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Polynomial Controllers for Linear Systems

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Abstract—In this paper, we study the problem of controlling a linear system with polynomial controls. We establish upper bounds of admissible polynomial controls in two special cases: systems with distinct eigenvalues and systems with all real eigenvalues. For certain other classes we show that the existence of such control of a minimum possible degree is equivalent to questions about the existence of multiple zeros for certain classes of entire functions.

1. INTRODUCTION

In this paper, we consider the problem of controlling an autonomous linear finite-dimensional system of the form

$$\dot{x} = Ax + bu, \tag{1}$$

where $x \in \Re^n$, between two arbitrary points in $\Re^n$. Necessary and sufficient conditions for the existence of a controller and construction of a controller were given by Kalman in the seminal paper [1] and the result is reproduced in almost every elementary control theory text. The standard controller is

$$u(t) = b'e^{-A't} \left(\int_0^T e^{-At} bb' e^{-A't} \, dt \right)^{-1} \left(e^{-AT} x(T) - x(0) \right), \tag{2}$$

and the solution is easily found by finding the point of minimum L^2 norm on the linear variety

$$e^{-AT} x(T) - x(0) = \int_0^T e^{-As} bu(s) \, ds. \tag{3}$$

Unfortunately, this control law is not very practical for most purposes. Since it optimizes a minimum fuel cost criterion, the control activity takes place as close as possible to 0 and the optimal maneuvers tend to be quite dramatic.

This raises the interesting question of just what control laws are contained in the solution set of the linear variety (2). There is a surprisingly small literature on this question. Brockett and Scaramuzzo [2] have studied this question in the context of controlling disk drives and have been

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granted a United States patent based on their results. It is easy to see that by adjusting the Hilbert space norm, a very large number of solutions can be obtained. These have very interesting properties and some of them are considered in papers by Zhang et al. [3,4]. The problem that we consider in this paper is the very simply posed problem: does there exist a polynomial control in the variety (3) and, if a polynomial does exist, what is the minimal degree of such a polynomial? This later problem, despite its simplicity, is surprisingly difficult to solve. In this paper, we give complete answers to the question in the case that the system matrix has distinct eigenvalues and in the case when the eigenvalues of the system matrix are real. We will show that in other cases, the problem of controllability with polynomials is equivalent to the existence of multiple zeros of certain entire functions, and in general the problem is equivalent to certain problems in the theory of moments.

The paper is organized as follows. The second section will give two basic reductions of the problem and we will show that every controllable linear system is controllable with a polynomial of sufficiently high degree. The first reduction is to reduce the problem to the calculation of the rank of a constructed matrix, and the second is to reduce the problem to the analysis of a certain class of entire functions. The first reduction will be used in Section 4 to prove that in the case of distinct eigenvalues the system is controllable with polynomials of degree n, and the second reduction will be used in Section 3 to show that the system is controllable with polynomials of degree $n-1$ in the case that the eigenvalues of the system are real. In the second section, we will show by example that polynomials of degree $n-1$ do not suffice if the eigenvalues are complex. In Section 5, we will show in the case the system matrix has one Jordan block that the question of controllability is equivalent to the existence of multiple zeros for a certain class of entire functions, and we will show that in general the problem is equivalent to a certain question in the theory of moments.

2. BASICS

We begin by considering the linear system of equation (1)

$$\dot{x} = Ax + bu,$$

with initial data $x(0)$. The solution is given by

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}bu(s)\,ds.$$

The problem of controllability is to determine conditions on T, $x(0)$, and $x(T)$ so that there exists a control $u(t)$ such that the equation

$$x(T) = e^{AT}x(0) + \int_0^T e^{A(T-s)}bu(s)\,ds$$

is satisfied. By multiplying both sides of the equation by e^{-AT} the problem is reduced to the form of equation (3). Thus, we see that the system is controllable in time T if and only if the linear operator

$$L(u) = \int_0^T e^{-As}bu(s)\,ds \tag{4}$$

is onto $\Re^n$. In this context, it is natural to specify the domain of L and to seek conditions under which the operator is onto. There are several classical necessary and sufficient conditions for the operator to be onto when the domain is, for example, the set of all entire functions. These conditions are due to Kalman [1].

THEOREM 2.1. *(See [1].) The following are equivalent.*

1. *There exists a control u such that*

$$x(T) = e^{AT}x(0) + \int_0^T e^{A(T-s)}bu(s)\,ds$$

is satisfied for all $x(T)$, $x(0)$, and all $T > 0$.

2. *The rank of the matrix $[b, Ab, A^2b, \ldots, A^{n-1}b]$ is n.*
3. *The matrix*

$$\int_0^T e^{-At}bb'e^{-A't}\,dt$$

is invertible.

The geometric properties of the matrix in item 3 are discussed at length in [5].

We begin by showing that conditions of the preceding theorem are equivalent to the condition that there exists a polynomial control which satisfies condition 1 of the theorem. For the sake of completeness, we include a proof.

PROPOSITION 2.1. *The linear system $\dot{x} = Ax + bu$ is controllable if and only if there exists a polynomial $p(t)$ such that*

$$x(T) = e^{AT}x(0) + \int_0^T e^{A(T-s)}bp(s)\,ds$$

is satisfied.

PROOF. Substituting the control law

$$\begin{aligned} u(t) &= b'e^{-At}\left(\int_0^T e^{-At}bb'e^{-A't}\,dt\right)^{-1}\left(e^{At}x(T) - x(0)\right) \\ &= \sum_{n=0}^{\infty} b'(-A)^n \left(\int_0^T e^{-At}bb'e^{-A't}\,dt\right)^{-1}\left(e^{At}x(T) - x(0)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \alpha_n t^n \end{aligned}$$

into the system equations, we see that

$$e^{-AT}x(T) - x(0) = \sum_{n=0}^{\infty} \alpha_n \int_0^T e^{-As}bs^n\,ds.$$

Since the left-hand side of the equation is arbitrary, we have that the set of vectors

$$\left\{\int_0^T e^{-As}bs^n\,ds : n = 0, 1, \ldots\right\}$$

spans $\Re^n$. Since $\Re^n$ is finite-dimensional, there exists a finite subset that spans, say $n < N$, and hence, there exist τ_is such that

$$\begin{aligned} e^{-AT}x(T) - x(0) &= \sum_{n=0}^{N} \tau_n \int_0^T e^{-As}bs^n\,ds \\ &= \int_0^T e^{-As}b\sum_{n=0}^{N}\tau_n s^n\,ds \\ &= \int_0^T e^{-As}bp(s)\,ds, \end{aligned}$$

where

$$p(s) = \sum_{n=0}^{N} \tau_n s^n.$$

We will see later that there are other ways of proving this fact, but this proof is very simple.

From the proof of the proposition, we obtain the first reduction of the problem which we state in the form of a proposition.

PROPOSITION 2.2. *The system $\dot{x} = Ax + bu$ is controllable with a polynomial control of degree N if and only if the matrix*

$$\left(\int_0^T e^{-As} b\, ds, \int_0^T e^{-As} bs\, ds, \ldots, \int_0^T e^{-As} bs^N\, ds \right)$$

has rank n.

From this we see that $N \geq n - 1$. Now consider the one-dimensional equation

$$\dot{x} = \lambda x + u.$$

The solution of the system at time T is

$$x(T) = e^{\lambda T} x(0) + \int_0^T e^{\lambda(T-s)} u(s)\, ds,$$

and if this system is controllable by a polynomial of degree 0, then we must have

$$e^{-\lambda T} x(T) - x(0) = \alpha \frac{e^{-\lambda T} - 1}{\lambda},$$

where $\alpha = u(t)$. So note that if $\lambda = i2\pi m/T$, then the right-hand side is 0, and hence, there will exist pairs of initial and terminal points which are not controllable. On the other hand, if λ is real, then the function $(e^{\lambda T} - 1)/\lambda$ is nonzero, and hence, by appropriate choice of α, the equation is always satisfied. We will now present an example that shows that even for systems with real coefficients polynomials of degree $n - 1$ may not suffice.

EXAMPLE 2.1. There exists a countable set of times T for which the system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

is not controllable with a polynomial control of degree 1.

DEMONSTRATION. The fundamental solution is

$$\exp At = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

We must show that the operator

$$L(u) = \int_0^T (\exp At) bu(t)\, dt$$

is not onto for controls of the form

$$u(t) = a + bt.$$

This is equivalent, by Proposition 2.2, to showing that the determinant

$$D(T) = \begin{vmatrix} \int_0^T \cos t\, dt & \int_0^T t \sin t\, dt \\ \int_0^T -\sin t\, dt & \int_0^T t\cos t\, dt \end{vmatrix}$$

has countably many zeros as a function of T. Doing the four integrations yields

$$\begin{aligned} \int_0^T \sin t\, dt &= 1 - \cos T, \\ \int_0^T \cos t\, dt &= \sin T, \\ \int_0^T t \sin t\, dt &= -T\cos T + \sin T, \\ \int_0^T t\cos t\, dt &= T \sin T - 1 + \cos T. \end{aligned}$$

Evaluating $D(t)$, we have

$$\begin{aligned} D(T) &= (\sin T)(T \sin T - 1 + \cos T) + (1 - \cos T)(-T\cos T + \sin T) \\ &= T - T\cos T. \end{aligned}$$

Setting this equal to 0, we have

$$\begin{aligned} T &= 0 \qquad \text{and} \\ T &= (2n+1)\frac{\pi}{2}. \end{aligned}$$

Thus, there is a zero in every interval of length π.

We now consider the linear operator L and reduce the problem of controllability to a functional question. We have the following proposition. Let $\mathcal{P}^m$ denote the linear space of polynomials of degree no greater than m.

Proposition 2.3. *The system $\dot{x} = Ax + bu$ is not controllable with a polynomial of degree less than or equal to N if and only if there exists a vector c such that*

$$c'L(u) = \int_0^T c'e^{-At}bu(t)\, dt = 0$$

when restricted to $\mathcal{P}^N$.

Proof. Since L is linear, the image of L with domain $\mathcal{P}^N$ is a linear subspace of $\Re^n$. If the system is not controllable, then L is not onto, and hence, there exists a vector c orthogonal to the image of L, which is just the conclusion of the proposition.

3. REAL EIGENVALUES

In this section, we consider the special case of real eigenvalues. We use a classical result of Pólya and Szegö to establish the result.

Theorem 3.1. *Consider the controllable system*

$$\dot{x} = Ax + bu, \qquad x \in \Re^n.$$

If the matrix A has real eigenvalues, then the system is controllable with a polynomial control of degree $n-1$.

PROOF. We will show that the operator

$$L(u) = \int_0^1 e^{-At} b u(t)\, dt$$

maps $\mathcal{P}^{n-1}$ onto $\Re^n$. Suppose not; then there exists a vector c' such that $c'L = 0$ and

$$c'L(u) = \int_0^1 c' e^{-At} b u(t)\, dt.$$

Since A has real eigenvalues, we can write the kernel

$$c' e^{-At} b = \sum_{i=1}^{k} p_i(t) e^{\lambda_i t},$$

where

$$\sum_{i=1}^{k} \deg p_i(t) + k = n.$$

We use the following fact from Pólya and Szegö [6].

LEMMA 3.1. *(See [6, Vol. II, p. 46, No. 75].) Let the polynomials $P_1(x), P_2(x), \dots, P_l(x)$ be not identically zero and of degree $m_1 - 1, m_2 - 1, \dots m_l - 1$, respectively, and let the real constants $a_1, a_2, \dots, a_l$ be distinct. The function $g(x) = P_1(x)e^{a_1 x} + \dots + P_l(x)e^{a_l x}$ has at most $m_1 + m_2 + \dots + m_l - 1$ real zeros.*

Using this result, $c' e^{-At} b$ has at most $n-1$ real zeros $\alpha_1, \dots, \alpha_{n-1}$. Let

$$u(t) = \prod_{i=1}^{n-1} (t - \alpha_i).$$

Then

$$c' e^{-At} b u(t)$$

does not change sign, and since $c'L = 0$, we must have $c' = 0$, and hence, L must be onto and the system is controllable with polynomial control of degree $n-1$.

4. DISTINCT EIGENVALUES

In this section we will show, using the first reduction, that if the system has distinct eigenvalues, then it is controllable with a polynomial control of degree n. Without loss of generality, we will assume that $T = 1$.

THEOREM 4.1. *Consider the controllable system*

$$\dot{x} = Ax + bu, \qquad x \in \Re^n.$$

If the system has distinct eigenvalues, then it is controllable with a polynomial control of degree no greater than n.

PROOF. Without loss of generality, we may assume that the matrix A is diagonal with eigenvalues $-\lambda_1, \dots, -\lambda_n$ and that the vector b has no entry equal to zero. We construct the $n \times n+1$ matrix H whose i^{th} column is the vector

$$h_i = \int_0^1 e^{-At} b t^i\, dt = \int_0^1 b_1 e^{\lambda_1 t} t^i\, dt, \dots, \int_0^1 b_n e^{\lambda_n t} t^i\, dt.$$

Now let

$$F(z) = \int_0^1 e^{zt}\,dt = \frac{e^z - 1}{z}.$$

Let

$$H = \begin{pmatrix} F(\lambda_1) & D(F)(\lambda_1) & \cdots & D^{n-1}(F)(\lambda_1) & D^n(F)(\lambda_1) \\ F(\lambda_2) & D(F)(\lambda_2) & \cdots & D^{n-1}(F)(\lambda_2) & D^n(F)(\lambda_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F(\lambda_{n-1}) & D(F)(\lambda_{n-1}) & \cdots & D^{n-1}(F)(\lambda_{n-1}) & D^n(F)(\lambda_{n-1}) \\ F(\lambda_n) & D(F)(\lambda_n) & \cdots & D^{n-1}(F)(\lambda_n) & D^n(F)(\lambda_n) \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_1 & \cdots & 0 \\ & \vdots & \\ 0 & \cdots & b_n \end{pmatrix}.$$

It is immediate that the system is controllable with polynomial control of degree n if and only if the rank of BH is n, and hence, if and only if the rank of H is n. Let

$$H(z) = \begin{pmatrix} F(\lambda_1) & D(F)(\lambda_1) & \cdots & D^{n-1}(F)(\lambda_1) & D^n(F)(\lambda_1) \\ F(\lambda_2) & D(F)(\lambda_2) & \cdots & D^{n-1}(F)(\lambda_2) & D^n(F)(\lambda_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F(\lambda_{n-1}) & D(F)(\lambda_{n-1}) & \cdots & D^{n-1}(F)(\lambda_{n-1}) & D^n(F)(\lambda_{n-1}) \\ F(\lambda_n) & D(F)(\lambda_n) & \cdots & D^{n-1}(F)(\lambda_n) & D^n(F)(\lambda_n) \\ F(z) & D(F)(z) & \cdots & D^{n-1}(F)(z) & D^n(F)(z) \end{pmatrix}.$$

We will show that the rank of $H(z)$ is n except at the points $z = \lambda_1$.

We assume that we have shown that for $k = 1$ and for all $k < n$ that the result is true. Now let $c' = (a_1, a_2, \ldots, a_{n-1}, -1)'$ and consider the system $c'H(z) = 0$.

We need the following lemma which is simply proved by induction.

Lemma 4.1. *The function*

$$F(z) = \frac{e^z - 1}{z}$$

satisfies the following differential recursions:

$$zDF(z) = (z-1)F(z) + 1,$$
$$zD^{k+2}F(z) = (z-k-1)D^{k+1}F(z) + kD^kF(z).$$

Now we use the last n columns of $c'H(z) = 0$ and substitute the values for $D^k(F)(z)$ into the last relation of Lemma 4.1. This gives a set of $n-1$ homogeneous equations in the $n-1$ unknown values of the a_is. Now these equations have a nontrivial solution if and only if the determinant of coefficients is 0. However, the coefficients are linear functions of z, and hence, the determinant is either a polynomial of degree $n-1$ in z or it is identically zero. If it is not identically zero, then we know that it is zero precisely when $z = \lambda_i$. Now if the determinant is identically zero, then every value of z leads to a nonzero solution in terms of the a_is, and this contradicts the fact that if the eigenvalues of the state matrix are real then the determinant has full rank.

This theorem then shows that the example of Section 2 is controllable with a polynomial of degree 2, but not with a polynomial of degree 1. It seems to not be feasible to generalize this theorem to the case when there are multiple eigenvalues.

5. ENTIRE FUNCTIONS

In this section, we show that in the case that the state matrix A has a single Jordan block, the problem of polynomial controllability reduces to a problem concerning zeros of a certain class of entire functions.

THEOREM 5.1. *A necessary and sufficient condition that*

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -z & 1 & \cdots & 0 \\ 0 & -z & \cdots & \\ \vdots & & & \\ 0 & \cdots & & 1 \\ 0 & \cdots & 0 & -z \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} u(t), \qquad x \in C^n, \tag{5}$$

is controllable with a polynomial control of degree n is that

$$F(z) = \int_0^T e^{zt} p(z)\, dt,$$

where $p(t)$ is a nonnegative polynomial of degree $n-1$, has no zeros of multiplicity $n+1$.

PROOF. By a change of time scale it suffices to take $T = 1$. This system matrix is $A = N + zI$ where N is nilpotent, and hence,

$$e^{At} = e^{zt} e^{Nt}.$$

The operator L is then of the form

$$L(u) = \int_0^1 e^{zt} e^{Nt} b u(t)\, dt.$$

Let $u \in \mathcal{P}^n$. Suppose L is not onto for some value z_0 of z. Then there exists a vector c such that $c'L = 0$ and also we have that $c'e^{Nt}b = p(t) \in \mathcal{P}^n$. We now have, since $c'L = 0$,

$$c'L(t^k) = \int_0^1 e^{z_0 t} t^k p(t)\, dt = 0, \qquad k = 0, \dots, n.$$

Now $c'L(t^k) = \frac{d^k}{dz^k} c'L(1)$, and hence, z_0 is a zero of multiplicity $n+1$.

Now suppose that there exists a polynomial $p(t)$ of degree $n-1$ such that

$$f(z) = \int_0^1 e^{zt} p(t)\, dt$$

has a zero, z_0 of multiplicity $n+1$. Then we have that

$$\frac{d^k}{dz^k} f(z) = \int_0^1 e^{zt} t^k p(t)\, dt$$

has a zero at z_0 for $k = 0, \dots, n$. Now there exists a vector c such that $p(t) = c'e^{Nt}b$, and hence,

$$\int_0^1 c'e^{Nt} b t^k\, dt = 0, \qquad k = 0, \dots, n.$$

Thus, $c'L(u) = 0$ for $u \in \mathcal{P}^n$. ∎

There is great deal known about the zeros of entire functions, but there is very little literature on the existence or distribution of multiple zeros of entire functions. The problem is, of course, that multiple zeros are unstable with respect to parameter change just as are multiple eigenvalues of the state matrix.

6. MOMENTS

In this section, we simply comment that the problem of polynomial controllability is a special problem from the theory of moments. Consider any function $f(t) = c'e^{At}b$. The moments of this function are the values

$$\int_0^1 f(t)t^k\,dt, \qquad k = 0, 1, \ldots.$$

A question of fundamental interest is to determine when can the function $f(t)$ be recovered from its sequence of moments. There is of course an enormous literature on this problem and its variations. The problem of polynomial controllability can be stated in the following manner.

PROPOSITION 6.1. *The system $\dot{x} = Ax + bu$ is controllable with a polynomial control of degree n if and only if for every function $f(t) = c'e^{At}b$ the first $n+1$ moments being zero implies that the function is identically zero.*

The proof of this proposition is just a restatement of the previous results.

7. CONCLUSIONS

The general question, what is the minimum degree of a polynomial input that can control the state of a linear system, remains open. This is an important question due to the following observations. First, polynomial interpolation is vastly simpler than exponential interpolation. Second, if the system to be controllable is stable, then the state asymptotically approaches polynomial functions provided that the inputs are polynomials. It is evident from Theorem 5.1 that this question is deeper than what appears at first sight, and it is closely related to open questions in the theory of complex variables regarding the existence and distribution of multiple zeros of an entire function.

REFERENCES

1. R. Kalman, On the general theory of control systems, In *Proceedings of the IFAC Congress*, pp. 2020–2030, Butterworths and Company, London, (1961).
2. J.A. Scaramuzzo, Jr. and R. Brockett, Band limited control of flexible structure using piecewise trigonometric input signals, United States Patent, Patent Number: 5,465,035; Date of Patent: Nov. 7, 1995.
3. Z. Zhang, J. Tomlinson and C.F. Martin, Splines and control theory, (submitted).
4. Z. Zhang, J. Tomlinson, C.F. Martin and P. Enqvist, Linear control theory, splines and interpolation, In *Computation and Control IV*, (Edited by J. Lund and K. Bowers), Birkhäuser, (1995).
5. L. Drager, C.F. Martin, R. Foote and J. Wolper, Controllability of linear systems, differential geometry of curves in Grassmanians, and the Riccati equation, *Acta Applicandae Mathematicae* **16**, 281–317, (1998).
6. G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Vol. 1 and 2, Springer-Verlag, New York, (1972).