JOURNAL OF

TIME SERIES ANALYSIS

A JOURNAL SPONSORED BY THE BERNOULLI SOCIETY FOR MATHEMATICAL STATISTICS AND PROBABILITY

VOLUME 16, No. 5, September 1995
ISSN 0143-9782

BLACKWELL Publishers

The **Journal of Time Series Analysis** is published six times a year in January, March, May, July, September and November by Blackwell Publishers, 108 Cowley Road, Oxford OX4 1JF, UK or 238 Main St., Cambridge, MA 01242, USA.

Editor: M. B. Priestley, Department of Mathematics, University of Manchester Institute of Science and Technology, PO Box 88, Manchester M60 1QD.

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ISSN 0143-9782. Printed on acid-free paper in England by Short Run Press, Exeter

AN APPLICATION OF THE SCHUR–COHN ALGORITHM TO TIME SERIES ANALYSIS

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First version received June 1994

Abstract. Standard least squares analysis of autoregressive moving-average (ARMA) processes with errors-in-variables entails the construction of a new set of parameters which are functions of the original ARMA parameters, and requires that derivatives of these new parameters of order three or less with respect to the ARMA parameters exist and be bounded. The boundedness of these derivatives in turn depends critically on the nonsingularity of a matrix $\boldsymbol{B}$ which is a function of the ARMA parameters via the new parameters in the model. A particular version of the classical Schur–Cohn algorithm enables us to establish this nonsingularity.

Keywords. Autoregressive moving-average models; errors-in-variables; Schur–Cohn algorithm; nonsingular matrix.

1. INTRODUCTION

Standard analysis of autoregressive moving-average (ARMA) models has been discussed in research articles and books during the last 25 years. An exhaustive bibliography on this subject appears in Anderson (1971), Box and Jenkins (1970) and Priestley (1981) among others. The problem of identifying and analyzing ARMA models with errors-in-variables is somewhat more complicated. (See Chanda (1994) for a substantive discussion on the subject.) The standard least squares analysis of time series data conforming to an ARMA model requires the creation of a new set of parameters which are functions of the autoregressive (AR) and moving-average (MA) parameters, and the variances of the errors-in-variables. One also needs the existence of the derivatives of these new parameters with respect to the ARMA and the variance parameters. The details are briefly discussed in Chanda (1994). A rigorous mathematical account of the solution of this problem, however, requires further elucidation.

The present paper deals with such details. What we establish in Section 2 is, basically, the nonsingularity of a matrix using the classical Schur–Cohn algorithm (see Theorem 43,1 in Marden (1966)). This will essentially prove the required existence of derivatives mentioned above.

*The first author was partially supported by Texas Advanced Research Program Grant #003644-125.

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To be more specific let us consider the following errors-in-variables stochastic process $\{X_t;\, t \in \mathbb{Z}\}$ (discussed in Chanda (1994)). Let

$$X_t = U_t + \varepsilon_t$$
$$\phi(B)(U_t - \mu) = \theta(B)\eta_t \tag{1.1}$$

where B is the backward shift operator, $\{U_t;\, t \in \mathbb{Z}\}$ is an ARMA sequence of unobservables, μ is a finite constant, $\{\varepsilon_t;\, t \in \mathbb{Z}\}$ is a sequence of independent and identically distributed (i.i.d.) random variables (r.v.'s) representing errors in measurement in $\{U_t\}$, $\{\eta_t\}$ is a sequence of i.i.d. r.v.'s with $E(\eta_1) = 0$, $E(\eta_1^2) = \sigma_\eta^2$, $E(\varepsilon_1) = 0$, $E(\varepsilon_1^2) = \sigma_\varepsilon^2$ and the two sequences $\{\varepsilon_t\}$ and $\{\eta_t\}$ are mutually independent. We assume, as usual, that the roots of $\phi(\xi) = 1 + \sum_{j=1}^{p} \phi_j \xi^j$ and $\theta(\xi) = 1 + \sum_{j=1}^{q} \theta_j \xi^j$ all lie outside the unit circle. We further assume that $\phi(\xi)$ and $\theta(\xi)$ are relatively prime to each other. By Lemma 2.1 in Chanda (1994) there exists a stationary sequence $\{\pi_t;\, t \in \mathbb{Z}\}$ of uncorrelated r.v.'s such that

$$\beta(B)\pi_t := \zeta_t$$
$$\zeta_t := \theta(B)\eta_t + \phi(B)\varepsilon_t \tag{1.2}$$

where $\beta(\xi) = 1 + \sum_{j=1}^{k} \beta_j \xi^j$ is a polynomial of degree $k = \max(p, q)$ with all its roots lying outside the unit circle. In fact, then

$$\delta\beta(\xi)\beta(\xi^{-1}) = \theta(\xi)\theta(\xi^{-1}) + \lambda\phi(\xi)\phi(\xi^{-1}) \tag{1.3}$$

where $\delta = \sigma_\pi^2/\sigma_\eta^2$, $\lambda = \sigma_\varepsilon^2/\sigma_\eta^2$ and $E(\pi_1^2) = \sigma_\pi^2$. Given $\theta(\xi)$ and $\phi(\xi)$ we select those k roots of $\xi^k\beta(\xi)\beta(\xi^{-1})$ which lie outside the unit circle and *define* $\beta(\xi)$ to have these k roots. Alternatively we compute β_j, $1 \leqslant j \leqslant k$, from the relations

$$\frac{\sum_{j=0}^{k-r} \beta_j \beta_{j+r}}{\sum_{j=0}^{k} \beta_j^2} = \frac{\sum_{j=0}^{q-r} \theta_j \theta_{j+r} + \lambda \sum_{r=0}^{p-r} \phi_j \phi_{j+r}}{\sum_{j=0}^{q} \theta_j^2 + \lambda \sum_{j=0}^{p} \phi_j^2} := \rho_r \qquad 1 \leqslant r \leqslant k. \tag{1.4}$$

Let $\boldsymbol{\gamma}^{\mathrm{T}} = [\gamma_1, \ldots, \gamma_a]$ where $a = p + q + 1$ if $p > q$ and $a = p + q$ (only a parameters on the right side of (1.4) are identifiable, λ being assumed known if $p \leqslant q$). Then it is easy to see that if we define

$$\boldsymbol{\Delta}_m^{\mathrm{T}} := \left[\frac{\partial \beta_1}{\partial \gamma_m}, \ldots, \frac{\partial \beta_k}{\partial \gamma_m}\right]$$
$$\boldsymbol{A} := [a_{ij}]_{1 \leqslant i,j \leqslant k} \qquad a_{ij} := \beta_{i+j} + \beta_{i-j} - 2\beta_i\rho_j$$

and

$$\boldsymbol{\delta}_m^{\mathrm{T}} := \sum_{j=0}^{k} \beta_j^2 \left[\frac{\partial \beta_1}{\partial \gamma_m}, \ldots, \frac{\partial \beta_k}{\partial \gamma_m}\right] \qquad 1 \leqslant m \leqslant a_j, \tag{1.5}$$

then relation (1.5) implies that the derivatives $\partial\beta_r/\partial\gamma_m$, $1 \leqslant r \leqslant k$, $1 \leqslant m \leqslant a$, exist if $\boldsymbol{A}$ is nonsingular. In fact, one can see quite clearly that if $\boldsymbol{A}$ is nonsingular then higher order derivatives $\partial\beta_r/\partial\gamma_m$, $\partial^2\beta_r/\partial\gamma_m\partial\gamma_{m'}$ and $\partial^3\beta_r/\partial\gamma_m\partial\gamma_{m'}p\gamma_{m''}$, $1 \leqslant r \leqslant k$, $1 \leqslant m, m', m'' \leqslant a$, also exist. Again if we set

$$\begin{aligned} \boldsymbol{B} &:= [\beta_{i+j} + \beta_{i-j}] \\ \boldsymbol{u}^{\mathrm{T}} &:= [\beta_1, \ldots, \beta_k] \\ \boldsymbol{v}^{\mathrm{T}} &:= -2[\rho_1, \ldots, \rho_k] \end{aligned} \tag{1.6}$$

then $\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ and $\boldsymbol{A}^{-1} = \boldsymbol{B}^{-1} - (\boldsymbol{B}^{-1}\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{B}^{-1}/(1 + \boldsymbol{v}^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{u}))$ if $\boldsymbol{B}^{-1}$ exists. Therefore, we need to prove that $\boldsymbol{B}$ is nonsingular. The nonsingularity of $\boldsymbol{B}$ was shown in Chanda (1994) for $k = 1, 2, 3$ to assure general applicability of the result. We shall now provide a formal proof of this nonsingularity for arbitrary k.

2. NONSINGULARITY OF B

We shall now state the following Lemma 2.1 (see Theorem 6.8b, p. 493, in Henrici (1974)).

LEMMA 2.1. *Let $f(\xi) = \sum_{j=0}^{s} a_j \xi^j$ with a_j, $0 \leqslant j \leqslant s$, real and not all zero. Define $f^*(\xi) := \xi^s f(\xi^{-1})$. Define the iterated Schur transforms $T^r f(\xi) := T\{T^{r-1} f(\xi)\}$, $r = 1, 2, \ldots, s$, with $Tf(\xi) = a_0 f(\xi) - a_s f^*(\xi)$ and set*

$$\gamma_r := T^r f(\xi)|_{\xi=0} \qquad r = 1, 2, \ldots, s. \tag{2.1}$$

Then all s zeros of $f(\xi)$ lie outside the unit circle if and only if

$$\gamma_r > 0 \qquad r = 1, 2, \ldots, s.$$

THEOREM 2.2. *Assume that the k roots of the polynomial $\beta(\xi)$ are all either (i) outside the unit circle or (ii) inside the unit circle. Then $\boldsymbol{B}$ is nonsingular.*

REMARKS. The result of Theorem 2.2 is sharp in the sense that if not all the roots of $\beta(\xi)$ are inside or outside the unit circle then the matrix $\boldsymbol{B}$ may be singular. Consider for example the case where $\beta(\xi) = (1 + c\xi)^2(1 + d\xi)$. Now choose $(c, d) \neq (0, 0)$ such that $1 + c^2 + 2cd(1 - c^2) - d^2c^2(1 - c^2) = 0$. Such a choice is possible if we let $d = [(1 - c^2 \pm \{2(1 + c^4)\}^{1/2}]/c(1 + c^2)$. Then it is easy to see that $|\boldsymbol{B}| = 1 + c^2 + 2cd(1 - c^2) - d^2c^2(1 - c^2) = 0$. For example, if $c = 0.5$ then $d = -1.1324$ or 3.5324.

PROOF. (i) Assume that the k roots of $\beta(\xi)$ all lie outside the unit circle. Set $\boldsymbol{B}_1 = [\beta_{i-j}]$, $\boldsymbol{B}_2 = [\beta_{i+j}]$ and $\boldsymbol{J} = [\delta_{k+1-i-j}]$, $1 \leqslant i, j \leqslant k$, where

$$\delta_r = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{if } r \neq 0 \end{cases}$$

In order to apply the matrix version of the Schur criterion in Marden (1966, pp. 198–201) we define

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2\boldsymbol{J} \\ \boldsymbol{J}\boldsymbol{B}_2 & \boldsymbol{B}_1^{\mathrm{T}} \end{bmatrix}. \tag{2.2}$$

We shall write $\tilde{\beta}(\xi) = \beta(\xi) + \beta_{k+1}\xi^{k+1}$ with $\beta_{k+1} = 0$. This introduces a zero at infinity for the polynomial $\tilde{\beta}(\xi)$. Then

$$\tilde{\beta}^*(\xi) = \sum_{j=0}^{k+1} \beta_j \xi^{k+1-j} \tag{2.3}$$

Note that $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$ and both $\boldsymbol{B}_2$ and $\boldsymbol{J}$ are symmetric matrices.

We now use Theorem 43,1 (Schur–Cohn criterion) on page 198 and Lemma 43,1 on page 199 (with $\delta_r = \gamma_r$) in Marden (1966) and conclude that

$$|\boldsymbol{G}| = \frac{\gamma_k}{\prod_{j=1}^{k-2} \gamma_j^{k-j-1}}.$$

Lemma 2.1 will then indicate that since $\gamma_r > 0$, $r = 1, 2, \ldots, k+1$, $|\boldsymbol{G}| > 0$.

Suppose now we assume that $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$ is singular. Then there exists $\boldsymbol{\alpha} \neq \boldsymbol{0}$ such that $\boldsymbol{B\alpha} = \boldsymbol{0}$. Since $\boldsymbol{J}^2 = \boldsymbol{I}$, this will imply that

$$\boldsymbol{G}\begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{J\alpha} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ (\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{J} + \boldsymbol{JB}_2)\boldsymbol{\alpha} \end{bmatrix}. \tag{2.4}$$

Now note that $\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{J}$ is a symmetric matrix. Therefore, $(\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{J} + \boldsymbol{JB}_2)\boldsymbol{\alpha} = (\boldsymbol{JB}_1 + \boldsymbol{JB}_2)\boldsymbol{\alpha} = \boldsymbol{JB\alpha} = \boldsymbol{0}$. Since $\boldsymbol{\alpha} \neq \boldsymbol{0}$ this will indicate that $\boldsymbol{G}$ is singular which is a contradiction. Result (i) of Theorem 2.1 is thus proved.

(ii) Now assume that the k roots of $\beta(\xi)$ all lie inside the unit circle. In this case, the polynomial $\xi^k\beta(\xi^{-1})/\beta_k = 1 + \sum_{j=1}^k \beta_j^*\xi^j$ (note that $\beta_k \neq 0$) has $\beta_j^* = \beta_{k-j}/\beta_k$, $1 \leq j \leq k$, with $\beta_0^* = 1$, and all its roots lie outside the unit circle. Thus an analogous argument to the above shows that $\boldsymbol{B}^* = [\beta_{i-j}^* + \beta_{i+j}^*]$ is nonsingular. To obtain nonsingularity of the matrix $\boldsymbol{B}$ as claimed we express $\boldsymbol{B}^*$ in terms of the β_j and conclude eventually that

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{J} & 1 \end{bmatrix}$$

and

$$\boldsymbol{B}^* = \beta_k^{-1}\begin{bmatrix} \boldsymbol{JC} & \boldsymbol{0} \\ \boldsymbol{\beta}^{\mathrm{T}} & \beta_k \end{bmatrix}$$

where $\boldsymbol{\beta}^{\mathrm{T}} = [1 \quad \beta_1 \quad \beta_2 \quad \ldots \quad \beta_k]$, $\boldsymbol{C} = [\beta_{i+j} + \beta_{i-j}]$, $1 \leq i, j \leq k-1$ and $\boldsymbol{J}$ is as defined earlier. Since $|\boldsymbol{B}^*| = \beta_k^{-k+1}|\boldsymbol{JC}| = \beta_k^{-k+1}|\boldsymbol{C}|$, $|\boldsymbol{B}| = |\boldsymbol{C}|$ and $\beta_k \neq 0$, part (ii) of Theorem 2.2 follows immediately.

ACKNOWLEDGEMENTS

The authors wish to thank the referee for suggestions which led to the present proof of part (ii) in Theorem 2.1. This proof is somewhat shorter than the one originally derived by the authors. Also we wish to thank R. S. Varga for some helpful comments.

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