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THE OMITTED AREA PROBLEM FOR UNIVALENT FUNCTIONS

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Let $U = \{z: |z| < 1\}$ and $S = \{f: f$ analytic and 1-1 in U, $f(z) = z + \cdots\}$. The omitted area problem was first posed by A. Goodman [7] in 1949, reiterated by T. MacGregor [10] in his survey article in 1972, then reposed in a much more general setting by D. Brannan [1]. It appears in Bernardi's survey article in [3] and in a few open problem sets since then [11], [5]. The problem is to find the maximum area of the intersection of the unit disk with the complement of $f(U)$ as f varies over the family S. If f is in S, let A_f be the area of $U \cap f(U)$ (so $\Pi - A_f$ is the omitted area). Further let $A^0 = \inf\{A_f: f \in S\}$. Goodman [7] showed that $.50\pi \leq A^0 \leq .77\pi$ and Goodman and Reich [8] improved this result to $.62\pi < A^0 < .77\pi$. Goodman's upper bound was obtained by computing the value of A_f where $f(U)$ is shown in Figure 1.

The two arcs are circular arcs with the inside arc chosen to maximize A_f for f in S.

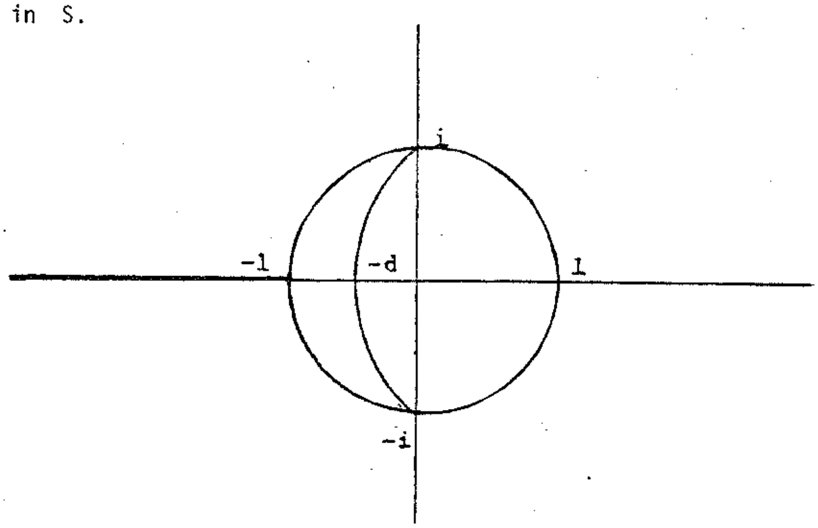

Figure 1

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In this paper we use symmetrization and subordination techniques, the Julia Variational Formula, and results on the boundary behavior of the derivative of the mapping function to prove the following result:

THEOREM. Let A_f = area of $f(U) \cap U$ for f in S and $A^0 = \inf\{A_f: f \in S\}$. If $A_{f_0} = A^0$, then $f_0(U)$ is circularly symmetric and has as its boundary (up to rotation) the negative reals up to -1, and arc λ of the ∂U that is symmetric about -1 and an arc γ, symmetric about the reals, lying in U and connecting the endpoints of λ. If $\partial f_0(U)$ is assumed to be a piecewise analytic curve then for $\gamma' = \gamma \cap \{U \setminus (-1,0)\} |zf_0'(z)|$ has a constant value c, $0 < c < 1$, on $f_0^{-1}(\gamma')$ and f' has a continuous extension to an open arc containing the closure of $f_0^{-1}(\gamma)$.

REMARK. We note that γ may contain slits along the reals.

PROOF. To assure the existence of an extremal function we observe from the definition of the kernel in the Carathéodory kernel convergence theory [6] that A_f is a lower semicontinuous functional over the class S, since the area may have a negative jump in the limit but not a positive jump. It follows from the existence of the minimum for l.s.c. functions on the compact set S that an extremal function f_0 exists, i.e., $A^0 = A_{f_0}$. We will use f_0 to designate the extension of the function to the boundary whenever appropriate. We let $D_0 = f_0(U)$ denote an extremal domain for this problem.

We now use symmetrization and subordination arguments to show that the ∂D_0 is as claimed. We recall the strict monotonicity property of the conformal mapping radius, m.r., for subordinating domains, i.e., if $D_1 \subsetneq D_2$ then m.r. $D_1 <$ m.r.D_2. We also use the symmetrization principle: i.e., if D^* is obtained from D by circular symmetrization, then m.r.$D \leq$ m.r.D^* with equality if and only if D^* is a rigid rotation of D. It follows from the definition of circular symmetrization that the area enclosed by any given circle centered at the origin is preserved by symmetrization. Let $R_{-1} = \{x: x < -1\}$. Now we claim that the set $C = \partial D_0 \cap \{z: |z| \geq 1\}$ must be $R_{-1} \cup \lambda$ (up to rotation) where λ is some arc of ∂U symmetric about -1. Indeed suppose that $C \neq R_{-1} \cup \lambda$. Upon symmetrization of D_0 with respect to the positive reals we obtain D_0^* with f_0^* and C^* appropriately defined. From the definition of symmetrization $\mathbb{C} \setminus C^* \subseteq \mathbb{C}^* \setminus \{R_{-1} \cup \lambda\}$ where λ is the longest arc of ∂U symmetric about -1 contained in C^*. Thus, the domain D, obtained by replacing C^* by $R_{-1} \cup \lambda$ would have larger m.r. Let d_0 be the point on the negative reals where $f_0^*(e^{i\theta})$ attains its minimum modulus. We could then define an f_2 in S corresponding to a domain D_2^* obtained by inserting a symmetric tear-drop shaped intrusion (with positive area) into D with its point at d_0 with the intrusion being of sufficient size so that the resulting symmetric domain

D^*_2 has m.r. one. The existence of such a D^*_2 is assured by a continuity argument noting that Koebe's 1/4 theorem guarantees that if the intrusion gets closer than 1/4 from the origin the resulting m.r. would then be less than one while m.r.$D^*_1 > 1$. Thus it follows that $A_{f_0^*} < A_{f_0}$, contradicting the extremal property of f_0.

It follows from the definition of symmetrization that the arc γ of ∂D_0 connecting the endpoints of γ lies in U, is symmetric about the reals and a parametrization of γ in the upper half plane would have monotonically decreasing modulus. We note that although γ may contain slits along the reals it follows from the extremal property of f_0 that γ cannot contain any slits along arcs of circles centered at the origin. Indeed, if it did, these slits, having zero area, could be removed by an argument similar to the one above.

Although the above results will assure that f has a continuous extension to γ, it does not assure any kind of smoothness of γ. We note at this point, the criticality of this statement. The author has been unable to overcome the barrier of having to assume sufficient smoothness of γ in order to proceed with the argument.

REMARK. The author has been notified that John L. Lewis has shown that the assumption of smoothness of γ is unnecessary. Indeed he has shown that ∂D_0 is piecewise analytic with left and right derivatives existing at each point.

We can then apply the Julia Variational Formula as used and described in detail by the author in his Transactions AMS paper [2] (see also [15]). For ε sufficient small and positive we let $\varepsilon\phi(w)$ be a normal variation in C^2 of the point w on the boundary. The varied function for $w = f(\zeta)$, $\zeta = e^{i\theta}$ is given up to $o(\varepsilon)$ (continuity arguments enable us to drop the $o(\varepsilon)$ term in the remaining argument) by

$$\hat{f}(z) = f(z) + \frac{\varepsilon z f'(z)}{2\pi} \int_0^{2\pi} \frac{\zeta+z}{\zeta-z} \frac{\phi(w)d\theta}{|\zeta f'(\zeta)|}, \quad z \in U.$$

If we let γ_1 be the set of smooth arcs where $\phi(w)$ is chosen to be nonzero, then it follows that the change in area is given by

$$\frac{\varepsilon}{2\pi} \int_{\gamma_1} \phi(w)|dw| = \frac{\varepsilon}{2\pi} \int_{f^{-1}(\gamma_1)} \phi(w)|\zeta f'(\zeta)|d\theta$$

while the change in m.r. is given by

$$\Delta \text{m.r.} = \frac{\varepsilon}{2\pi} \int_{f^{-1}(\gamma_1)} [\phi(w)/|\zeta f'(\zeta)|]d\theta.$$

Using this one can vary the boundary γ by locally pushing in one place and pushing out in another, or vice-versa, in such a way as to preserve the m.r.

while changing the area under certain conditions. More explicitly, if we let w_1 be a point on γ' where for $w_1 = f(\zeta_1) = f(e^{i\theta_1})$, $f'(\zeta)$ exists, is nonzero, and has nonconstant modulus in a neighborhood of w_1 along γ_1 and if we choose sufficiently small arcs w_0w_1 and w_1w_2 on γ with $w_0 = f(e^{i\theta_0})$ and $w_2 = f(e^{i\theta_2})$ such that $|f'(\zeta)| - |f'(z)|$, $f(\zeta) \in w_0w_1$, $f(z) \in w_1w_2$, is of constant sign, then the change in area would be given by

$$\frac{\varepsilon}{2\pi}\int_{\theta_0}^{\theta_1} \frac{\phi(w)}{|\zeta f'(\zeta)|}\,|\zeta f'(\zeta)|^2 d\theta \;-\; \frac{\varepsilon}{2\pi}\int_{\theta_1}^{\theta_2} \frac{\phi(w)}{|zf'(z)|}\,|zf'(z)|^2 d\theta. \tag{1}$$

Since $\phi/|f'|$ does not change sign in the respective intervals, a mean value theorem for integrals assures that there exist ζ_3, z_3, with $f(\zeta_3) \in w_0w_1$ and $f(z_3) \in w_1w_2$ such that the quantity in (1) becomes

$$\frac{\varepsilon}{2\pi}\,|\zeta_3 f'(\zeta_3)|^2 \int_{\theta_0}^{\theta_1} \frac{\phi(w)}{|\zeta f'(\zeta)|}\,d\theta - \frac{\varepsilon}{2\pi}\,|z_3 f(z_3)|^2 \int_{\theta_1}^{\theta_2} \frac{\phi(w)}{|zf'(z)|}\,d\theta.$$

Then with the appropriate sign on $\phi(w)$ this is less than

$$\frac{\varepsilon}{2\pi}\max[|f'(\zeta_3)|^2, |f'(z_3)|^2]\Delta m.r.$$

By continuity θ_0, θ_2, and ϕ can be chosen so that $\Delta m.r. = 0$. It follows that if there exists a place on $f_0^{-1}(\gamma')$ where $|\zeta f'(\zeta)|$ is nonconstant, then a locally varied function $\hat{f}_0$ can be defined such that $A_{\hat{f}_0} < A_{f_0}$, contradicting the extremal property of f_0.

We exclude here the possibility that $f_0(z) \equiv z$ since A_{f_0} is a maximum, not minimum, area. Then we observe that γ' cannot contain an arc, say B, of a circle with center at a, or a straight line segment, say ℓ. Indeed, along B it would follow for $z = e^{i\theta}$ that $\partial[\log|f_0(z)-a|]/\partial\theta = \mathrm{Im}\{zf_0'(z)/[f_0(z)-a]\} = 0$ and along ℓ that $\arg zf_0'(z)$ would be constant. These conditions along with $|zf_0'(z)|$ being constant would force $zf_0'(z)/(f_0(z)-a)$ and $zf_0'(z)$ to be constant along B and ℓ, respectively. Thus, using Privalov's uniqueness theorem [6], f_0 would be a Möbius transformation on U, contradicting the extremal property of f_0. We note that the observations made in this paragraph do not need a priori the assumption of smoothness on the boundary because the sections of the boundary, if they were to exist, being circles or straight lines could be varied as described.

We now study the behavior of f_0' on the boundary. It follows from the Schwarz reflection principle that f_0' has an analytic extension across the piecewise analytic boundary of D_0 [Bieberbach, 4, pp. 152] with the exception of a finite number of points where derivatives may not exist. We shall study the points on γ where f_0' may not be continuous.

We show that f' has a continuous extension to a neighborhood of γ. For the purpose of contradiction, assume γ contains a corner at $f_0(\zeta_0)$ of opening $\alpha\pi$, for some α, $0 \leq \alpha < 1$ or $1 < \alpha \leq 2$. In the case $\alpha > 0$ it follows from Lichtenstein's theorem [14] as proved by Warshawski in [14] that for z in a neighborhood N of ζ_0

$$df_0'(z)/dz = (z - \zeta_0)^{\alpha-1}\, h(z)$$

where h is continuous and nonzero in N. It follows that $|\zeta f_0'(\zeta)|$ cannot be piecewise constant in a neighborhood of $f(\zeta_0)$ along $|\zeta| = 1$ for $\alpha\pi$, $0 < \alpha < 2$, $\alpha \neq 1$. Thus γ has no corners of this type even at the endpoints. The case when $\alpha = 0$, i.e., when γ has an interior cusp, must be considered separately because of the utilization of the $\omega^{1/\alpha}$ mapping in proofs of Lichtenstein's theorem. Indeed there exists a function, say g, in S with $|\zeta g'(\zeta)|$ having two different constant values along the boundary in a neighborhood of a cusp. In particular the function defined by

$$g(z) = \int_0^z \exp\{\frac{i}{\beta}\log\frac{1+\zeta}{1-\zeta}\}d\zeta$$

is in S for $\beta \geq 2$ by Becker's criterion [see 12] and has $|\zeta g'(\zeta)| = e^{\pm\pi/2\beta}$ for ζ in a neighborhood of 1 along $|\zeta| = 1$. We note that the phenomenon occurring here is that the boundary of $g(U)$ spirals into a cusp.

We will now show that if a cusp were to exist on γ then the angular derivative of the function would have to approach infinity. We then show that the conditions obtained for γ will not let this happen. We need the Carathéodory-Lelong-Ferrand result on angular derivatives. A function f analytic in U has the angular derivative a at $\zeta \in \partial U$ if $f'(z) \to a$ as $z \to \zeta$, $z \in A$ for every Stolz angle A at ζ. Then C-L-F [12] proved the following:

THEOREM A. Let f and g be analytic and univalent in U and let $f(U) \subset g(U)$. Let there exist a Jordan arc Γ ending at $\zeta \in \partial U$ such that

$$\phi(z) = g^{-1}(f(z)) \to \zeta \text{ as } z \to \zeta,\ \ z \in \Gamma.$$

If $f'(\zeta)$ exists and is finite, then $g'(\zeta)$ exists and $f'(\zeta) = 0$ implies $g'(\zeta) = 0$.

REMARK. Since the proof uses the computation, assuming that $\zeta = 1$,

$$\frac{g(1)-g(\phi(x))}{1-\phi(x)} = \frac{1-x}{1-\phi(x)}\,\frac{f(1)-f(x)}{1-x} \to \beta f'(1) \quad (x \to 1-0)$$

where $0 \leq \beta < \infty$, it follows that under the conditions of the hypothesis if $g'(z) \to \infty$ then $f'(z) \to \infty$ as $z \to \zeta$ along Γ.

To apply Theorem A we need to prescribe the Jordan arc Γ and a superordinating function g with the appropriate properties. Since γ has strictly monotonically decreasing modulus, if we let γ_1 be an arc of $\{w: |w| = |f_0(\zeta_0)|\}$ lying inside $f_0(U)$ ending at $f_0(\zeta_0)$, then we can let $\Gamma = f_0^{-1}(\gamma_1)$. From the circular symmetry of D_0 and thus the geometry of γ and γ_1, there exist two sufficiently short straight line segments, ℓ_1, ℓ_2, lying in $U \setminus f_0(U)$ ending at $f_0(\zeta_0)$ and forming a corner of opening $\varepsilon\pi$, $0 < \varepsilon < 1$. (We note that if γ could spiral the existence of ℓ_1 and ℓ_2 could not be assured.) From the simple connectivity of $f_0(U)$, ℓ_1 and ℓ_2 can be connected to the point -1 by two curves λ_1, λ_2 lying in $U \setminus f_0(U)$ so that $\{x: x \leq -1\} \cup \lambda_1 \cup \lambda_2 \cup \ell_1 \cup \ell_2$ bounds a simply connected domain Ω with corresponding mapping function g such that $g(U) = \Omega \supset f_0(U)$. Since ℓ_1 and ℓ_2 are straight line segments it is clear that $g'(z) \to \infty$ as $z \to \zeta$, $z \in \Gamma$. It follows from the remark following Theorem A that f would have an infinite angular derivative at an interior cusp on γ.

To show that the condition obtained for γ would not allow for a cusp, we use a sequence of classical results. First consider the function $u(z) = \operatorname{Re} \log f_0'(z)$. Since f_0 is in S there exists an analytic branch of the log defined for f_0' so that $u(z)$ is an harmonic function with its harmonic conjugate given by a branch of $v(z) = \operatorname{Im} \log f_0'(z)$. Since f_0 is circularly symmetric, Jenkin's results in [9] show that $zf_0'(z)/f_0(z)$ and f_0 define typically real functions upon renormalization, so that $\log f_0'(z) = \log[zf_0'(z)/f_0(z)] + \log[f_0(z)/z]$ has its imaginary part bounded. Thus $v(z) = \operatorname{Im} \log f'(z)$ defines a function in h_p, $1 \leq p < \infty$, the class of functions $u(r,\theta)$ that are harmonic in U and have $\int_0^{2\pi} |u(r,\theta)|^p d\theta < \infty$ for $0 < r < 1$. From Riesz's theorem [6] it follows that since v is in h_p for $p > 1$, its harmonic conjugate u is in h_p for $p > 1$. Thus, we have by [6], that u has a Poisson integral representation given by

$$u(z) = u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} g(t)P(r,t)dt, \quad 0 \leq r < 1 \tag{2}$$

for the Poisson kernel $P(r,t)$ where $g(t)$ is a boundary value for u at each point of continuity of g. However, we have the following result from Tsuji [13], at points of discontinuity of g.

LEMMA 3. Suppose g is discontinuous at t_0, such that $g(t_0 + 0)$, $g(t_0 - 0)$ exist. Let $\ell_\psi(e^{it_0})$ be a segment through e^{it_0}, making an angle $\psi (0 < \psi < \pi)$ with the positive tangent of $|z| = 1$ at e^{it_0}. If $z \to e^{it_0}$ along $\ell_\psi(e^{it_0})$, then u defined by (2) has the property that

$$u(z) \to g(t_0 + 0) + (\psi/\pi)(g(t_0 + 0) - g(t_0 - 0)).$$

Hence with $|\zeta f'(\zeta)|$ being piecewise constant, the angular derivative, although not existing as an explicit limit, could not become infinite as z approached any point on $f_0^{-1}(\gamma)$. Therefore γ has no cusps or corners as claimed.

To complete the proof of the theorem we need to find the range for the constant value $c = |zf_0'(z)|$ for $z \in f_0^{-1}(\gamma)$. We observe from Jenkins' characterization of circularly symmetric functions in [9] that $f_0(z) = z + a_2z^2 + \ldots$ being circularly symmetric assures that $g(z) = [zf_0'(z)/f_0(z) - 1]/a_2 = z + \ldots$ defines a typically real function. Thus $g(z) = [z/(1-z^2)]/p(z)$, where p has positive real part and real coefficients. Since a function that is circularly symmetric with respect to the positive reals takes its minimum modulus at $z = -r$, $0 < r < 1$, we have $(-r)f_0'(-r)/f_0(-r) > 0$ while $[(-r)f_0'(-r)/f_0(-r) - 1]/a_2 = [-r/(1-r^2)]p(-r) < 0$. Letting $r \to 1$ we have that

$$0 < \left|\frac{zf_0'(z)}{f_0(z)}\right|_{z=-1} = \frac{-f_0'(-1)}{f_0(-1)} \leq 1.$$

Thus, since Schwarz's lemma assures that $|f_0(-1)| \neq 1$, we have that for $z \in f^{-1}(\gamma) \Rightarrow c = |zf_0'(z)| = |-f_0'(-1)| \leq |f_0(-1)| < 1$ and the theorem is proved.

This problem has been solved independently by John Lewis using different methods.

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