An application from partial sums of e^z to a problem in several complex variables

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Abstract

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Let $B_r^n = \{z \in \mathbb{C}^n : |z| < r\}$, where $|\cdot|$ is the Euclidean norm, and for $X \subset \mathbb{C}^n$, let $\mathscr{H}X$ denote the closed convex hull of X in $\mathbb{C}^n$. In 1990, Graham showed that if f is a normalized holomorphic map from B_1^n into $\mathbb{C}^n$, and if f is either an open map or a polynomial map, then there is a sharp, uniform constant a, a given by $ae^{1+a} = 1$, such that $\mathscr{H}f(B_1^n) \supset B_a^n$. Graham posed the question to find, for normalized polynomial maps f of degree m, the best constant a_m so that $\mathscr{H}f(B_1^n) \supset B_{a_m}^n$. We answer this question and obtain, for each m, the sharp constant

$$a_m = a + \frac{a \ln m}{2(1+a)m} + \frac{a \ln\{\sqrt{2\pi}\,(1+a)/a\}}{(1+a)m} + o\left(\frac{1}{m}\right), \quad m \to \infty.$$

We also note that this solution extends an old result of Pólya and Szegő.

Keywords: Holomorphic maps in several complex variables; zeros of the partial sums of e^z; the Szegő curve.

In this short note, we show how a numerically motivated result on the zeros of the partial sums of e^z in classical one complex variable function theory can be used to answer a question arising in several complex variables. To put the problem in context, we introduce the following

notations. With $\mathbb{C}^n := \{z = (z_1, z_2, \ldots, z_n): z_k \in \mathbb{C}$ for $k = 1, 2, \ldots, n\}$, let $|z|^2 := \sum_{k=1}^n |z_k|^2$ and $B_r^n := \{z \in \mathbb{C}^n: |z| < r\}$, with $B_r := B_r^1$, $B := B_1^1$ and $B^n := B_1^n$. We consider the classes

$$\mathscr{S} := \{f: f \text{ is analytic and one-to-one on } B, \text{ with } f(0) = 0 \text{ and } f'(0) = 1\},$$

$$\mathscr{S}^* := \{f \in S: f(B) \text{ is starlike with respect to } z = 0\},$$

$$\mathscr{K} := \{f \in \mathscr{S}: f(B) \text{ is convex}\}.$$

Classical results (cf. [6]) obtained at the beginning of this century are as follows.

Distortion theorem:

$$\frac{(1 - |z|)^{p-1}}{(1 + |z|)^{p+1}} \leqslant |f'(z)| \leqslant \frac{(1 + |z|)^{p-1}}{(1 - |z|)^{p+1}}, \quad z \in B; \tag{1}$$

Growth theorem:

$$\frac{|z|}{(1 + |z|)^p} \leqslant |f(z)| \leqslant \frac{|z|}{(1 - |z|)^p}, \quad z \in B; \tag{2}$$

Koebe Covering theorem:

$$f(B) \supset B_{1/2^p}, \tag{3}$$

where $p = 2$, for all f in $\mathscr{S}$ and in $\mathscr{S}^*$. All three theorems are sharp when $p = 2$ for the function, known as the Koebe function, defined by

$$f(z) := \frac{z}{(1 - z)^p}. \tag{4}$$

Corresponding results hold for functions $f \in \mathscr{K}$ with the exponent $p = 1$ in (1)–(4).

In 1933, in the appendix of Montel's book on Univalent Function Theory, Cartan [5] posed the question: Can the Distortion, Growth and Koebe Covering theorems be extended to one-to-one functions f which are biholomorphic on B^n and normalized by $f(0) = 0$ and $Jf(0) = I$, where Jf denotes the Jacobian of f and I denotes the identity matrix? (The analogue of the Distortion theorem for f in (1) would give bounds for the modulus of the determinant of the Jacobian f.) Cartan also explicitly asked if there were extensions to the convex and starlike subclasses of maps from B^n into $\mathbb{C}^n$. In that appendix, he gave several examples of polynomial maps on $\mathbb{C}^n$. Since then, polynomial maps have been studied extensively in the literature of several complex variables.

That the three theorems cannot be directly extended to arbitrary biholomorphic maps from B^n into $\mathbb{C}^n$ has been known for some time (cf. [7,9]). As a simple counterexample, consider the map $F(z_1, z_2) := (z_1, z_2 e^{\alpha z_1})$ for $\alpha \in \mathbb{R}$, so that F is a normalized, one-to-one and biholomorphic mapping from B^2 into $\mathbb{C}^2$. It can be verified that for a suitable choice of α, each of the

Theorem A. *Let f be a normalized biholomorphic map from* B^2 *into* $\mathbb{C}^2$ *with* $f(B^2)$ *a convex domain. Then, there exists a positive constant c such that*

$$\frac{(1-|z|)^{c-3/2}}{(1+|z|)^{c+3/2}} \leqslant |\det Jf| \leqslant \frac{(1+|z|)^{c-3/2}}{(1-|z|)^{c+3/2}}, \quad z \in B^2,$$

where the constant c satisfies

$$\tfrac{3}{2} \leqslant c < 1.71 \ldots . \tag{5}$$

In [2], it was conjectured that $c = \tfrac{3}{2}$. In [1], the next result was established.

Theorem B. *Let f be a normalized biholomorphic map from* B^n *into* $\mathbb{C}^n$ *with* $f(B^n)$ *a starlike domain. Then,*

$$\frac{|z|}{(1+|z|)^2} \leqslant |f(z)| \leqslant \frac{|z|}{(1-|z|)^2}, \quad z \in B^n,$$

and

$$f(B^n) \supset B^n_{1/4},$$

Furthermore, the above inequalities and inclusion are all sharp.

Although many of the classical results in $\mathbb{C}^1$ follow from each other, the situation in $\mathbb{C}^n$ for $n \geqslant 2$ is different, where new techniques of proof must be used, and the results derived appear to be independent of each other.

Theorems A and B were also obtained and improved later in [8,13].

A further result, answering Cartan's question, was obtained by Graham in an interesting paper [9], where he proved the following.

Theorem C. *Let f be a normalized holomorphic map from* B^n *into* $\mathbb{C}^n$ *with either f being an open map or f being a polynomial map. Let a be the unique positive constant satisfying* $ae^{1+a} = 1$, *so that* $a = 0.278\ 46 \ldots$. *If* $\mathscr{H}X$ *denotes the closed convex hull of X, then* $\mathscr{H}f(B^n) \supset B^n_a$.

Note that if $f(B^n)$ is convex, then $f(B^n) \supset B^n_a$.

If we define the degree of a polynomial mapping $\mathbb{C}^n$ into $\mathbb{C}^n$ to be the maximal coordinate degree, then in [9], Graham posed the following question.

Question. If f is a polynomial map of degree m, m a fixed integer, can an $a = a_m$ be found such that

$$\mathscr{H}f(B^n) \supset B^n_{a_m}?$$

A major tool used by Graham is a set of ideas introduced in [14]. It is shown in [14] that for each positive integer m, there exists a positive constant a_m and a set $\{\alpha_j\}_{j=1}^m$, where $\alpha_j \in \mathbb{C}$ for $j = 1, 2, \ldots, m$, such that

(i) $\qquad |\alpha_j| \leq \dfrac{1}{a_m}$,

(ii) $\qquad \dfrac{1}{m} \displaystyle\sum_{j=1}^m \alpha_j = 1,$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (6)

(iii) $\qquad \displaystyle\sum_{j=1}^m \alpha_j^k = 0, \quad k = 2, 3, \ldots, m, \text{ if } m > 1.$

Together, (ii) and (iii) imply that, for any holomorphic f mapping B into $\mathbb{C}^n$, the following representation formula is valid:

$$\frac{1}{m} \sum_{j=1}^m f(\alpha_j z) = f(0) + zf'(0) + O(|z|^{m+1}), \tag{7}$$

for $|z| < a_m$. The idea of the proof of [14] for the existence of $\{\alpha_j\}_{j=1}^m$ was to observe that the α_j's satisfying (ii) and (iii) are the zeros of the monic polynomials p_m, defined by $p_m(z) := z^m P_m(-1/z)$, where $P_m(z) = \sum_{k=1}^m (mz)^k/k!$. If $s_m(z) := \sum_{j=0}^m z^j/j!$ denotes the familiar partial sum of e^z, then $P_m(z) = s_m(mz)$. As shown in [1], if f is a polynomial of degree m, then $\mathscr{H}f(B)$ contains the ball $f(0) + zf'(0)$, $|z| < a_m$. Thus, an answer can be given to Graham's question by determining explicit bounds for the zeros of $P_m(z)$.

In 1924, Szegő [12] defined a simple closed curve

$$D_\infty := \{z \in \mathbb{C}: |z e^{1-z}| = 1 \text{ and } |z| \leq 1\}, \tag{8}$$

lying in the closed disk $\bar{B}$. D_∞ has become known as the *Szegő curve* for the polynomials P_m. If $\{z_{k,m}\}_{k=1}^m$ denotes the zeros of P_m, it is known from the Eneström–Kakeya theorem (cf. [15, Chapter 4]) that $\{z_{k,m}\}_{k=1}^m \subset \bar{B}$ for each $m \geq 1$, and that $\{z_{k,m}\}_{k=1}^m \subset B$ for any $m > 1$. Clearly, the infinite set of all such zeros $\{z_{k,m}\}_{k=1,m=1}^{m,\infty}$ must possess at least one accumulation point in $\bar{B}$. In [12], Szegő showed that each accumulation point must lie on D_∞, and, conversely, that each point of D_∞ is an accumulation point of these zeros. Subsequently, it was shown in [3] that all of these zeros lie *outside* of D_∞. These facts are illustrated in Fig. 1.

Carpenter et al. [4] considered the problem of accurately estimating the zeros of P_m. They introduced the arc defined by

$$D_m := \left\{ z \in \mathbb{C}: |z e^{1-z}|^m = \frac{m! \, e^m}{m^m} \left| \frac{1-z}{z} \right|, \ |z| \leq 1 \text{ and } |\arg z| \geq \cos^{-1}\left(\frac{m-2}{m}\right) \right\}, \tag{9}$$
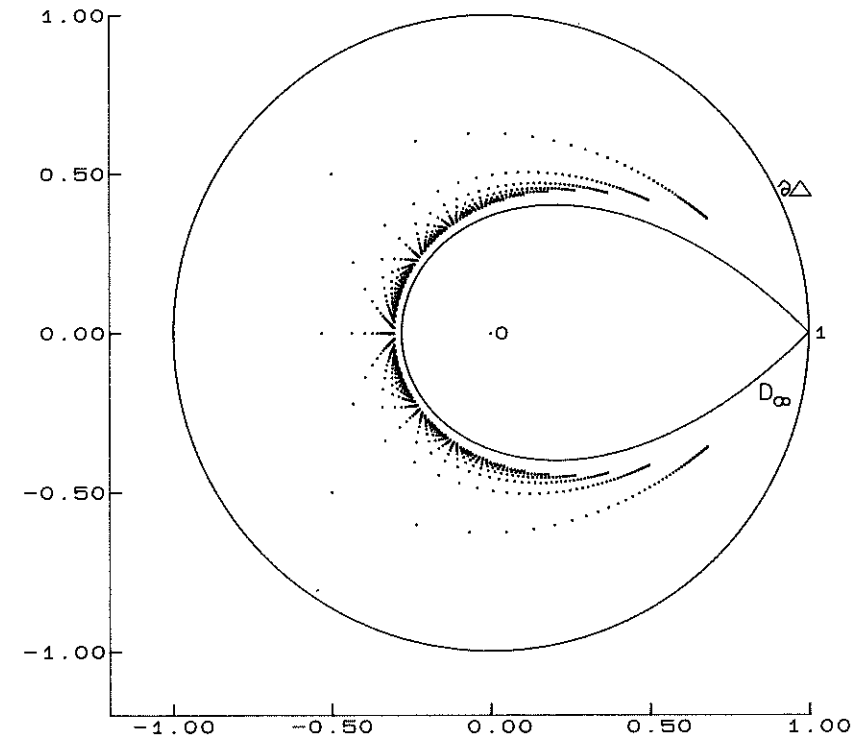
for each $m = 1, 2, \ldots$.

Fig. 1. Zeros of $\{P_n(z)\}_{n=1}^{40}$.

natural analog of Buckholtz's result [3] for D_∞), then an answer to Graham's question would have followed. Indeed, numerical computations did suggest this, noting that in [4] the zeros of P_{27} appeared, up to plotting accuracy, to lie *on* the arc D_{27} (see Fig. 2). However, in an attempt to obtain more precise information about the zeros of P_m relative to D_m, it was discovered that the zeros of P_m do *not* all lie outside D_m for every $m \geqslant 1$. Indeed, it was shown in [16] that there exists a positive integer m_0, such that at least one zero of P_m does not lie outside D_m for every $m > m_0$. A direct calculation indicates the rather surprising outcome that $m_0 = 96$. The size of m_0 necessitated great precision in calculating the zeros. Brent's MP package was used, with 120 significant digits, for these calculations.

Because of this result, it was natural to ask if a *modification* $\hat{D}_m$ of the arc D_m could be found for which all of the zeros of P_m would be *outside* of $\hat{D}_m$, for *each* $m \geqslant 1$. This was done recently in [16] where, for each $m \geqslant 1$, the arc $\hat{D}_m$ was defined by

$$\hat{D}_m := \left\{ z \in \mathbb{C} : |z e^{1-z}|^m = \frac{m! \, e^m}{m^m} \left| \frac{1 - \operatorname{Re} z}{z} \right|, \, |z| \leqslant 1 \text{ and} \right.$$

$$\left. |\arg z| \geqslant \cos^{-1}\left(\frac{m-2}{m} \right) \right\}, \tag{10}$$
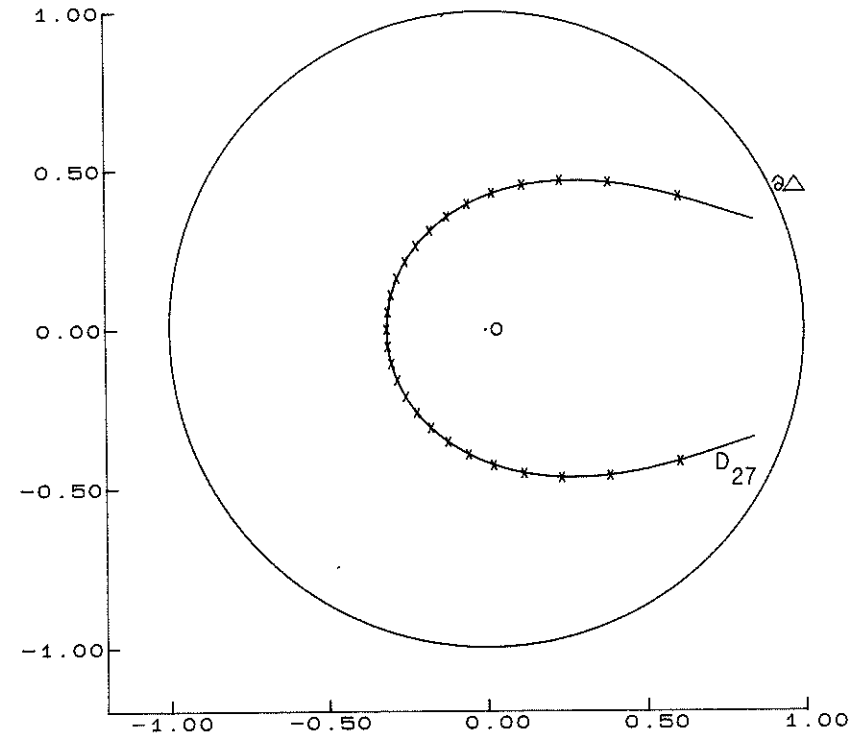
Fig. 2. D_{27} and the zeros of $P_{27}(z)$.

It is easy to verify by differentiation that

$$e^{-z}s_m(z) = 1 - \frac{1}{m!} \int_0^z \zeta^m e^{-\zeta} \, d\zeta, \tag{11}$$

and, on replacing z by mz and ζ by $m\zeta$ and recalling that $s_m(mz) =: P_m(z)$, we have

$$e^{-mz}P_m(z) = 1 - \frac{m^{m+1}}{m!} \int_0^z \zeta^m e^{-m\zeta} \, d\zeta = 1 - \frac{m^{m+1}}{m!} e^{-m}I_m(z), \tag{12}$$

where

$$I_m(z) := \int_0^z \left(\zeta e^{1-\zeta}\right)^m \, d\zeta.$$

Using integration along the radial path $\zeta = \rho e^{i\theta}$, $0 \leqslant \rho \leqslant r$, for the integral $I_m(z)$, we obtain

$$|I_m(z)| \leqslant \int_0^r \left(\rho e^{1-\rho \cos \theta}\right)^m \, d\rho. \tag{13}$$
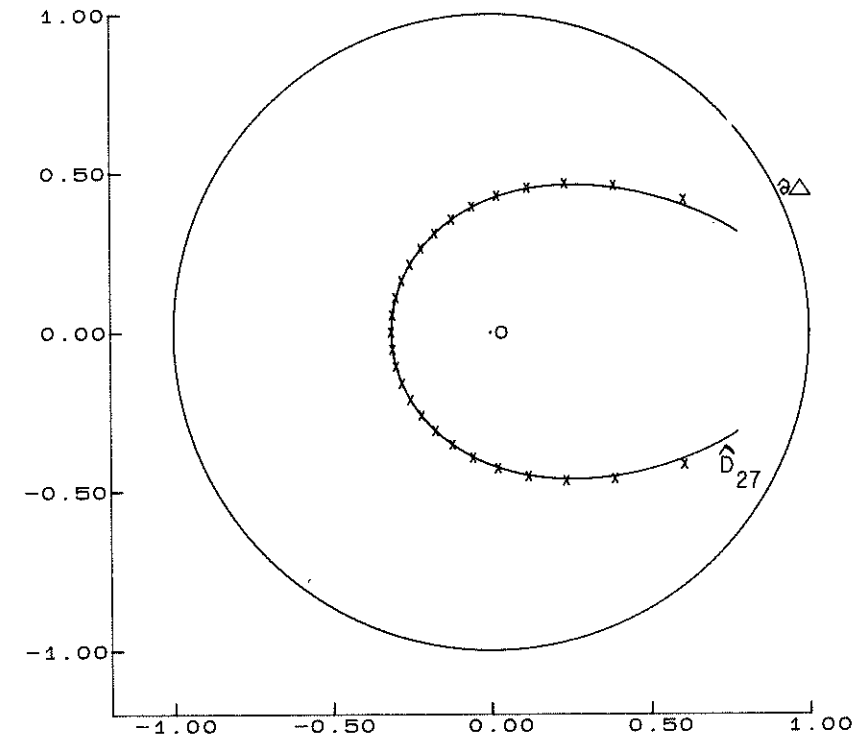
Fig. 3. $\hat{D}_{27}$ and the zeros of $P_{27}(z)$.

with careful use of the sign of cos θ. Since $\mu/(1 + \mu)$ is strictly increasing, it follows that

$$|I_m(z)| < \frac{r(re^{1-r\cos\theta})^m}{m(1 - r\cos\theta)} = \frac{|z||ze^{1-z}|^m}{m(1 - \text{Re } z)}, \quad 0 < r = |z| < 1, \; m = 1, 2, \dots .$$

Thus, if $z_{k,m}$ is a zero of P_m, then

$$\frac{m^{m+1}}{m! \, e^m} I_m(z_{k,m}) = 1$$

from (12), so that (13) gives

$$\frac{m^m |z_{k,m}| |z_{k,m}e^{1-z_{k,m}}|^m}{m! \, e^m (1 - \text{Re } z_{k,m})} > 1.$$

Therefore, all the zeros of P_m lie outside $\hat{D}_m$ for all $m \geqslant 1$, as claimed.

We now determine the point of $\hat{D}_m$ which is closest to the origin, i.e.,

$$r_m := \min\{|z|: z \in \hat{D}_m\}, \quad m = 1, 2, \dots . \tag{14}$$

since, by definition, $\rho(\phi)$ satisfies $\rho(\phi) \leqslant 1$, which implies that $1 - \rho(\phi) \cos \phi \geqslant 0$. On differentiating (15) as a function of ϕ, a straightforward calculation gives that

$$\frac{\mathrm{d}\rho(\phi)}{\mathrm{d}\phi} = -\frac{\rho^2(\phi) \sin \phi [m(1 - \rho(\phi) \cos \phi) - 1]}{m(1 - \rho(\phi) \cos \phi)^2 + 1}. \tag{16}$$

The definition in (10) implies that $\cos \phi \leqslant (m - 2)/m$, showing that the quantity in brackets in (16) is at least unity for any $m \geqslant 1$. Thus, for $z = \rho(\phi)\mathrm{e}^{\mathrm{i}\phi}$ on $\hat{D}_m$ in the open upper half-plane, i.e., for $\cos^{-1}((m - 2)/m) \leqslant \phi < \pi$, the derivative $\mathrm{d}\rho(\phi)/\mathrm{d}\phi$ in (16) is *negative*. This establishes that the point of $\hat{D}_m$, closest to the origin in the closed upper half-plane, is the intersection of the arc $\hat{D}_m$ with the negative real axis, and, as the arc $\hat{D}_m$ is, from (10), clearly symmetric about the real axis, the same is true for all of $\hat{D}_m$.

Returning to Graham's question, an a_m, affirmatively answering Graham's question, is defined by $a_m := r_m$ where

$$\left(r_m \mathrm{e}^{1 + r_m}\right)^m = \frac{m!\, \mathrm{e}^m}{m^m} \left(\frac{1 + r_m}{r_m}\right), \quad m = 1, 2, \ldots . \tag{17}$$

We remark that if a is the unique positive constant satisfying $a\mathrm{e}^{1 + a} = 1$, so that $a = 0.278\,46\ldots$, then it can be shown from (17) that

$$r_m = a + \frac{a \ln m}{2(1 + a)m} + \frac{a \ln\{\sqrt{2\pi}\,(1 + a)/a\}}{(1 + a)m} + \mathrm{o}\left(\frac{1}{m}\right), \quad m \to \infty. \tag{18}$$

We finally show that (18) is *sharp* in the following sense. It is well known that the partial sum $s_n(z)$ of e^z has exactly one negative real zero if n is an odd positive integer, and that it has no real zeros if n is an even positive integer. If $\{z_k(n)\}_{k=1}^n$ denotes the set of zeros of $s_n(z)$ and if we write $z_k(n) := r_k(n)\mathrm{e}^{\mathrm{i}\theta_k(n)}$ where $|z_k(z)| = r_k(n)$ and where $0 < |\theta_k(n)| \leqslant \pi$, then from (11) we have

$$n! = \mathrm{e}^{\mathrm{i}(n+1)\theta_k(n)} \int_0^{r_k(n)} u^n \mathrm{e}^{-u \cos \theta_k(n)} \mathrm{e}^{-\mathrm{i}u \sin \theta_k(n)} \, \mathrm{d}u, \quad n = 1, 2, \ldots . \tag{19}$$

On taking moduli in (19),

$$n! \leqslant \int_0^{r_k(n)} u^n \mathrm{e}^{-u \cos \theta_k(n)} \, \mathrm{d}u, \quad n = 1, 2, \ldots,$$

so that

$$n! < \int_0^{r_k(n)} u^n \mathrm{e}^u \, \mathrm{d}u, \quad \text{for any } \theta_k(n) \text{ with } 0 < |\theta_k(n)| < \pi. \tag{20}$$

For n odd, say $n := 2l + 1$, let $\tilde{r}_{2l+1} > 0$ be such that $s_{2l+1}(-\tilde{r}_{2l+1}) = 0$. From (19) and (20), we have

$$(2l + 1)! = \int^{\tilde{r}_{2l+1}} u^{2l+1} \mathrm{e}^u \, \mathrm{d}u < \int^{r_k(2l+1)} u^{2l+1} \mathrm{e}^u \, \mathrm{d}u.$$

for any zero $z_k(2l + 1)$ with $|\theta_k(2l + 1)| \neq \pi$. This shows that, for *each* odd positive integer m, the unique negative real zero of $P_m(z) = s_m(mz)$ is the *closest* zero of $P_m(z)$ to the origin. In other words, the very *best* choice of a_m, for every $m = 2l + 1$ odd, in Graham's problem is just $\tilde{r}_{2l+1}$. Now, in [11, II. Abschnitt, Exercise 215], it is shown that $\tilde{r}_{2l+1}$ satisfies (18), with m replaced by $2l + 1$. In this sense, the expression in (18), which holds for all integers m, is sharp and extends, to the case of even integers, the result of [11].

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