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Open Problems and Conjectures in Complex Analysis

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Introduction

This article surveys some of the open problems and conjectures in complex analysis that the author has been interested in and worked on over the last several years. They include problems on polynomials, geometric function theory, and special functions with a frequent mixture of the three. The problems that will be discussed and the author's collaborators associated with each problem are as follows:

1. Polynomials with nonnegative coefficients

We first discuss a series of conjectures which have as one of their sources the work of Rigler, Trimble and Varga in [66]. In [66] these authors considered two earlier papers by Beauzamy and Enflo [23] and Beauzamy [22], which are connected with polynomials and the classical Jensen inequality. To describe their results, let

$$p(z)=\sum_{j=0}^{m}a_jz^j=\sum_{j=0}^{\infty}a_jz^j,\quad \text{where}\ \ a_j=0,\ j>m,$$

be a complex polynomial ($\not\equiv 0$), let d be a number in the interval $(0,1)$, and let k be a nonnegative integer. Then (cf [22], [23]) p is said to have concentration d of degree at most k if

$$\sum_{j=0}^{k}|a_j|\geq d\sum_{j=0}^{\infty}|a_j|. \tag{1}$$

Beauzamy and Enflo showed that there exists a constant $\hat{C}_{d,k}$, depending only on d and k, such that for any polynomial p satisfying (1), it is true that

$$\frac{1}{2\pi}\int_0^{2\pi}\log|p(e^{i\theta})|d\theta-\log\left(\sum_{j=0}^{\infty}|a_j|\right)\geq\hat{C}_{d,k}. \tag{2}$$

In the case of $k=0$ in (2) the inequality is equivalent to the Jensen inequality [23],

$$\frac{1}{2\pi}\int_0^{2\pi}\log|p(e^{i\theta})|d\theta\geq\log|a_0|.$$

Rigler, etc., in [66] considered the extension of this inequality from the class of polynomials to the class of H^∞ (cf. Duren [36]) functions. For $f\in H^\infty$ the functional

$$J(f):=\frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{i\theta})|d\theta-\log\left(\sum_{j=0}^{\infty}|a_j|\right)$$

can be well-defined and is finite. They let

$$C_{d,k}=\inf\{J(f):f\in H^\infty\ \text{and}\ f(z)=\sum_{j=0}^{\infty}a_jz^j(\not\equiv 0)\ \text{satisfies (1)}\}. \tag{3}$$

For a (fixed) $d\in(0,1)$ and a (fixed) nonnegative integer k, it was shown that there exists an unique positive integer n (dependent on d and k) such that

$$\frac{1}{2^n}\sum_{j=0}^{k}\binom{n}{j}\leq d<\frac{1}{2^{n-1}}\sum_{j=0}^{k}\binom{n-1}{j}.$$

For this n, set

$$\rho = \frac{\binom{n-1}{k}}{\sum_{j=0}^{k} \binom{n-1}{j} - d2^{n-1}} - 1.$$

With these definitions the following conjecture was made in [66].

Conjecture 1. *Let $C_{d,k}$ be defined by (3). Then*

$$C_{d,k} = \log\left(\frac{\rho}{(\rho+1)2^{n-1}}\right). \tag{4}$$

In [66] Conjecture 1 was verified for $k = 0$ and for the subclass of Hurwitz polynomials, i.e., those polynomials with real coefficients and having all their zeros in the left half-plane. In order to verify the conjecture for the entire class an interim step was suggested. This step was one of the motivations for the following problem which was solved recently by this author and others in [10]. Let p be a real polynomial with nonnegative coefficients. Can a conjugate pair of zeros be factored from p so that the resulting polynomial still has nonnegative coefficients? We gave an answer to one proposed choice for factoring out a pair of zeros. Fairly straightforward arguments show that if the degree of the polynomial is less than 6 then a conjugate pair of zeros of *greatest real part* can be factored out and the resulting polynomial will still have non-negative coefficients. However, the example

$$p(z) = 140 + 20z + z^2 + 1000z^3 + 950z^4 + 5z^5 + 20z^6$$

shows that the statement is not true for arbitrary polynomials with non-negative coefficients. A large amount of computer data had suggested the following:

Conjecture 2. *The nonnegativeness of the coefficients of a real polynomial is preserved upon factoring out a conjugate pair of zeros of smallest positive argument in absolute value.*

Interestingly this last conjecture also arose quite independently in the work of Brian Conrey in analytic number theory in his work on one of Polya's conjectures. Conrey announced Conjecture 2 at the annual West Coast Number Theory Conference in December 1987. The conjecture was communicated to this author by the number theorist Ron Evans. Indeed Evans, using a large amount of computer evidence, has generated a closely related conjecture which we include.

Conjecture 3. *If a polynomial of degree $2n$ has zeros*

$$e^{i(t+a_k)} \text{ and } e^{-i(t+a_k)}, \quad k = 1, 2, \cdots n,$$

where the a_k lie between 0 and π, then all the coefficients are nondecreasing functions of t for small $t > 0$ provided the coefficients are all nonnegative for $t = 0$.

A special case of Conjecture 3 where the zeros on the upper semicircle are equally spaced would be of special interest. Although Conjecture 2 was verified in [10] the techniques do not appear applicable to Conjecture 3.

2. The center divided difference of polynomials

Another series of polynomial problems was generated in classical number theory by the work of Evans and Stolarsky in [37]. Given a polynomial p and a real number λ define $\delta_\lambda(p)$, the center divided difference of p, by

$$\delta_\lambda(p) = \begin{cases} \dfrac{p(x+\lambda)-p(z-\lambda)}{2\lambda}, & \lambda \neq 0, \\ p'(z), & \lambda = 0. \end{cases}$$

We did a study of the behavior of the $\delta_\lambda(p)$ as a function of λ in [11]. A number of classical results of Walsh and Obrechkoff and of Kuipers [50] give some information about the zeros of $\delta_\lambda(p)$ as a function of λ. Let $W[p]$ equal the width of the smallest vertical strip containing the zeros of p. It follows from the classical work that

$$W[\delta_\lambda(p)] \leq W[p]$$

and that the diameter of the zero set of $\delta_\lambda(p)$ approaches ∞ as $|\lambda|$ approaches ∞. The Gauss-Lucas theorem shows that

$$W[p'] \leq W[p].$$

It was shown in [11] that

$$W[\delta_\lambda(p)] \leq W[p'] \tag{5}$$

and the conditions on p when equality holds in (5) are given. We were also able to prove that

$$W[\delta_\lambda(p)] = O(1/\lambda) \quad \text{as } |\lambda| \to \infty.$$

The numerical work done by the number theorists had suggested,

Conjecture 4. *$W[\delta_\lambda(p)]$ monotonically decreases to zero as $|\lambda| \to \infty$.*

In that direction it was shown in [11] that

$$W[\delta_{2\lambda}(p)] \leq W[\delta_\lambda(p)] \tag{6}$$

for all positive λ and conditions for equality in (6) were found. In addition, if the zero set of p is symmetric about a vertical line then

$$W[\delta_\lambda(p)] = 0 \text{ for all } \lambda \geq W[p']. \tag{7}$$

However, an example was given of a polynomial p_ε, that contradicts Conjecture 4 at least for some λ. The polynomial p_ε has its zero set symmetric about the imaginary axis and has the property that for small ε, $W[\delta_1(p_\varepsilon)] = 0$ and $W[\delta_\lambda(P_\varepsilon)] = 0$ for $\lambda \geq \sqrt{1+2\varepsilon} = W[p_\varepsilon']$ while $W[\delta_\lambda(p_\varepsilon)] > 0$ for

$$1 < \lambda < \sqrt{1+2\varepsilon}.$$

Thus conjecture 4 needs to be modified to read

Conjecture 5. *$W[\delta_\lambda(p)]$ monotonically decreases to zero for $\lambda > W[p']$.*

The original question that motivated the number theorist's interest in this problem was the determination of the zeros of $\delta_\lambda(p_N)$ where

$$p_N(z) = \prod_{k=-N}^{N} (z - k).$$

Also occuring in their work were the iterates, $\delta^{(n)}$ of δ defined inductively by

$$\delta_\lambda^{(n)}(p_N) = \delta_\lambda[\delta_\lambda^{(n-1)}(p_N)]$$

with

$$\delta_\lambda^{(1)}(p_N) = \delta_\lambda(p_N).$$

The numerical work had suggested

Conjecture 6. *All nonreal zeros of $\delta_\lambda^{(n)}(p_N)$ are purely imaginary for all λ and all n.*

Conjecture 6 has been verified in [11] for $n = 1$. Indeed, an interesting problem, with other ramifications in number theory, see Stolarsky [71], would be to characterize those polynomials for which $\delta_\lambda^{(n)}$ has only real and pure imaginary roots.

3. Digital filters and zeros of interpolating polynomials

Some interesting problems arise when classical complex analysis techniques are applied to digital filter theory.

Polynomials to be used in interpolation of digital signals are called interpolating polynomials. These polynomials may require modification to assure convergence of their reciprocals on the unit circle. Such modifications provide the opportunity to apply classical analysis theory as was done by the author, Ford, and Wang in [12].

A real function, g, defined for all values of the real independent variable time, t, is called a signal. A digital signal, γ, is a real sequence, $\{\gamma_m : -\infty < m < \infty\}$, consisting of equally spaced values or samples, $\gamma_m = g(m\Delta t)$, from the signal, g, with a time increment or sample interval, Δt. Thus, the independent variable for digital signals such as γ is sample time, $m\Delta t$, or simply sample number, m.

The signal, g, is studied in terms of its classical Fourier transform, G, as a function of real frequency, ω. The digital analog of the Fourier transform consists of the study of a sequence such as γ in terms of its Z-transform, which is defined to be the power series, Γ, having γ_m as the coefficient of z^m. Frequency's digital analog comes from evaluation of Z-transforms such as Γ on the unit circle with the negative of the θ in $z = e^{i\theta}$ referred to as frequency. If the coefficients in Γ are used without any actual evaluation of $\Gamma(z)$ or g is used without computation of G, such use is said to be in the time domain. But if $\Gamma(z)$ is used with evaluation for some z of unit modulus or G is used, such use is said to be in the frequency domain.

Signals are based on even functions in a number of applications. This restricts digital signals to self-inversive cases meaning that $\Gamma(z) = \Gamma(z^{-1})$ for $z \neq 0$. Equivalently, γ is a symmetric sequence meaning that $\gamma_m = \gamma_{-m}$ for all m.

A second signal, f, with Fourier transform, F, poses as a filter of the signal, g, if the convolution integral, $g * f$, of g and f is considered. Of course, the Fourier transform of $g * f$ is the product of the Fourier transforms, G of g and F of f. The discrete analogy consists of the product of Z-transforms, Γ and Φ, where the latter refers to the power series with the sample, $\Phi_m = f(m\Delta t)$, taken from the filter, f, as the coefficient of z^m.

Reduction of certain frequencies is a fundamental aim in the application of a filter, f, to a function, g. This can involve the definition of f by the requirement that $F(\omega)$ be a constant, c, for $|\omega| < \omega_0$ but zero otherwise. If so, c can be chosen so that

$$f(t) = \operatorname{sinc} \omega_0 t, \tag{8}$$

where sinc is defined by

$$\operatorname{sinc} x = \frac{\sin x}{x}. \tag{9}$$

These equations illustrate that the definition of a real signal is determined from the specifications of its Fourier transform. Similarly, digital signals are often defined by the specification of Z-transforms.

The Fourier transform, F, of the f in (8) is referred to as a frequency window since it has compact support in frequency. Application of such a window to a signal, g, is known as a frequency windowing. These problems concern discrete time windowing. This consists of the scaled truncation of an infinite sequence such as γ to obtain a finite sequence of the form $\{c_m\gamma_m : -L < m < L\}$ wherein the finite sequence, $\{c_m : -L < m < L\}$, is referred to as a time window.

Suppose a given digital signal, $\{b_k : -\infty < k < \infty\}$, is such that b_k is understood to correspond to the time, $kN\Delta t$, with the sample interval, $N\Delta t$, where N is a natural number such that $N > 1$. If this digital signal is to be compared with digital signals based on the smaller sample interval, Δt, the given digital signal must be interpolated to the smaller sample interval, Δt. For example, insertion of $N-1$ zeros between every b_k and b_{k+1}, followed by multiplication of the Z-transform of the result by the interpolating series, P_N, defined by

$$P_N(z) = 1 + \sum_{m=1}^{\infty} (z^m + z^{-m}) \operatorname{sinc} \frac{m\pi}{N}, \tag{10}$$

leads to

$$A(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \left(\sum_{j=-\infty}^{\infty} b_j z^{jN} \right) P_N(z). \tag{11}$$

Since the coefficient of z^{kN}, a_{kN}, in A comes from products of b_j and $\operatorname{sinc}(m\pi/N)$ such that $kN = jN \pm m$, it follows that $m \equiv 0 \ (\mathrm{mod} N)$, $\operatorname{sinc}(m\pi/N)$=0 for nonzero m, and $a_{kN} = b_k$. Thus, A is an interpolation of the given B with coefficients, b_j.

A major goal is to study possible alternatives to the interpolation used in (10) in terms of truncation of the interpolating series in (11). In practice one truncates P to obtain the interpolating polynomial, $P_{N,L}$ defined by

$$P_{N,L}(z) = z^{L-1}\left(1 + \sum_{m=1}^{L-1}(z^m + z^{-m}) \operatorname{sinc} \frac{m\pi}{N}\right), \tag{12}$$

where $N > 1$.

To assure stability and accuracy of evaluation it is important that alternative P's have *no* zeros on the unit circle. It is shown in [12] that all of the zeros of $P_{N,L}$ are of unit modulus when $L \leq N$ and examples are given showing that when $L > N + 1$ almost any combination of zeros inside, on, and outside the unit circle can occur. A number of classical results are then combined to give sharp conditions on real sequences $\{c_m : 1 \leq m \leq \infty\}$ so that the function $P^*_{N,L}$ defined by

$$P^*_{N,L}(z) = z^{L-1}\left[1 + \sum_{m=1}^{L-1}(z^m + z^{-m})c_m \operatorname{sinc} \frac{m\pi}{N}\right] \tag{13}$$

has no zero of unit modulus. In particular, in order to define a useful test to determine if a specific sequence of numbers will work for the c_m's in (13) the following theorem was proved in [12].

Theorem 1. *If a real sequence,* $\{b_m : 0 \leq m < L, b_0 = 1\}$ *is such that*

$$\begin{vmatrix} 1 & b_1 & \cdots & & b_{k-1} & b_k \\ b_1 & 1 & b_1 & \cdots & & b_{k-1} \\ \vdots & & & & & \\ b_{k-1} & \cdots & & b_1 & 1 & b_1 \\ b_k & b_{k-1} & \cdots & & b_1 & 1 \end{vmatrix} \geq 0$$

for $0 < k < L$, *let*

$$c_m = b_m\left(1 - \frac{2\log L}{L}\right)^m$$

define the coefficients in (13). Then $P^*_{N,L}$ *has no zero of unit modulus.*

A number of the standard "windows" that occur in the engineering literature are then shown to be just special cases of those defined in Theorem 1, including the very generalized Hamming window and the Hanning window. (see Rabniner and Gold's book, *Theory and Application of Digital Signal Processing.*)

The distribution of zeros and the orthogonality property of the sinc functions determine the interpolating properties in (11) and enables the classical results to be applied. Thus one can ask, can the sinc functions be replaced by more general orthogonal functions, e.g., Jacobi polynomials, to create a more general setting in which many more applications can be found? Discussions with several engineers have suggested this.

4. Omitted values problems

We now discuss a number of open problems in geometric function theory. Let

$$\Delta_r = \{z : |z| < r\}, \text{ with } \Delta_1 = \Delta.$$

Let S denote the class of univalent functions f in Δ normalized by $f(0) = 0$ and $f'(0) = 1$. The problem of omitted values was first posed by Goodman [38] in 1949, restated by MacGregor [57] in his survey article in 1972, then reposed in a more general setting by Brannan [5] in 1977. It also appears in Bernardi's survey article [24] and has appeared in several open problem sets since then including [27],[40] and [60].

For a function f in S, let $A(f)$ denote the Lebesgue measure of the set $\Delta\backslash f(\Delta)$ and let $L(f,r)$ denote the Lebesgue measure of the set $\{\Delta\backslash f(\Delta)\}\bigcap\{w : |w| = r\}$ for some fixed $r, 0 < r < 1$. Two explicit problems posed by Goodman and by Brannan were to determine

$$A = \sup_{f \epsilon S} A(f), \tag{14}$$

and

$$L(r) = \sup_{f \epsilon S} L(f,r). \tag{15}$$

Goodman [38] showed that $.22\pi < A < .50\pi$. The lower bound which he obtained was generated by a domain of the type shown in Figure 1.

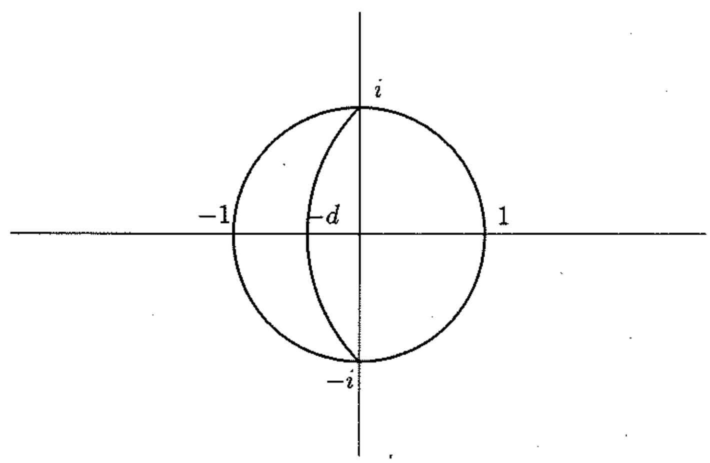

Figure 1

Later, Goodman and Reich [39] gave an improved upper bound of $.38\pi$ for A. Using variational methods developed by the author in [6] and some deep results of Alt and Caffarelli [4] in partial differential equations for free boundary problems, a geometric description for an extremal function for A was given by the author in [9] and by Lewis in [54]. This can be described as follows: There is an f_0 in S with $A = A(f_0)$ such that $f_0(\Delta)$ is circularly symmetric with respect to the positive real axis, i.e., it has the property that for $0 < r < 1$,

$$\frac{\partial}{\partial\theta}|f_0(re^{i\theta})| \text{ and } \frac{\partial}{\partial\theta}|f_0(re^{-i\theta})| \leq 0, \text{ for } 0 < \theta < \pi$$

(cf. Hayman [44]). Moreover the boundary of $f_0(\Delta)$ consists of the negative real axis up to -1, an arc γ of the unit circle that is symmetric about -1 and an arc λ lying in Δ, except for its endpoints. The arc λ is symmetric about the reals, connects the endpoints of γ and has monotonically decreasing modulus in the closure of the upper half disc. These results follow by standard symmetrization methods. Much deeper methods are needed to show (as in [9] and in [54]) that f_0 has a piecewise analytic extension to λ with f_0' continuous on $f_0^{-1}(\lambda)$ and $|f_0'(f_0^{-1}(w))| \equiv c < 1$ for all $w \in \lambda \bigcap \{\Delta \backslash (-1,1)\}$. Using these properties of f_0 it was shown by the author and Pearce in [19] that by "rounding the corners" in certain gearlike domains a close approximation to the extremal function could be obtained. This gives the best known lower bound of

$$.24\pi < A.$$

The upper bound is conceptually harder since it requires an estimate on the omitted area of each function in S. Indeed, it appears difficult to use the geometric description of f_0 to calculate A directly. However, an indirect proof was used by the author and Lewis in [17] to obtain the best known upper bound of

$$A < .31\pi.$$

Open problem. *Show that f_0 is unique and determine A explicitly.*

For the class S^* of functions in S whose images are starlike with respect to the origin, the problem of determining the corresponding

$$A^* = \sup_{f \in S^*} A(f)$$

has been completely solved by Lewis in [54]. The extremal function $f_1 \in S^*$ defined by

$$A^* = A(f_1) \cong .235\pi$$

is unique (up to rotation). The boundary of $f_1(\Delta)$ has two radial rays projecting into Δ with their end points connected by an arc λ_1 that is symmetric about the reals and has $|f_1'(\zeta)| \equiv c_1$ for all $\zeta \in f_1^{-1}(\lambda_1)$.

The problem of determining $L(r)$ in (15) was solved by Jenkins in [47] where he proved that for a fixed $r, 1/4 < r < 1$,

$$L(r) = 2r \arccos(8\sqrt{r} - 8r - 1).$$

The extremal domain in this case is the circular symmetric domain (unique up to rotation) having as its boundary the negative reals up to $-r$ and a single arc of $\{w : |w| = r\}$ symmetric about the point $-r$.

The corresponding problem for starlike functions of determining $L^*(r) = \sup_{f\in S^*} L(f,r)$ was solved by Lewandowski in [53] and by J. Stankiewicz in [70]. The extremal domain in that case is the circularly symmetric domain (unique up to rotation) having as its boundary two radial rays and the single arc of $\{w : |w| = r\}$ connecting their endpoints. An explicit formula for the mapping function in this case was first given by Suffridge in [72].

For the class S^c of functions in S whose images are convex domains the corresponding problem of determining

$$A^c(r) = \sup_{f\in S^c} A(f,r) \tag{16}$$

and

$$L^c(r) = \sup_{f\in S^c} L(r,v). \tag{17}$$

where $A(f,r)$ denotes the Lebesgue measure of $\Delta_r/f(\Delta)$, presents some interesting difficulties. One particular difficulty is that the basic tool of circular symmetrization used in the solution to each of the previous determinations is no longer useful. The example of starting with the convex domain bounded by a square shows that convexity is not always preserved under circular symmetrization. However, Steiner symmetrization (cf. Hayman [44]) can still be used in certain cases such as sectors. Another difficulty is the introduction of distinctly different extremal domains for different ranges of r. Since every function in S^c covers a disk of radius $1/2$ (cf. Duren [36]) r needs only to be considered in the interval $(1/2, 1)$. Waniurski has obtained some partial results in [74]. He defined r_1 and r_2 to be the unique solutions to certain transcendental equations where $r_1 \approx .594$ and $r_2 \approx .673$. If $F_{\pi/2}$ is the map of Δ onto the half plane $\{w : \mathrm{Re}\, w > -1/2\}$ and F_α maps Δ onto the sector

$$\left\{w : \left|\arg\left(w + \frac{\pi}{4\alpha}\right)\right| < \alpha\right\}$$

whose vertex, $v = -\pi/4\alpha$, is located inside the disk, then

$$\begin{aligned} A^c(r) &= A(F_{\pi/2}, r) \text{ for } 1/2 < r < r_1, \\ L^c(r) &= L(F_{\pi/2}, r) \text{ for } 1/2 < r < r_1, \end{aligned}$$

and

$$L^c(r) = L(F_\alpha, r) \text{ for } r_1 < r < r_2.$$

This author had announced in his survey talk on open problems in complex analysis at the 1985 *Symposium on the Occasion of the Proof of the Bieberbach Conjecture* the following conjecture:

Conjecture 7. *The extremal domains in determining $A^c(r)$ and $L^c(r)$ will be half-planes, symmetric sectors and domains bounded by singles arcs of $|w| = r$ along with tangent lines to the endpoints of these arcs, the different domains depending on different ranges of r in $(1/2, 1)$.*

This conjecture was also made independently by Waniurski at the end of his paper [74] in 1987.

Another conjecture that was announced at the *Symposium on the Proof of the Bieberbach Conjecture* arose out of this author and Pearce's work on the omitted values problem. A significant part of characterizing the extremal domains for $A^c(r)$ and $L^c(r)$ in (16) and (17) via the variational method developed in [6] would be the verification of the following:

Conjecture 8. *If* $f \in S^c$ *then*

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{f'(re^{i\theta})}\right|d\theta \le \sup_{z\in\Delta}\left|\frac{z}{f(z)}\right|. \tag{18}$$

Using standard integral means notation this is equivalent to showing that the smallest c such that

$$\mathcal{M}_1\left[1/f'\right] \le c\mathcal{M}_\infty\left[z/f(z)\right] \tag{19}$$

holds is $c = 1$. Well known results (cf. Duren [36], pp. 214) on integral means show that the smallest c for all functions in S is two, while unpublished results of the author and Pearce show that the smallest c for the class of functions starlike of order 1/2 [cf Goodman [40]] (a slightly larger class than S^c) is $c = 4/\pi$. It was also shown that equality holds in (18) for all domains bounded by regular polygons and it was conjectured that equality holds for those convex domains bounded by single arcs of $\{w : |w| = r\}$ and tangent lines at the endpoints of these arcs. Verification of Conjecture 8 would give an interesting geometric inequality. Let a convex curve Γ have length L and have its minimum distance from the origin be denoted by d. An application of the isoperimetric inequality along with the conjecture would imply

$$\sqrt{\frac{2d\pi}{L}} \le \sqrt{\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|f'(e^{i\theta})|}} \le \frac{L}{2d\pi}. \tag{20}$$

We note that the normalization for the functions f in S^c would force the first and last terms in inequality (20) to go to one as d goes to one.

Determining explicit values for $A^c(r)$ and $L^c(r)$ would involve computing the map that takes Δ onto the convex domains bounded by an arc of $\{w : |w| = r\}$ along with the two tangent lines at the endpoints of this arc. The function defining this map involves the quotient of two hypergeometric functions (cf. Nehari, [62]). In particular an extensive verification shows that the function g as shown in Figure 2

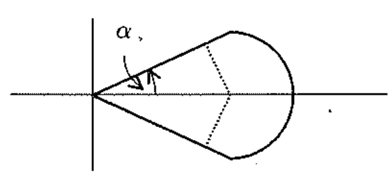

Figure 2

is given by

$$g(z) = \frac{z_2F_1\left(\frac{2\alpha-1}{4}, \frac{2\alpha+3}{4}, 1+\alpha; z\right)}{_2F_1\left(\frac{2\alpha+1}{-4}, \frac{3-2\alpha}{4}, 1-\alpha; z\right)}.$$

A difficulty arises when determining the explicit preimage of the center of the circle so that g can be renormalized to the mapping function f in S taking Δ onto a domain whose boundary circle is centered at the origin.

5. Möbius transformations of convex mappings

Another problem on convex mappings originated from a question of J. Clunie and T. Sheil-Small. If $f \in S$ and $w \notin f(\Delta)$, then the function

$$\hat{f} = f/(1 - f/w) \tag{21}$$

belongs again to S. The transformation $f \to \hat{f}$ is important in the study of geometric function theory. It is useful in the proofs of both elementary and not so elementary properties of S.

If F is a subset of S, let

$$\hat{F} = \{\hat{f} : f \in F, w \in \mathbb{C}^* \backslash f(\Delta)\}.$$

Here $\mathbb{C}^* = \mathbb{C} \bigcup \{\infty\}$. Since we admit $w = \infty$, it is clear that $F \subset \hat{F} \subset S$.

If F is compact in the topology of locally uniform convergence, then so is $\hat{F}$. If F is rotationally invariant, that is, $f_a(z) = e^{ia} f(e^{-ia} z)$ belongs to F whenever f does, then $\hat{F}$ is also rotationally invariant. It is an interesting question to ask which properties of F are inherited by $\hat{F}$. Since $\hat{S} = S$, this question is trivial for S.

In [20] and [21] the author and Schober considered the class S^c of convex mappings. Simple examples show that $\hat{S}^c$ is strictly larger than S^c. Since the coefficients of functions in S^c are uniformly bounded (by one), J. Clunie and T. Sheil-Small had asked whether the coefficients of functions in $\hat{S}^c$ have a uniform bound. The affirmative solution of this problem was given by R.R. Hall [42].

Open Question. Find the best uniform bound as well as the individual coefficient bounds for $\hat{S}^c$.

In [20] the variational procedure developed in [17] is applied to a class of extremal problems for $\hat{S}^c$. If $\lambda : \hat{S}^c \to \mathbb{R}$ is a continuous functional that satisfies certain admissibility criteria, it was shown that the problem

$$\max_{\hat{S}^c} \lambda$$

has a relatively elementary extremal function $\hat{f}$. More specifically, it was shown that $\hat{f}$ either is a half-plane mapping $f(z) = z/(1 - e^{ia} z)$ or is generated through (21) by a parallel strip mapping $f \in S^c$.

The class of functionals considered in [20] contain the second-coefficient functional $\lambda(\hat{f}) =$ Re a_2 and the functionals $\lambda(\hat{f}) =$ Re $\Phi(\log \hat{f}(z)/z)$ where Φ is entire and z is fixed. The latter functionals include the problems of maximum and minimum modulus ($\Phi(w) = \pm w$). In general, the extremal strip domains $f(\Delta)$ need not be symmetric about the origin. This adds a nontrivial and interesting character to the problems.

A sharp estimate for the second coefficient of functions in $\hat{S}^c$ is given explicitly in the following result. Surprisingly, the answer is not an obvious one.

Theorem 2. *If $\hat{f}(z) = z + a_2 z^2 + \cdots$ belongs to $\hat{S}^c$, then*

$$|a_2| \le \frac{2}{x_0} \sin x_0 - \cos x_0 \approx 1.3270$$

where $x_0 \approx 2.0816$ is the unique solution of the equation

$$\cot x = \frac{1}{x} - \frac{1}{2} x$$

in the interval $(0, \pi)$. Equality occurs for the functions $e^{-i\alpha} \hat{f}(e^{i\alpha} z)$, $\alpha \in \mathbb{R}$, where $\hat{f}(z) = f(z)/[1 - f(z)/f(1)]$ and f is the vertical strip mapping defined by

$$f(z) = \frac{1}{2i \sin x_0} \log \frac{1 + e^{i x_0} z}{1 + e^{-i x_0} z}. \tag{22}$$

We make the following:

Conjecture 9. *The extremal functions for maximizing $|a_n|$ over $\hat{S}^c$ are the vertical strip mappings defined by (22) where a different x_0 is needed for each n.*

In [21] the Koebe disk, radius of convexity, and sharp estimates for the coefficient functional $|t a_3 + a_2^2|$, for t in a certain interval, were found for functions in the class $\hat{S}^c$ Also, in [3], R.M. Ali found sharp upper and lower bounds for $|f(z)|$ for $\hat{f}$ in $\hat{S}^c$.

6. Robinson's 1/2 conjecture

A conjecture that has been open for more than 40 years is Robinson's 1/2 conjecture. Let $\mathcal{A}$ denote the class of analytic functions on Δ. For a subclass X (possibly a singleton) of $\mathcal{A}$ let $r_S(X)$ denote the minimum radius of univalence over all functions f in X.

For a function f in S define the operator $\Theta : S \to \mathcal{A}$ by

$$\Theta f = (zf)'/2.$$

In 1947, in [67], R. Robinson considered the problem of determining $r_S[\Theta(S)]$ which will be denoted by r_0. He observed that for each f in S, $[zf]' \neq 0$ for $\Delta_{1/2}$ and noted that for the Koebe function, $k, k(z) = z(1-z)^{-2}$,

$$r_S(k) = r_{S^*}(k) = 1/2$$

which implies $r_0 \le 1/2$. Robinson made

Conjecture 10. *If $f \in S$ then $(zf)'/2$ is univalent in Δ_r for $0 < r \le 1/2$, i.e. $r_0 = 1/2$.*

He was able to show that $.38 < r_{S^*}[\Theta(S)] \le r_0$. Little or no progress was made directly on the study of the operator Θ following Robinson's work until Livingston in [56] proposed a shift for the setting of the problem from the full class S to subclasses of S. He showed that Θ preserved many of the well-known subclasses of S. e.g., S^* and S^c. Livingston's work renewed interest in the study of Θ. Numerous papers by various authors followed (see [13]) connecting the operator Θ to various subclasses of S. It was shown by the author and Kellogg in [13] that most of these results follow directly from the Ruscheweyh-Sheil-Small theory on Hadamard convolutions. However, for the entire class S, if appears that the easily obtained lower bound of approximately .41 is the most that can be obtained from the convolution methods. Thus Conjecture 10 is still open. Although Bernardi had suggested that $r_{S^*}[\Theta(S)] = 1/2$ might even be true, in [7], it was shown that $r_{S^*}[\Theta(S)] < .445$, while Pearce proved in [64] that $.435 < r_{S^*}[\Theta(S)]$. In [8] the author proved that $.490 < r_0 \le .50$ using the Grunsky inequalities. The closeness, but non sharpness, of this result has intrigued a number of people in the field. Robinson's conjecture and the progress on this problem appeared in A. W. Goodman's book, *Univalent Functions* [40], and in [27].

7. Campbell's conjecture on a majorization- subordination result

A conjecture relating majorization and subordination was made by Campbell in [34]. Let f, F, and w be analytic in Δ_r. f is said to be majorized by F, denoted by $f \ll F$, in Δ_r if $|f(z)| \leqq |F(z)|$ in Δ_r. f is said to be subordinate to F, denoted by $f \prec F$, in Δ_r if $f(z) = F(w(z))$ where $|w(z)| \leqq |z|$ in Δ_r.

Majorization-subordination theory began with Biernacki who showed in 1936 that if $f'(0) \geqq 0$ and $f \prec F(F \in S)$ in Δ, then $f \ll F$ in $\Delta_{1/4}$. In the succeeding years Goluzin, Tao Shah, Lewandowski and MacGregor examined various related problems (for greater detail see [33]).

In 1951 Goluzin showed that if $f'(0) \geqq 0$ and $f \prec F(F \in S)$ then $f' \ll F'$ in $\Delta_{0.12}$. He conjectured that majorization would always occur for $|z| < 3 - \sqrt{8}$ and this was proved by Tao Shah in 1958.

In a series of papers [32,33,34], D. Campbell extended a number of the results to the class $\mathcal{U}_\alpha$ of all normalized locally univalent $(f'(z) \neq 0)$ analytic functions in Δ with order $\leqq \alpha$ where $\mathcal{U}_1 = S^c$, the class of convex functions in S. In particular in [34] he showed that if $f'(0) \geqq 0$ and $f \prec F(F \in \mathcal{U}_\alpha)$ then $f' \ll F'$ in $|z| < \alpha + 1 - (\alpha^2 + 2\alpha)^{1/2}$ for $1.65 \leqq \alpha < \infty$ where $\alpha = 2$ yields $3 - \sqrt{8}$. Note that $\alpha = 1$ yields $2 - \sqrt{3}$, the radius of convexity for S. Campbell's proof breaks down for $1 \leqq \alpha < 1.65$ because of two different bounds being used for the Schwarz function with different ranges of α. Nevertheless, he made the following:

Conjecture 11. *If $f'(0) \geq 0$ and $f \prec F$ $(F \in \mathcal{U}_\alpha)$ then $f' \ll F'$ for $|z| < \alpha + 1 - (\alpha^2 + 2\alpha)^{1/2}$.*

In [14] the author and Kellogg combined Ruscheweyh's subordination result [68], variational methods, and some tedious computations to verify the conjecture for $\alpha = 1$, i.e., it is shown that if $f'(0) \geqq 0$ and $f \prec F(F \in S^c)$ in Δ then $f' \ll F'$ for $|z| < 2 - \sqrt{3}$.

8. Krzyż's conjecture for bounded nonvanishing functions

Another conjecture that has been investigated by a large number of function theorists is Krzyż's conjecture. Let B denote the class of functions defined by $f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$ for which $0 < |f(z)| < 1$ for $z \in \Delta$. In 1968 in [49] J. Krzyż posed the fundamental problem of determining for $n \geq 1$

$$A_n = \sup_{f \in B} |a_n|.$$

That $A_1 = 2/e$ dates back to 1932 (see Levin [51]) and appears explicitly in Hummel, etc. [46] and Horowitz [45]. That $A_2 = 2/e$ appears in [46] and $A_3 = 2/e$ in [65]. For a fairly complete history of this problem see [46] or Brown [31]. These results suggest what has become known as the Krzyż Conjecture,

Conjecture 12. *$A_n = 2/e$, for all $n \geq 1$, , with equality only for the functions*

$$K_n(z) = \exp\left[\frac{z^n + 1}{z^n - 1}\right] = \frac{1}{e} + \frac{2}{e} z^n + \cdots$$

and its rotations $e^{iu} K_n(e^{iv} z)$.

A_n is to equal the apocryphal Pondiczery constant, named by Boas in [25]. A sharp uniform bound less than one is expected. However, the bound $2/e \approx .7357\cdots$, is somewhat surprising in view of the fact that the best uniform estimate known to date is

$$|a_n| \leq 1 - \frac{1}{3\pi} + \frac{4}{\pi} \sin\left(\frac{1}{12}\right) = 0.9998772\cdots$$

given by D. Horovitz in 1978 in [45].

The open problem of Krzyż's Conjecture is stated in *A.* Goodman's book "Univalent Functions" [40, page 83]. De Branges' recent solution to the Bieberbach Conjecture gave hope to solving many of these type problems. However, not withstanding the amount of effort by several function theorists to solve the corresponding coefficient problem, Conjecture 12 still remains open.

A related conjecture made by Ruscheweyh upon verification would give a much improved uniform estimate for A_n. Consider $f(z) = \exp[-\lambda p(z)]$ for $\lambda > 0$ and $p \in P$ where

$$P = \{p : p(z) = 1 + p_1 z + \cdots, \operatorname{Re} p(z) > 0, |z| < 1\}.$$

Then consider the following: For $0 < r < 1$, choose $x = x(r)$ such that

$$i\frac{J_1\left(ix\frac{2r}{1-r^2}\right)}{J_0\left(ix\frac{2r}{1-r^2}\right)} = r \quad (J_0, J_1 \text{ are Bessel Functions})$$

and define

$$F(r) = x(r)e^{-x(r)\frac{1+r^2}{1-r^2}} J_0\left(ix(r)\frac{2r}{1-r^2}\right). \tag{23}$$

Ruscheweyh conjectured that for any positive integer $n, A_k \geq 0, |\xi_k| = 1, k = 1, 2, \cdots, n$ and

$$p(z) = \sum_{k=1}^{n} A_k \left(\frac{1+\zeta_k z}{1-\zeta_k z}\right),$$

Conjecture 13.

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{Re}p(re^{i\varphi})}\mathrm{Re}\{p(re^{i\varphi})\}d\varphi \leq F(r^n), \tag{24}$$

with F defined in (23).

We have shown by using the Legendre polynomial expansion for Bessel functions that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-x\mathrm{Re}\left\{\frac{1+r^n e^{in\varphi}}{1-r^n e^{in\varphi}}\right\}}\mathrm{Re}\left\{\frac{1+r^n e^{in\varphi}}{1-r^n e^{in\varphi}}\right\} d\varphi = e^{-x\frac{1+r^{2n}}{1-r^{2n}}} J_0\left(ix\frac{2r^n}{1-r^{2n}}\right). \tag{25}$$

Equation (25) shows that the estimate (24) would be sharp for fixed r for $\tilde{p}$ defined by

$$\tilde{p}(z) = x(r^n)\frac{1+z^n}{1-z^n}.$$

Upon verification of Conjecture 13 it can be shown that

$$|a_n| \leq \frac{2}{n}\frac{F(r^n)}{r^{n-1}(1-r^2)}, 0 < r < 1. \tag{26}$$

Choosing $r^2 = (n-1)/(n+1)$ in (26) it would follow that

$$|a_n| \leq \lim_{k\to\infty} \frac{2}{k}\frac{F\left[\left(\frac{k-1}{k+1}\right)^{k/2}\right]}{k\left(\frac{k-1}{k+1}\right)^{\frac{k-1}{2}}\frac{2}{k+1}} = eF\left(\frac{1}{e}\right) \approx .869 \tag{27}$$

by numerical calculations.

9. A conjecture for bounded starlike functions

A conjecture that was made by this author in [6] in 1975 was recently disproved with computer methods by Pearce leaving the problem now as one that probably can only be done numerically. The conjecture involved coefficient estimates for bounded starlike functions in S. Define, for a fixed $M \geq 1$,

$$S_M = \left\{ f \in S : f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, |f(z)| \leq M, z \in \Delta \right\}.$$

The fact that $|a_2|$ is maximized in S_M by the function mapping onto Pick's domain of the disk Δ_M minus a single radial slit has been known since 1917 [see Goodman, vol. I, p.38]. In the early sixties Tammi [73] used Schiffer's variational methods to determine the explicit extremal domains for maximizing the first few coefficients in S_M. In particular he proved that the extremal domains for maximizing $|a_3|$ in S_M are as shown in Figure 3 for the different values of M.

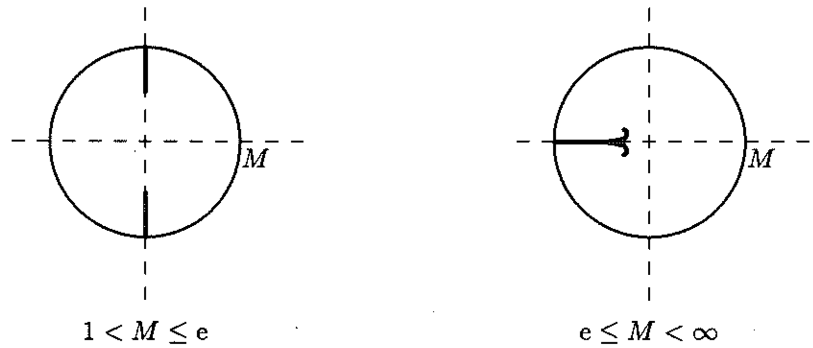

Figure 3

There is a difficulty in modifying Schiffer's variational methods to allow for preservation of both boundedness and starlikeness at the same time. Also the fact that the forked slit domains occurring for $M > 3$ are no longer starlike suggested the need for a local variational technique that preserved these properties. This was developed by combining the Julia Variational formula with the Loewner Theory in [6] and in [17]. Let

$$S^*_M = \{ f \in S_M : f(\Delta) \text{ is starlike with respect to the origin} \}.$$

It was shown in [6] that the extremal domain maximizing $|a_3|$ in S^*_M is the disc Δ_M minus at most two symmetric radial slits. Define D_M as Δ_M minus two symmetric radial slits where 2θ is the angle between the 2 slits. Let $A_3(M, \theta)$ be the third coefficient for the function in S^*_M mapping Δ onto the domain D_M.

From the properties of the extremal domains in the class S_M, along with initial computations and the observation that $A_3(3, 0) = A_3(3, \pi/2) = 8/9$ led this author to the following:

Conjecture 14. *For all* $f \in S^*_M$

(28) $$|a_3| \leq A_3(M, \pi/2), \quad 1 < M \leq 3,$$

$$(29) \qquad |a_3| \leq A_3(M,0), \quad 3 \leq M < \infty.$$

It follows from Tammi's results that (28) holds for $1 < M \leq e$ and it was shown by the author and Lewis in [16] that (29) holds for $5 \leq M < \infty$. Verifying Conjecture 14 for $e < M < 5$ remained an open problem. This conjecture was announced at the 1978 Brockport Conference and appeared in the open problem set in the proceedings [60] for that conference. It was announced by J. Lewis at the 1980 Canterbury Conference and appeared in the open problem set in its proceedings, [27]. It was also announced at the 1985 *Symposium on the Proof of the Bieberbach Conjecture.*

Motivated by the observation that the domain D_M is indeed a "gearlike" domain and now having the computer software available, Pearce was able to compute $A_3(3,\theta)$ and discovered that $A_3(3,\theta)$, as a function of θ from 0 to $\pi/2$, was convex downward, i.e., it took its *minimum* at the endpoints.

Thus Conjecture 14 was false. Indeed further computations shows that there exists a $\theta(M), 0 < \theta(M) < \pi/2$, such that, for some $M_0 > 0$,

$$\max\left[A_3(M,\theta), A_3(M,\pi/2)\right] < A_3(M,\theta(M))$$

for $2.83 < M < M_0 < 5$.

10. A. Schild's 2/3 conjecture

Another long standing conjecture that was proved false was the 2/3 conjecture. Let $r_1 = r_1(f)$ be the radius of convexity of f, i.e. $r_1(f) = \sup\{r : f(\Delta_r) \text{ is a convex domain}\}$. Put $d^* = \min\{|f(z)| : |z| = r_1\}$ and $d = \inf|\beta|$ for which $f(z) \neq \beta$. In 1953 in [69], A. Schild conjectured that $d^*/d \geq 2/3$ for all functions $f \in S^*$. Here equality holds for the Koebe function $k(z) = z(1+z)^{-2}$. Schild noted that $d^*/d \geq r_1 \geq 2 - \sqrt{3}$ and proved the conjecture for p symmetric functions, $p \geq 7$. He also showed for a certain class of circularly symmetric functions that $d^*/d \geq .49$. Lewandowski in [52], proved the conjecture true for certain subclass of S^*. In [58], McCarty and Tepper obtained the best known lower bound of .38 for all starlike functions. The conjecture was shown false by the author and Lewis in [15] by giving two explicit counter examples.

The first example is given simply by the two slit map defined by

$$f(z) = \frac{z}{(1-z)^{\alpha}(1+z)^{2-\alpha}},$$

where α is sufficiently near 0. It was noted that if d is computed as a function of α, then $\alpha'(d) \to +\infty$ as $\alpha \to 0$ so that a minimal value of .656 for d^*/d was obtained for this function at $\alpha \approx .03$.

The second example is a more complicated function that maps Δ onto the circularly symmetric domain shown in Figure 4.

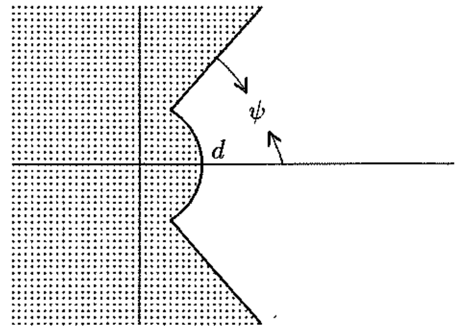

Figure 4

An explicit formula for this function g_a, determined by Suffridge in [72], is given by

$$\log\frac{g_a(z)}{z} = 2b\log\left\{\left[\left(\frac{1+2az+z^2}{(1-z)^2}\right)^{1/2} + b\frac{1+z}{1-z}\right]\frac{1}{(1+b)}\right\}$$

$$+2\log\frac{2}{\left[(1+2az+z^2)^{1/2}+1+z\right]},$$

where $a = 2b^2 - 1$ and $d = \left[(1+b)^{1+b}(1-b)^{1-b}\right]^{-1}$ with $\psi = \pi(1-b)$. A close approximation to the minimum of d^*/d for this function is 0.644 given by $a \approx 0.89$. Also $\psi \approx .03\pi$ for this minimum value. The author's work suggests:

Conjecture 15.

$$\inf_{f\in S^*} d^*/d = \min_a \{d^*/d \text{ for } g_a\} \approx .644\cdots$$

11. Brannan's coefficient conjecture for certain power series

An innocent looking, but not so trivial, conjecture was made by Brannan in 1973 in [26] on the coefficients of a specific power series. The problem originated in the Brannan, Clunie, Kirwan paper [28] (later completed by a Aharanov and Friedland in [1]) solving the coefficient problem for functions of bounded boundary rotation. Consider the coefficients in the expansion

$$\frac{(1+xz)^\alpha}{(1-z)^\beta} = \sum_{n=0}^{\infty} A_n^{(\alpha,\beta)}(x)z^n, \quad |x|=1, \alpha>0, \beta>0.$$

Brannan posed the problem as to when

$$\left|A_n^{(\alpha,\beta)}(x)\right| \le A_n^{(\alpha,\beta)}(1). \tag{30}$$

He gave a short elegant proof that (30) held if $\beta = 1$ and $\alpha \geq 1$. However, he showed that for $\beta = 1, 0 < \alpha < 1$, (30) did *not* hold for the even coefficients and that for $x = e^{i\theta}$, (30) held for odd coefficients in a sufficiently small neighborhood of $\theta = 0$. He also noted that for $0 < \alpha = \beta < 1, |A_3^{(\alpha,\alpha)}(x)| \leq A_3^{(\alpha,\alpha)}(1)$. By using the expansion

$$A_n^{(\alpha,\beta)}(x) = \frac{(\beta)(\beta+1)\cdots(\beta+n-1)}{n!}\ {}_2F_1(-n,\alpha,1-\beta;-x)$$

and the properties of ${}_2F_1$, the hypergeometric function, this author has shown that

(i) (30) holds for $\alpha \leq \beta, \beta + \alpha \geq 1$ and $|A_n^{(\alpha,\beta)}(x)| < A_n^{(\alpha,\beta)}(1)$ for $|x| = 1, x \neq 1$ and $n = 1, 2, 3, \cdots$.

(ii) $|A_{2n+1}^{(\alpha,1)}(x)| \leq A_{2n+1}^{(\alpha,1)}(1), n = 1, 2, 3, \cdots$ for $0 < \alpha < \alpha + \epsilon, 1 - \delta < \alpha < 1$ for ϵ, δ sufficiently small and positive, and

(iii) $|A_3^{(\alpha,\beta)}(x)| \leq A_3^{(\alpha,\beta)}(1), 0 < \alpha < \beta < 1$.

In [61], D. Moak has shown that (30) holds for $\alpha \geq 1, \beta \geq 1$. Milcetich, in [59], has recently shown that (30) holds for $n = 5, \beta = 1$ and $2 < \alpha < n$ but does *not* hold for non integer α's less than $n - 1, \beta$ near zero, for odd $n \geq 3$. The basic

Conjecture 16. $|A_{2n+1}^{(\alpha,1)}(x)| \leq A_{2n+1}^{(\alpha,1)}(1)$

is still open.

12. Polynomial approximations using a differential equation model

Another conjecture on special functions arose out of the author's and L. Reichel's work on polynomial approximations using a differential equation model. Given equidistant data (x_i, y_i) with $x_i = 1 - (2i-1)/M$, the problem is to *best* fit a polynomial of given degree $N - 1$ to M data points. A comparison is used between the discrete norm $\|\cdot\|_D$, defined by

$$\|f\|_D^2 = \frac{1}{M}\sum_1^M |f(x_i)|^2$$

and the continuous norm, $\|\cdot\|_c$, defined by

$$\|f\|_c = \frac{1}{M}\max_{x\in I}|f(x)|.$$

Gram polynomials $\{\varphi_j\}$ are used where they are orthonormal in the discrete norm with an expansion for p given by

$$p(x) = \sum_j \alpha_j \varphi_j(x),$$

so that $\|p\|_D^2 = \sum \alpha_j^2$. These are defined recursively by

$$\varphi_N(x) = 2x\alpha_{N-1}\varphi_{N-1} - (\alpha_{N-1}/\alpha_{N-2})\varphi_{N-2}(x), \tag{31}$$

where

$$\alpha_N = \frac{M}{N}\left(\frac{N^2 - 1/4}{M^2 - N^2}\right)^{1/2}.$$

The asymptotics as M and $N \to \infty$ are studied by letting $\tau = N/\sqrt{M}$ and $x = 1 - \zeta/M$. Then the recurrence relation in (31) can be used to obtain

$$\varphi_N - 2\varphi_{N-1} + \varphi_{N-2} = \left[\tau^2 - (1/4\tau^2) - 2\zeta\right]\varphi_N/M + o(1).$$

This in turn can be used to obtain the differential equation model:

$$\varphi''(t) = \left[t^2 - (1/4t^2) - 2\zeta\right]\varphi(t), \tag{32}$$

where $t = \tau - 1/\sqrt{2M}$ and the initial condition as $t \to 0$ is defined by

$$\varphi_N\left[1 - \zeta/M\right] = \sqrt{2\sqrt{M}}\sqrt{t} + O(1/M),$$

i.e., $\varphi(t) \approx \sqrt{t}(t \to 0)$. A normalization is made by $\varphi(t) \approx \varphi_n/\sqrt{2\sqrt{M}}$ where ζ is an odd positive integer if and only if x is a grid point. The solution to (32) is given by

$$\varphi(t) = t^{1/2}e^{-t^2/2}\ {}_1F_1\left(\frac{1-\zeta}{2}, 1; t^2\right),$$

where ${}_1F_1$ is Kummer's confluent hypergeometric function (see Gradshteyn and Ryzhik [41]).

To find error estimates for least square approximates by these polynomials an application of Brass's result in [29] can be used that gives error estimates for least square norms in terms of the uniform sup norm. But in order to apply this result all the $\varphi_N(1 - \zeta/M)$'s must have their sup norms occur at the right end point of the interval $[-1, 1]$. An extensive computer analysis suggested that this does occur. What is needed then is to verify

Conjecture 17. *For all $\zeta > 0$ and real t we have*

$${}_1F_1\left(\frac{1-\zeta}{2}, 1, t^2\right) \leq {}_1F_1\left(1/2, 1, t^2\right).$$

Indeed, by converting to the Whittaker functions $M_{\kappa,\mu}(x)$ see [41], for a more convenient range of variables the conjecture is equivalent to showing that

$$M_{\kappa,0}(x) \leq M_{0,0}(x) \quad \text{for all } \kappa \geq 0 \text{ and } x \geq 0.$$

We have verified Conjecture 17 for the regions dotted in Figure 5.

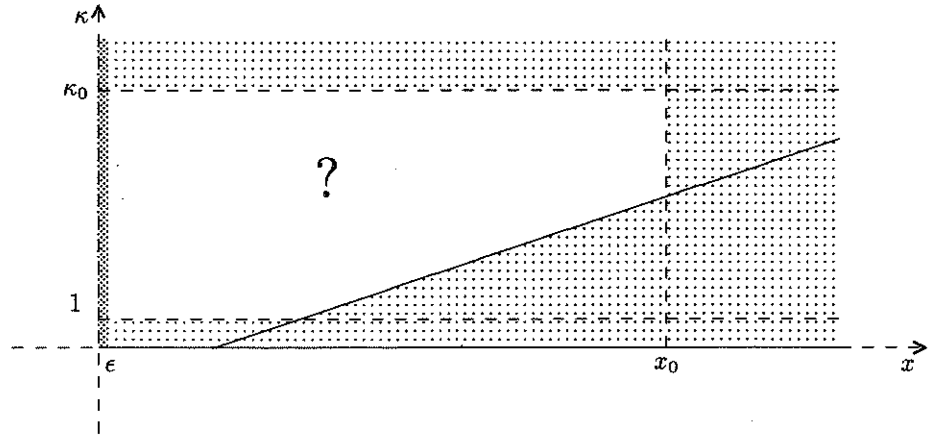

Figure 5

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