha+2}}{\sqrt{x^{\alpha+2}/(\alpha+2) - x^{\alpha+1}/(\alpha+1) + 1/(\alpha+1)(\alpha+2)}}.$$

The numerator of $(\partial G_\alpha/\partial x)(x, 1)$ simplifies to

$$x^\alpha \left( -x^{\alpha+3} + \frac{\alpha+3}{\alpha+1} x^{\alpha+2} - \frac{\alpha+3}{\alpha+1} x + 1 \right)$$

and a derivative analysis shows that the values of the interior expression decrease from 1 to 0 as $x$ increases from 0 to 1. Hence $(\partial G/\partial x)(x, 1) > 0$ for $0 < x < 1$ and the result follows. ∎

The proof of inequality (1.3) is now immediate. From (2.6) it is clear that $G_\alpha(x, x) = 1$ for $0 \leqslant x \leqslant 1$. This is the set where $G_\alpha(x, y)$ takes its global maximum value. According to Lemma 3.1, $G_\alpha(x, y)$ decreases as $y$ increases from the diagonal, where $y = x$, to the upper boundary, where $y = 1$. And from Lemma 3.2, the surface increases with $x$ along the upper boundary. Since $G_\alpha(x, y) = G_\alpha(y, x)$ we conclude that the minimum of $G_\alpha(x, y)$ on the rectangle $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ is given by

$$G_\alpha(0, 1) = \sqrt{\frac{\alpha+1}{2(\alpha+2)}}.$$

Our thanks to George Gasper for pointing out that this proof is valid for $\alpha > -1$, not just for $\alpha = 0, 1, 2....$

To prove inequality (1.4), we again suppose that

$$1 = \int_0^1 u(t)^p\, t^\alpha\, dt,$$

anduse representation (2.2) to conclude that

$$1 \leqslant \int_0^1 \left( \int_0^1 K(t, x)^p\, t^\alpha\, dt \right)^{1/p} f(x)\, dx.$$

Much as before, we find that

$$\int_0^1 u(t)\, t^\alpha\, dt \geqslant \operatorname*{Min}_{0\leqslant x\leqslant 1} \frac{\int_0^1 K(x, t)\, t^\alpha\, dt}{(\int_0^1 K(x, t)^p\, t^\alpha\, dt)^{1/p}}. \tag{3.13}$$

Let $T(x)$ denote this ratio. A calculation based on (2.1) shows that

$$T(x) = \frac{1}{(\alpha+1)(\alpha+2)} \frac{1-x^{\alpha+1}}{[(1-x^p)\, x^{\alpha+1}/(p+\alpha+1) + \int_x^1 (1-t)^p\, t^\alpha\, dt]^{1/p}}. \tag{3.14}$$

Clearly,

$$T(0) = \frac{1}{(\alpha+1)(\alpha+2)} \frac{1}{B(p+1, \alpha+1)^{1/p}} \tag{3.15}$$

and a derivative analysis shows that $T'(x) > 0$ for $0 < x < 1$. Hence $T(0) = \operatorname{Min}_{0\leqslant x\leqslant 1} T(x)$ and (1.4) follows.

## 4. PROOF OF THEOREM 2

Starting with (2.12) and (2.14), a short calculation yields an explicit representation for (2.16):

$$H(x, y) = \begin{cases} \dfrac{2-(x+2)\, e^{-x} - xe^{-y}}{(p \int_0^x t^{p-1} e^{-t}\, dt)^{1/p}\, (q \int_0^y t^{q-1} e^{-t}\, dt)^{1/q}}, & 0 \leqslant x \leqslant y < \infty \\ \dfrac{2-(y+2)\, e^{-y} - ye^{-x}}{(p \int_0^x t^{p-1} e^{-t}\, dt)^{1/p}\, (q \int_0^y t^{q-1} e^{-t}\, dt)^{1/q}}, & 0 \leqslant y \leqslant x < \infty. \end{cases} \tag{4.1}$$

Note that $H(x, y)$ is not symmetric unless $p = q = 2$. By (2.15), the theorem will follow if we can establish that

$$\inf_{0\leqslant x,\, y<\infty} H(x, y) = \operatorname{Min}\{\Gamma(p+1)^{-1/p}, \Gamma(q+1)^{-1/q}\}. \tag{4.2}$$

We intend to do this by showing, first that $H(x, y)$ has no local minima in the quadrant $0 \leqslant x, y < \infty$. Then, by analyzing the functions $H(x, \infty)$ and $H(y, \infty)$, we will conclude that the infimum of $H(x, y)$ is the minimum of $H(0, \infty)$, $H(\infty, \infty)$, $H(\infty, 0)$ and, finally, that the minimum of these three values is

$$\operatorname{Min}\{\Gamma(p+1)^{-1/p}, \Gamma(q+1)^{-1/q}\}.$$

In the following $p$ and $q$ denote conjugate indices

$$1 < p, q < \infty, \qquad p^{-1} + q^{-1} = 1.$$

LEMMA 4.1. *$H(x, y)$ has no local minima in the regions $0 \leqslant x < y < \infty$ and $0 \leqslant y < x < \infty$.*

*Proof.* First consider the region $0 < x < y < \infty$. Except for a fixed positive factor that depends on $x$ only, the numerator $N_1(y)$ of $\partial H(x, y)/\partial y$ is

$$N_1(y) = aq^2 e^{-y} \int_0^y t^{q-1} e^{-t}\, dt - qy^{q-1} e^{-y}(b - ae^{-y}), \tag{4.3}$$

where

$$a = x \qquad \text{and} \qquad b = 2 - (x+2)\, e^{-x}. \tag{4.4}$$

Clearly $N_1(0) = 0$, and since $b > 0$ if $x > 0$ and $q - 1 > 0$, $N_1(y)$ is eventually negative. On the diagonal (4.3) reduces to

$$N_1(x) = qxe^{-x} \left\{ q \int_0^x t^{q-1} e^{-t}\, dt - 2x^{q-2}[1 - (x+1)\, e^{-x}] \right\}. \tag{4.5}$$

If $q = 2$, the last factor is zero; otherwise a derivative analysis shows that

$$\begin{aligned} N_1(x) &< 0 \quad \text{if} \quad q > 2,\ 0 < x < \infty, \\ N_1(x) &= 0 \quad \text{if} \quad q = 2,\ 0 < x < \infty, \\ N_1(x) &> 0 \quad \text{if} \quad 1 < q < 2,\ 0 < x < \infty. \end{aligned} \tag{4.6}$$

Next consider the function $N_2(y)$,

$$N_2(y) = q^{-1} e^y N_1(y) = aq \int_0^y t^{q-1} e^{-t}\, dt - y^{q-1}(b - ae^{-y}). \tag{4.7}$$

For $0 < x \leqslant y < \infty$ this function is strictly decreasing because

$$N_2'(y) = (q-1)\, y^{q-2}[aye^{-y} - (b - ae^{-y})] < 0. \tag{4.8}$$

To see this, use (4.4) to conclude that

$$[aye^{-y} - (b - ae^{-y})]_{y=x} = -\int_0^x t^2 e^{-t}\, dt < 0;$$

then observe that

$$\frac{d}{dy}\left[aye^{-y} - (b - ae^{-y})\right] = -aye^{-y} < 0.$$

Consequently, the function $N_1(y)$ is also strictly decreasing for $0 < x \leqslant y < \infty$. So, we see from (4.6) that for $q \geqslant 2$, $N_1(y) < 0$ if $0 < x \leqslant y < \infty$. We can therefore conclude that $\partial H(x, y)/\partial y < 0$ if $0 < x < y < \infty$ and $2 \leqslant q$.

In case $1 < q < 2$ it follows from (4.8), (4.7), and the last inequality in (4.6) that $N_1(x) > 0$, that $N_1(y)$ decreases if $y > x$, and that $N_1(y)$ is eventually negative. Consequently there exists a $y_0$, $y_0 > x$, such that $N_1(y) > 0$ if $x < y < y_0$ and $N_1(y) < 0$ if $y_0 < y$. Therefore, if $1 < q < 2$, $H(x, y)$ has no local minimum in $0 < x < y < \infty$.

A similar but more elementary analysis shows that

$$H(0^+, y) = \frac{1 - e^{-y}}{(q \int_0^y t^{q-1} e^{-t}\, dt)^{1/q}}$$

strictly decreases from 1 to $\Gamma(q+1)^{-1/q}$ as $y$ increases from 0 to $\infty$.

We have shown that $H(x, y)$ has no local minima in $0 \leqslant x < y < \infty$. An identical analysis, based on $\partial H(x, y)/\partial x$, shows that $H(x, y)$ has no local minima in $0 \leqslant y < x < \infty$. ∎

Lemma 4.1 also rules out the diagonal $y = x$ as the location of a local minimum of $H$. For if $2 \leqslant q < \infty$ and $0 \leqslant x < y < \infty$, $(\partial H/\partial y)(x, y) < 0$; but if $1 < q < 2$, then $2 < p < \infty$ and we examine the sector $0 \leqslant y < x < \infty$ where $\partial H(x, y)/\partial x < 0$. Therefore we conclude that the infimum of $H(x, y)$ is to be found among the infima of the functions

$$H(x, \infty) = \frac{2 - (x+2)\, e^{-x}}{\Gamma(q+1)^{1/q}\, (p \int_0^x t^{p-1} e^{-t}\, dt)^{1/p}}, \qquad 0 \leqslant x < \infty \tag{4.9}$$

and

$$H(\infty, y) = \frac{2 - (y+2)\, e^{-y}}{\Gamma(p+1)^{1/p}\, (q \int_0^x t^{q-1} e^{-t}\, dt)^{1/q}}, \qquad 0 \leqslant y < \infty. \tag{4.10}$$

LEMMA 4.2. (a) *If* $2 \leqslant p < \infty$, *then,*

$$\inf_{0 \leqslant x < \infty} H(x, \infty) = \operatorname{Min}\{\Gamma(q+1)^{-1/q}, 2\Gamma(p+1)^{-1/p}\, \Gamma(q+1)^{-1/q}\}.$$

(b) *If* $2 \leqslant q < \infty$, *then*,

$$\inf_{0 \leqslant y < \infty} H(y, \infty) = \operatorname{Min}\{\Gamma(p+1)^{-1/p}, 2\Gamma(p+1)^{-1/p}\,\Gamma(q+1)^{-1/q}\}.$$

*Proof.* It suffices to establish (a). Note that

$$\lim_{x \to 0^+} H(x, \infty) = \Gamma(q+1)^{-1/q}$$

and

$$\lim_{x \to \infty} H(x, \infty) = 2\Gamma(p+1)^{-1/p}\,\Gamma(q+1)^{-1/q}.$$

The numerator of $H'(x, \infty)$ is the function

$$M_1(x) = p^2(x+1)\,e^{-x} \int_0^x t^{p-1}e^{-t}\,dt - px^{p-1}e^{-x}[2-(x+2)\,e^{-x}]. \quad (4.11)$$

If $p=2$ this expression is positive and $H(x, \infty)$ increases for $0 < x < \infty$. So suppose $2 < p < \infty$. Starting at $M_1(0) = 0$, $M_1(x)$ is positive for small $x$ and, since $p-1 > 1$, eventually negative as $x \to \infty$. Hence $H(x, \infty)$ initially increases and decreases ultimately to its asymptotic value. Thus (a) would follow if we could demonstrate the existence of an $x_0 > 0$ such that $M_1(x) > 0$ if $0 < x < x_0$ and $M_1(x) < 0$ if $x_0 < x$.

To obtain more precise information consider the function

$$M_2(x) = p^{-1}e^x M_1(x). \quad (4.12)$$

Its derivative

$$M_2'(x) = p \int_0^x t^{p-1}e^{-t}\,dt - (p-1)\,x^{p-2} \int_0^x t^2 e^{-t}\,dt$$

is also positive for $x$ near 0 and, since $p-2 > 0$, eventually negative as $x \to \infty$. Further, any positive $x$ where

$$M_2''(x) = e^{-x}x^{p-3}\{-(p-1)x^3 + [p+(p-1)(p-2)]x^2 + 2(p-1)(p-2)(x+1) - 2(p-1)(p-2)e^x\}$$

vanishes must also be a solution of

$$\frac{e^x - x - 1}{x^2} = -\frac{1}{2(p-2)}x + \frac{1}{2}\left[1 + \frac{p}{(p-1)(p-2)}\right].$$

At $x=0$, the function on the left has the value $\frac{1}{2}$ and strictly increases with $x$, whereas the linear function on the right has a value greater than $\frac{1}{2}$ at $x=0$ and strictly decreases with $x$. Hence this last equation has exactly one positive solution $x_2$; from this we infer that $M_2''(x)>0$ if $0<x<x_2$ and $M_2''(x)<0$ if $x_2<x<\infty$. This, in turn, shows that $M_2'(x)$, starting from $M_2'(0)=0$, increases in $0<x<x_2$, decreases in $x_2<x<\infty$, and eventually becomes negative beyond some point $x_1$. Accordingly, $M_2(x)$, also starting from $M_2(0)=0$, increases in $0<x<x_1$, decreases in $x_1<x<\infty$, and, as noted above, eventually becomes negative at some point $x_0$. Since $M_2(x)>0$ in $0<x<x_0$ and $M_2(x)<0$ in $x_0<x<\infty$, the same statement applies to $M_1(x)$ by (4.12). Finally, from the definition of $M_1(x)$, $H(x, \infty)$ increases in $0<x<x_0$ and decreases in $x_0<x<\infty$. ∎

LEMMA 4.3. (a) *If* $1<p<2$, $\inf_{0\leqslant x<\infty} H(x, \infty)=\Gamma(q+1)^{-1/q}$.

(b) *If* $1<q<2$, $\inf_{0\leqslant y<\infty} H(\infty, y)=\Gamma(p+1)^{-1/p}$.

*Proof.* Assertions (a) and (b) are identical. Working with (a) we note, from (4.9), that $\lim_{x\to 0^+} H(x, \infty)=\Gamma(q+1)^{-1/q}$. Hence it is only necessary to show that

$$f(x)=(2-(x+2)\,e^{-x})^p-p\int_0^x t^{p-1}e^{-t}\,dt\geqslant 0 \tag{4.13}$$

for $1<p<2$ and $0\leqslant x<\infty$. Clearly $f(0)=0$. Application of the inequality $e^{-x}\leqslant 1/(1+x)$ in

$$f'(x)=p(2-(x+2)\,e^{-x})^{p-1}\,(x+1)\,e^{-x}-px^{p-1}e^{-x}$$

shows that

$$f'(x)\geqslant px^{p-1}e^{-x}[(x+1)^{2-p}-1]\geqslant 0.$$

This establishes (4.13) and (a) follows. ∎

The thrust of these last two lemmas is that

$$\inf_{0\leqslant x,\,y<\infty} H(x, y)$$

is the minimum of the three numbers

$$\Gamma(p+1)^{-1/p}, \qquad \Gamma(q+1)^{-1/q}, \qquad 2\Gamma(p+1)^{-1/p}\,\Gamma(q+1)^{-1/q}. \tag{4.14}$$

However, it can never happen that

$$\Gamma(p+1)>2^p \qquad \text{and} \qquad \Gamma(q+1)>2^q$$

for conjugate indices. For one of the indices, say $p$, satisfies $1<p\leqslant 2$ and

in this range $\Gamma(p+1)\leqslant 2$ and $2<2^p$. This means that the last number in (4.14) is never the minimum of the three numbers listed; hence

$$\inf_{0\leqslant x,\, y<\infty} H(x, y)=\operatorname{Min}\{\Gamma(p+1)^{-1/p}, \Gamma(q+1)^{-1/q}\}.$$

This completes the proof.

## 5. Remarks

It is worth pointing out that inequality (1.3) of Theorem 1 implies Theorem 2 in the case $p=2$. Since $u(t)$ is concave if and only if $u(1-t)$ is concave, (1.3) may be written in the form

$$\int_0^1 u(t)\, v(t)(1-t)^n\, dt$$
$$\geqslant \sqrt{\frac{n+1}{2(n+2)}}\left(\int_0^1 u^2(t)(1-t)^n\, dt\right)^{1/2}\left(\int_0^1 v^2(t)(1-t)^n\, dt\right)^{1/2}, \tag{5.1}$$

$n$ a positive integer.

Now let $u$ and $v$ be nonnegative and concave in $0\leqslant t<\infty$. Fix $x$, $x>0$, and replace $u$ and $v$ in (5.1) by $u(nt/x)$ and $v(nt/x)$. The charge of variable $nt/x=y$ gives

$$\int_0^{n/x} u(y)\, v(y)\left(1-\frac{xy}{n}\right)^n dy$$
$$\geqslant \sqrt{\frac{n+1}{2(n+2)}}\left(\int_0^{n/x} u^2(y)\left(1-\frac{xy}{n}\right)^n dy\right)^{1/2} \tag{5.12}$$
$$\times\left(\int_0^{n/x} v^2(y)\left(1-\frac{xy}{n}\right)^n dy\right)^{1/2}.$$

Letting $n\to\infty$, we obtain

$$\int_0^\infty u(y)\, v(y)\, e^{-xy}\, dy$$
$$\geqslant \frac{1}{\sqrt{2}}\left(\int_0^\infty u^2(y)\, e^{-xy}\, dy\right)^{1/2}\left(\int_0^\infty v^2(y)\, e^{-xy}\, dy\right)^{1/2}. \tag{5.3}$$

Finally, we observe that the method used in improving Theorems 1 and

2 is clearly limited by computational difficulties; hence the absence of a weighted $L^p$-version,

$$\int_0^1 u(t)\, v(t)\, t^\alpha\, dt \geqslant C_{p,q}^{(\alpha)} \left( \int_0^1 u(t)^p\, t^\alpha\, dt \right)^{1/p} \left( \int_0^1 v(t)^q\, t^\alpha\, dt \right)^{1/q},$$

of the Bellman–Weinberger inequality (1.2) and other rather obvious extensions. However, we have not been able to craft our proofs with other methods that offer the potential of greater elegance. For example, a proof that reveals the very essence of inequality (1.1) can be based on monotone rearrangements [12, Chap. X]. Also, in [13], Karlin and Ziegler illustrate the role of generalized convexity in deducing inequalities of Favard [9], Berwald [3], and Borell [5].

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