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Möbius Transformations of Starlike Mappings

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A variational method is developed for a family of normalized Möbius transformations of starlike mappings. It is applied to a class of functionals with the result that extremal functions have an elementary form. As a special case, the radius of starlikeness of this family is determined to be approximately 0.6759.

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1. INTRODUCTION

Let S denote the familiar class of normalized analytic univalent functions f defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the unit disk $\mathbb{D} = \{z: |z| < 1\}$. The starlike subclass S^* consists of those functions $f \in S$ such that $f(\mathbb{D})$ is starlike with respect to the origin.

If f is in S and $w \notin f(\mathbb{D})$, then the function

$$\hat{f} = f/(1 - f/w) \tag{1.1}$$

belongs again to S. The elementary transformation $f \to \hat{f}$ is important in the study of univalent functions. It is a useful technique in the proofs of many properties of S.

If F is a subset of S, let

$$\hat{F} = \{\hat{f}: f \in F \text{ and } w \in \mathbb{C}^* \setminus f(\mathbb{D})\}$$

where $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. It is clear that $F \subset \hat{F} \subset S$ and $\hat{\hat{F}} = \hat{F}$. If F is compact in the topology of locally uniform convergence, then so is $\hat{F}$. If F is rotationally invariant (i.e., $e^{-i\alpha} f(e^{i\alpha} z)$ belongs to F whenever $f \in F$ and $\alpha \in \mathbb{R}$), then $\hat{F}$ is also rotationally invariant. It is an interesting question to ask which properties of F are inherited by $\hat{F}$. Since $\hat{S} = S$, this question is trivial for S.

In [2], [3] we studied the class $\hat{K}$ where K is the convex subclass of S. We developed a variational procedure for $\hat{K}$ and applied it to extremal problems that included the coefficient functionals a_2 and $ta_3 + a_2^2$, the point-evaluation functionals $\mathrm{Re}\{\varphi(\log[\hat{f}(z)/z])\}$ where φ is entire, the Koebe disk, and the radius of convexity.

In this article we shall consider the class $\hat{S}^*$. Simple examples show that $\hat{S}^*$ is strictly larger than S^*. The nth coefficient problem for $\hat{S}^*$ is solved by de Branges' proof of the Bieberbach conjecture since the extremal function $k(z)=z(1-z)^{-2}$ is starlike. On the other hand, there are many interesting problems in function theory for which the extremal function over S is not starlike or even in $\hat{S}^*$. We shall restrict our attention to such problems.

The purpose of this article is to develop a variational method extending the procedure in [2] to a class of extremal problems for $\hat{S}^*$. In [2] the procedure basically involved local variations of only circular arcs. In the current setting we shall need a more general method involving a combination of the Löwner theory to vary tips of slits and the Julia variational formula to vary circular slits in the plane. For our purposes this framework appears to be superior to other methods since it permits us to preserve certain geometric properties when the family of mappings does not seem to admit a useful structural formula.

We shall apply our method to the functional $\lambda: \hat{S}^* \to \mathbb{R}$ defined by $\lambda(\hat{f}) = \operatorname{Re}\{\varphi(\log[z\hat{f}'(z)/\hat{f}(z)])\}$ for any fixed nonconstant entire function φ and any fixed $z \in \mathbb{D}$. We shall show that the problem

$$\max_{\hat{S}^*} \operatorname{Re}\{\varphi(\log[z\hat{f}'(z)/\hat{f}(z)])\}$$

has a relatively elementary extremal function $\hat{f}$. More specifically, in Section 4 we shall show that $\hat{f}$ is generated through (1.1) by a mapping $f \in S^*$ with at most two radial slits and, furthermore, that w is the tip of one of the slits. As an application, in Section 5 we indicate the numerical solution to the problem $\max_{\hat{S}^*} \arg[z\hat{f}'(z)/\hat{f}(z)]$. This permits us to determine numerically the radius of starlikeness of the class $\hat{S}^*$ to be approximately 0.6759. Indeed, this was our motivation in beginning this study.

Our methods are applicable to other functionals as well. For example, it is even easier to study the point-evaluation functional $\lambda(\hat{f}) = \operatorname{Re}\{\varphi(\log[\hat{f}(z)/z])\}$. It came as a surprise for us that our extended methods enabled us to show that a class of functionals involving even the derivative leads to such elementary extremal functions. Normally, extremal functions for such functionals possess several more parameters.

2. VARIATION OF SLIT MAPPINGS

In this section we shall discuss how the Löwner theory and the Julia variational formula can be used to produce variations within a class of functions.

Let f belong to S and map $\mathbb{D}$ onto a domain Ω whose boundary Γ contains a piecewise analytic slit T with endpoint w_0. Let $w_0 = f(e^{i\theta_0})$. We shall construct from f a new function by either extending or shortening T in a specific manner or by displacing T locally about w_0.

First, to extend T, let T_1 be an analytic slit containing T with new endpoint w_1. Let γ be the arc in $\mathbb{D}$ such that $f(\gamma) = T_1 \setminus T$. Parametrize γ by $w = w(t)$, $t_1 \leqslant t < 0$, and let $w(0) = \lim_{t \nearrow 0} w(t) = e^{i\theta_0}$. Denote by $g(\cdot, t)$ the function that maps $\mathbb{D}$ conformally onto $\mathbb{D}$ minus the arc $w([t,0))$ with normalizations $g(0,t)=0$ and $g'(0,t)>0$. Then $g(z,0)=z$ and we may choose our parametrization (cf. [6]) so that $g'(0,t)=e^t$ and that g satisfies the Löwner differential equation in the form

$$\frac{\partial g(z,t)}{\partial t}=z\,\frac{\partial g(z,t)}{\partial z}\,\frac{1+e^{-i\theta(t)}z}{1-e^{-i\theta(t)}z}. \tag{2.1}$$

The function θ is a continuous function of t, and $g(e^{i\theta(t)},t)$ is the point $w(t)$. Consider the function $F(\cdot,t)$ defined by the composition $F(z,t)=e^{-t}f(g(z,t))$. It is clear that $F(\cdot,t)$ is in S for each $t\in[t_1,0]$. Furthermore, with the help of (2.1) we see that F has the asymptotic expansion

$$\begin{aligned} F(z,t)&=f(z)-\left[f(z)-f'(z)\frac{\partial g}{\partial t}(z,0)\right]t+o(t)\\ &=f(z)-\left[f(z)-zf'(z)\frac{1+e^{-i\theta_0}z}{1-e^{-i\theta_0}z}\right]t+o(t) \end{aligned} \tag{2.2}$$

as $t\nearrow 0$. That the remaining terms are $o(t)$, uniformly on compact subsets of $\mathbb{D}$, follows from Tsuji's work in [7, ch. IX].

Second, to shorten the slit, let T be parameterized by $w=w(t)$, $0<t\leqslant t_2$, with $w(0)=\lim_{t\searrow 0} w(t)=e^{i\theta_0}$. Denote by $g(\cdot,t)$ the function that has the normalizations $g(0,t)=0$ and $g'(0,t)>0$ and maps $\mathbb{D}$ conformally onto $f(\mathbb{D})$ except that the slit whose endpoint was at $w(0)$ is now shortened so that its endpoint is at $w(t)$. Then $g(z,0)=f(z)$ and we may choose our parametrization so that $g'(0,t)=e^t$ and that g satisfies a Löwner differential equation in the form (2.1). It is evident that the function $F(\cdot,t)$ defined by $F(z,t)=e^{-t}g(z,t)$ is in S for each $t\in[0,t_2]$, and the asymptotic expansion (2.2) is again valid as $t\searrow 0$.

Thus we have defined variations of f within S with the expansion (2.2) as $t\to 0$. Negative t's correspond to extending the slit, and positive t's to shortening it.

An additional variation will be necessary for our purposes. In the rest of this section we will study the effect of displacing a slit locally near an endpoint. To do this, we consider the slit $T\subset\partial\Omega$ as having two sides. The sides of T are viewed as the images under f of distinct arcs γ_1 and γ_2 on $\partial\mathbb{D}$ with common endpoint $e^{i\theta_0}$, the preimage of the tip of T.

Let $\mathbf{n}(w)$ be the unit exterior normal to the boundary of Ω at $w=f(e^{i\theta})$, where $e^{i\theta}$ is in $\gamma=\gamma_1\cup\gamma_2$. Let $\varphi(w)$ be a real-valued, continuous, piecewise continuously differentiable function on $\Gamma=\partial\Omega$ that vanishes outside a compact subset of T and at points of nonanalyticity of T, including its endpoint. We permit φ to be defined differently on the two sides of T, but whenever $f(z)=f(\zeta)$ for $z\in\gamma_1$ and $\zeta\in\gamma_2$, we require that $\varphi\circ f(z)=-\varphi\circ f(\zeta)$. Thus if ε is positive and sufficiently small, then $w^*=w+\varepsilon\varphi(w)\mathbf{n}(w)$ maps Γ homeomorphically onto an arc Γ^*, which is the boundary of a domain Ω^*. It follows from Julia's work [4] that the Riemann mapping function g^* of $\mathbb{D}$ onto Ω^*, with $g^*(0)=0$ and $g^{*\prime}(0)>0$, is given by

$$g^*(z)=f(z)+\frac{\varepsilon zf'(z)}{2\pi i}\int_\Gamma\frac{\zeta+z}{\zeta-z}\,\frac{\varphi(\omega)\mathbf{n}(\omega)}{[\zeta f'(\zeta)]^2}\,d\omega+o(\varepsilon) \tag{2.3}$$

as $\varepsilon \searrow 0$. Here $\omega = f(\zeta)$ and the $o(\varepsilon)$ term is uniform for z in compact subsets of $\mathbb{D}$. In our development we shall apply this formula to obtain variations of certain curvilinear regions.

It will be convenient to denote

$$d\psi = \frac{\varphi(\omega)\mathbf{n}(\omega)}{i[\zeta f'(\zeta)]^2}\, d\omega = \frac{\varphi(\omega)}{|\zeta f'(\zeta)|}\, d\vartheta \tag{2.4}$$

where $\zeta = e^{i\vartheta}$. Then it is apparent that $d\psi$ is real and that formula (2.3) becomes

$$g^*(z) = f(z) + \frac{\varepsilon z f'(z)}{2\pi} \int_\Gamma \frac{\zeta + z}{\zeta - z}\, d\psi + o(\varepsilon).$$

Since

$$g^{*\prime}(0) = 1 + \frac{\varepsilon}{2\pi} \int_\Gamma d\psi + o(\varepsilon),$$

the function $f^*(\cdot) = g^*(\cdot)/g^{*\prime}(0)$ belongs again to S and has the asymptotic form

$$f^*(z) = f(z) + \frac{\varepsilon}{2\pi}\left[z f'(z) \int_\Gamma \frac{\zeta + z}{\zeta - z}\, d\psi - f(z) \int_\Gamma d\psi \right] + o(\varepsilon). \tag{2.5}$$

3. VARIATIONS FOR A DENSE SUBSET OF $\hat{S}^*$

For $n = 1, 2, 3, \ldots$ let S_n^* consist of those mappings f in S^* whose complement consists of at most n radial slits. By discretely approximating the measure in the representation

$$f(z) = z \exp\left\{ -2 \int_{|\eta|=1} \log(1 - \eta z)\, d\mu \right\}, \qquad \int_{|\eta|=1} d\mu = 1,$$

for functions in S^*, it is easy to see that $\bigcup_{n=1}^{\infty} S_n^*$ is dense in S^*, and it follows that $\bigcup_{n=1}^{\infty} \hat{S}_n^*$ is dense in $\hat{S}^*$. Furthermore, the sets S_n^* and $\hat{S}_n^*$ are compact.

Functions $\hat{f}$ in $\hat{S}_n^*$ map $\mathbb{D}$ onto the complement of at most n circular slits in the Riemann sphere $\mathbb{C}^*$. In particular, if $\hat{f} = f/(1 - f/w_0)$, then the slits of $\mathbb{C} \setminus \hat{f}(\mathbb{D})$ all lie on circles through the point $w_1 = -w_0$ and the origin (see Figure 1). In fact, these properties characterize functions in $\hat{S}_n^*$. That is, if $g \in S$ and if $\mathbb{C}^* \setminus g(\mathbb{D})$ consists of at most n slits which lie on circles through a point $w_1 \notin g(\mathbb{D})$ and the origin, then $f = g/(1 - g/w_1)$ belongs to S_n^* and so $g = \hat{f} = f/(1 - f/w_0)$, $w_0 = -w_1$, belongs to $\hat{S}_n^*$.

In the rest of this section we shall construct variations within $\hat{S}_n^*$ by rotating a circular (or linear) slit that passes through the common point w_1 to a slit lying on a nearby circle that passes through w_1 and the origin. After renormalizing, the variations will be of the form (2.5).

Let $f \in \hat{S}_n^*$ and $\Omega = f(\mathbb{D})$, and assume that $\Gamma = \partial\Omega$ contains the circular arc T lying on the circle C passing through origin and the point w_1. Let $\overline{AB}$ and $\overline{BD}$ be the two

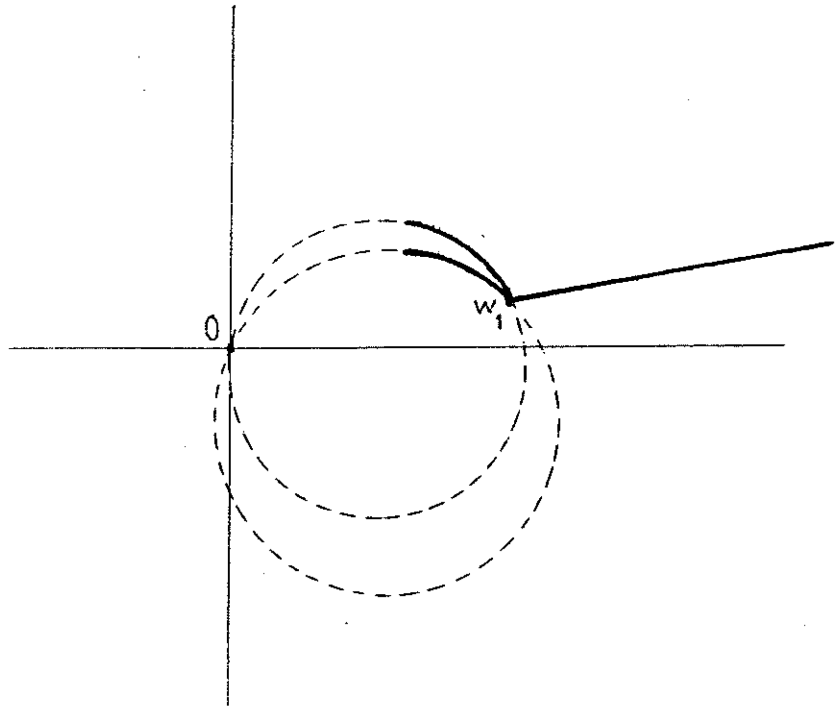

FIGURE 1.

sides of T and γ_{AB} and γ_{BD} the corresponding arcs on $\partial\mathbb{D}$ such that $f(\gamma_{AB}) = \overline{AB}$ and $f(\gamma_{BD}) = \overline{BD}$. For convenience of notation, for $w \in T$ let $w = f(z) = f(\zeta)$ where $z \in \gamma_{AB}$ and $\zeta \in \gamma_{BD}$.

We shall construct a variation $w^* = w + \varepsilon\varphi(w)\mathbf{n}(w)$ of T by defining φ at $w = f(z) = f(\zeta) \in \Gamma$ so that $\varphi \circ f(z) = -\varphi \circ f(\zeta)$. The sign of φ will be determined by the direction in which T is to be moved. For $w \in \Gamma \setminus T$ the function φ will be zero. Note that the circle C on which T lies belongs to the family of circles passing through the origin and w_1. The variations to be constructed will be obtained by displacing the slit T to slits T^* lying on neighboring circles through the origin and w_1, by considering an adjustment at the endpoint, and finally, by renormalizing.

To define φ explicitly, let C^* be a circle through the origin and w_1 a point that is sufficiently close to C. Let the ray from the center of circle C through B intersect C^* at B^*. The points of T^* are constructed in the same fashion. Then the new slit T^* lies on C^* and connects w_1 to B^*. The varied boundary Γ^* is then $(\Gamma \setminus T) \cup T^*$, and the varied domain is $\Omega^* = \mathbb{C} \setminus \Gamma^*$. However, in order to obtain a variational formula of the form (2.3), an additional construction is necessary. Its purpose is to make φ vanish continuously also at the tip of T. Let B_1^* be a point on C^* near B^* such that the ray from the center of C intersects the slit T. Let T_1^* be the circular slit lying in T^* that connects w_1 to B_1^*. Let L_1^* be the straight line segment joining B_1^* to B. Now to each point $w = f(z) = f(\zeta)$ on T we associate a point w^* on the ray through w from the center of C. The point $w^* = w^*(w)$ is the intersection of this ray with the slit T_1^* if defined, and otherwise it is the intersection of this ray with the segment L_1^*. On the rest of Γ define $w^*(w) = w$.

If C^* is sufficiently close to C, then the normal displacement in one direction is given by

$$\varepsilon\varphi(w) = \begin{cases} |w^*(w) - w| = |w^*[f(z)] - f(z)| & \text{for} \quad z \in \gamma_{AB} \\ -|w^*(w) - w| = -|w^*[f(\zeta)] - f(\zeta)| & \text{for} \quad \zeta \in \gamma_{BD} \\ 0 & \text{on the rest of } \Gamma. \end{cases}$$

The signs are reversed for displacement in the opposite direction. The resulting φ is continuous, piecewise continuously differentiable, and vanishes outside of T and at the endpoints of T. Thus Julia's variational formula (2.3) applies to the mapping function g_0^* onto the varied domain. However, the normalized function $g_0^*(\cdot)/g_0^{*\prime}(0)$ does not necessarily belong to $\hat{S}_n^*$ because of the adjustment just made near the endpoint in order to use Julia's formula. Nevertheless, if g^* denotes the mapping onto the desired domain Ω^* before the adjustment, then the argument used in [1, pp. 348–356] shows that $g^* - g_0^* = o(\varepsilon)$ uniformly in compact subsets of $\mathbb{D}$ as $\varepsilon \searrow 0$ (see also [2, p. 63]). Consequently, the function f^* defined by $f^*(z) = g^*(z)/g^{*\prime}(0)$ does belong to $\hat{S}_n^*$ and also admits an asymptotic development of the form (2.5).

4. MAIN THEOREMS AND PROOF

We now state and prove our main results. For a fixed $z \in \mathbb{D}$, let the functional λ be defined for $\hat{f} \in \hat{S}^*$ by

$$\lambda(\hat{f}) = \operatorname{Re}\{\varphi(\log[z\hat{f}'(z)/\hat{f}(z)])\} \tag{4.1}$$

where φ is a given nonconstant entire function. To be consistent, we choose the branch of the logarithm for which $\log[\varkappa\hat{f}'(\varkappa)/\hat{f}(\varkappa)]$ vanishes at $\varkappa = 0$. First, we shall consider the problem

$$\max_{\hat{S}_n^*} \lambda. \tag{4.2}$$

Since λ is continuous and $\hat{S}_n^*$ is compact, an extremal function exists within $\hat{S}_n^*$. The following lemma describes its properties.

LEMMA 4.1 *Suppose that $n \geq 2$, and that λ assumes its maximum over $\hat{S}_n^*$ at $\hat{f} = f/(1 - f/w)$, $f \in S_n^*$. Then f belongs to S_2^*. That is, f maps $\mathbb{D}$ onto the complement of at most two radial slits, and furthermore, w is at the tip of one of the slits.*

As a consequence of Lemma 4.1 there is a common solution to the problem (4.2) for all $n \geq 2$. Since λ is continuous and $\bigcup_{n=1}^{\infty} \hat{S}_n^*$ is dense in $\hat{S}^*$, it follows that this same function solves the problem

$$\max_{\hat{S}^*} \lambda. \tag{4.3}$$

Notice, however, that this limiting procedure does not prevent the possibility of additional extremal functions in $\hat{S}^* \setminus \bigcup_{n=1}^{\infty} \hat{S}_n^*$. Before proving Lemma 4.1 we record this consequence.

THEOREM 4.2 *Let φ be a nonconstant entire function and z a fixed point of $\mathbb{D}$. Then the maximum value for the functional* $\operatorname{Re}\{\varphi(\log[z\hat{f}'(z)/\hat{f}(z)])\}$ *over $\hat{S}^*$ is assumed at a function $\hat{f} = f/(1 - f/w)$ where f is in S_2^*. That is, f maps $\mathbb{D}$ onto the complement of at most two radial slits, and furthermore, w is at the tip of one of the slits.*

Proof of Lemma 4.1 In order to verify the result, we need to show that if $\hat{f}$ is an extremal function for the problem (4.2), then $\hat{f}$ maps $\mathbb{D}$ onto the complement of a slit made up of a radial ray from some point w_1 to infinity and, possibly, an arc from w_1 that lies on some circle through w_1 and the origin. In particular, the slit has only one finite endpoint. We will show even more, namely, that the preimage of this tip has to satisfy a certain condition.

To determine this condition for an extremal function $\hat{f}$, we assume that $\mathbb{C}\setminus\hat{f}(\mathbb{D})$ contains an arc T with endpoint $w_0 = f(e^{i\theta_0})$, lengthen or shorten T as described in Section 2, and consider its effect on the functional λ. If T is varied along its circle, then the resulting normalized function $F(\cdot, t)$ remains in $\hat{S}_n^*$, and from (2.2) it has the asymptotic form

$$F(z,t) = \hat{f}(z) - \left[\hat{f}(z) - z\hat{f}'(z)\frac{1 + e^{-i\theta_0}z}{1 - e^{-i\theta_0}z}\right]t + o(t)$$

as $t \to 0$ through both positive and negative values. Its logarithmic derivative is of the form

$$\frac{zF'(z,t)}{F(z,t)} = q(z) + z\left[q(z)\frac{1 + e^{-i\theta_0}z}{1 - e^{-i\theta_0}z}\right]' t + o(t)$$

where $q(z) = z\hat{f}'(z)/\hat{f}(z)$. Let

$$K(z,\zeta) = z\left[q(z)\frac{1 + \bar{\zeta}z}{1 - \bar{\zeta}z}\right]' \Big/ q(z) = \frac{2\bar{\zeta}z}{(1 - \bar{\zeta}z)^2} + \frac{zq'(z)}{q(z)}\frac{1 + \bar{\zeta}z}{1 - \bar{\zeta}z}. \tag{4.4}$$

Then we have

$$\log\frac{zF'(z,t)}{F(z,t)} = \log q(z) + K(z, e^{i\theta_0})t + o(t),$$

$$\varphi\left(\log\frac{zF'(z,t)}{F(z,t)}\right) = \varphi(\log q(z)) + \varphi'(\log q(z))K(z, e^{i\theta_0})t + o(t),$$

and

$$\operatorname{Re}\left\{\varphi\left(\log\frac{zF'(z,t)}{F(z,t)}\right)\right\} = \lambda(\hat{f}) + \operatorname{Re}\{\varphi'(\log q(z))K(z, e^{i\theta_0})\}t + o(t)$$

as $t \to 0$. Since λ is a maximum at $\hat{f}$, it follows that

$$\operatorname{Re}\{\varphi'(\log q(z))K(z, e^{i\theta_0})\} = 0. \tag{4.5}$$

It will be useful to have a little more information about one of the factors in the expressions derived from (4.4). For this purpose we use the fact that $\hat{S}_n^*$ is rotationally invariant. Thus the rotation $e^{-i\alpha}\hat{f}(e^{i\alpha}z)$ is a competitor with the extremal function $\hat{f}$. As a consequence, $\operatorname{Re}\{\varphi(\log[e^{i\alpha}z\hat{f}'(e^{i\alpha}z)/\hat{f}(e^{i\alpha}z)])\}$ is a maximum for $\alpha=0$. This implies that

$$0=\frac{\partial}{\partial\alpha}\operatorname{Re}\{\varphi(\log q(e^{i\alpha}z))\}\bigg|_{\alpha=0}=\operatorname{Re}\{\varphi'(\log q(z))izq'(z)/q(z)\}.$$

That is, we may write

$$\varphi'(\log q(z))zq'(z)/q(z)=R(z) \tag{4.6}$$

where $R(z)$ is real at the point z for which the functional λ is defined.

Next we shall show that an extremal function $\hat{f}$ has the property that $\mathbb{C}\backslash\hat{f}(\mathbb{D})$ contains at most two tips. To do so, we shall show that the condition (4.5) can hold in at most two points $\zeta=e^{i\theta_0}$ on the unit circle. Indeed, let $\xi=\bar{\zeta}z$ and use (4.4) and (4.6). Then

$$\begin{aligned}\operatorname{Re}\{\varphi'(\log q(z))K(z,\zeta)\}&=\operatorname{Re}\left\{\varphi'(\log q(z))\frac{2\xi}{(1-\xi)^2}\right\}+R(z)\operatorname{Re}\left\{\frac{1+\xi}{1-\xi}\right\}\\&=\frac{1-|\xi|^2}{|1-\xi|^2}\left[\operatorname{Re}\left\{2\varphi'(\log q(z))\frac{\xi-|\xi|^2}{(1-|\xi|^2)(1-\xi)}\right\}+R(z)\right].\end{aligned}$$

As ζ traverses the unit circle, ξ traverses the circle $|\xi|=r$ where $r=|z|$. At the same time, the Möbius transform $\dfrac{\xi-r^2}{(1-r^2)(1-\xi)}$ also traverses a circle. If $\varphi'(\log q(z))\neq 0$, it is a consequence that the factor $\operatorname{Re}\left\{2\varphi'(\log q(z))\dfrac{\xi-|\xi|^2}{(1-|\xi|^2)(1-\xi)}\right\}+R(z)$ has at most two zeros, and so $\operatorname{Re}\{\varphi'(\log q(z))K(z,\zeta)\}$ has the same property. Therefore $\mathbb{C}\backslash\hat{f}(\mathbb{D})$ contains at most two tips. We shall relax the assumption that $\varphi'(\log q(z))\neq 0$ later.

Now we suppose for the purpose of contradiction that an extremal function $\hat{f}$ has the property that $\mathbb{C}\backslash\hat{f}(\mathbb{D})$ contains two tips. In this case we shall make a variation of $\hat{f}$ as in Section 3 by displacing one of the circular slits through the common point w_1 to a nearby circular arc lying on another circle through w_1 and the origin. Let the slit be denoted by $T=\hat{f}(\gamma_1)=\hat{f}(\gamma_2)$ where $\gamma=\gamma_1\cup\gamma_2$ lies on the unit circle, $\gamma_1\cap\gamma_2=\{e^{i\theta_0}\}$, and $\hat{f}(e^{i\theta_0})$ is the tip of T. By making this variation and then renormalizing, it follows from (2.5) that the resulting function F_ε will belong again to S_n^* and will have the asymptotic form

$$F_\varepsilon(z)=\hat{f}(z)+\frac{\varepsilon}{2\pi}\left[z\hat{f}'(z)\int_\gamma\frac{1+\bar{\zeta}z}{1-\bar{\zeta}z}\,d\psi-\hat{f}(z)\int_\gamma d\psi\right]+o(\varepsilon)$$

as $\varepsilon\searrow 0$. The direction of the displacement is incorporated into $d\psi$ by the sign of φ in the definition (2.4) of $d\psi$. To determine the effect on the functional λ, we find as before

$$\frac{zF_\varepsilon'(z)}{F_\varepsilon(z)} = q(z) + \frac{\varepsilon z}{2\pi}\int_\gamma \left[q(z)\frac{1+\bar{\zeta}z}{1-\bar{\zeta}z}\right]' d\psi + o(\varepsilon),$$

$$\log\frac{zF_\varepsilon'(z)}{F_\varepsilon(z)} = \log q(z) + \frac{\varepsilon}{2\pi}\int_\gamma K(z,\zeta)\,d\psi + o(\varepsilon),$$

and

$$\operatorname{Re}\left\{\varphi\left(\log\frac{zF_\varepsilon'(z)}{F_\varepsilon(z)}\right)\right\} = \lambda(\hat{f}) + \operatorname{Re}\left\{\varphi'(\log q(z))\frac{\varepsilon}{2\pi}\int_\gamma K(z,\zeta)\,d\psi\right\} + o(\varepsilon)$$

where $q(z) = z\hat{f}'(z)/\hat{f}(z)$ and $K(z,\zeta)$ is defined in (4.4). Since λ is a maximum at $\hat{f}$, it follows that

$$\operatorname{Re}\left\{\varphi'(\log q(z))\int_\gamma K(z,\zeta)\,d\psi\right\} \leqslant 0. \tag{4.7}$$

From (4.5) we know that $\operatorname{Re}\{\varphi'(\log q(z))K(z,\zeta)\}$ vanishes at both points ζ corresponding to tips of slits, and by the previous paragraph, it cannot vanish elsewhere. As a consequence, $\operatorname{Re}\{\varphi'(\log q(z))K(z,\zeta)\}$ has a constant sign for $\zeta \in \gamma_1$ and a constant, but opposite, sign for $\zeta \in \gamma_2$. However, by varying in one direction or the other, that is, by choosing the sign of φ properly on γ_1 and necessarily opposite on γ_2, we can force each term in

$$\operatorname{Re}\left\{\varphi'(\log q(z))\int_\gamma K(z,\zeta)\,d\psi\right\} = \operatorname{Re}\left\{\varphi'(\log q(z))\int_{\gamma_1} K(z,\zeta)\,d\psi\right\} + \operatorname{Re}\left\{\varphi'(\log q(z))\int_{\gamma_2} K(z,\zeta)\,d\psi\right\}$$

to be positive, in violation of (4.7). This completes the proof of Lemma 4.1, except to relax the assumption that $\varphi'(\log q(z)) \neq 0$.

The argument given by W. E. Kirwan in [5] would imply that $\varphi'(\log q(z)) \neq 0$ if the family $\hat{S}_n^*$ were closed under the operation $f \to f_\zeta$ where $f_\zeta(z) = f(\zeta z)/\zeta$ for $0 < |\zeta| < 1$ and $f_0(z) = z$. Unfortunately, it is not. However, the family $\hat{S}^*$ is closed under this operation. Therefore extremal functions for the problem (4.3) have the corresponding property that $\varphi'(\log q(z)) \neq 0$. If $\varphi'(\log q(z))$ were zero for extremal functions to infinitely many of the problems (4.2), then by normal families and continuity this condition would persist for $\hat{S}^*$ and yield a contradiction. Thus Lemma 4.1 is proved for all n sufficiently large. But since the families $\hat{S}_n^*$ are increasing, the lemma is true also for all $n \geqslant 2$.

The same method applies to functionals of the form $\operatorname{Re}\{\varphi(\log[\hat{f}(z)/z])\}$. Since the proof is even easier, we omit it and only state the result.

THEOREM 4.3 *Let φ be a nonconstant entire function and z a fixed point of $\mathbb{D}$. Then the maximum value for the functional* $\operatorname{Re}\{\varphi(\log[\hat{f}(z)/z])\}$ *over $\hat{S}^*$ is assumed at a function $\hat{f} = f/(1 - f/w)$ where f is in S_2^*. That is, f maps $\mathbb{D}$ onto the complement of at most two radial slits, and furthermore, w is at the tip of one of the slits.*

5. APPLICATIONS

In this section we choose $\varphi(\varkappa) = -i\varkappa$ and apply Theorem 4.2 to the functional $\arg[z\hat{f}'(z)/\hat{f}(z)]$. Since this functional is rotationally invariant, it is no loss of generality to assume that $z \in \mathbb{D}$ is real and positive. That is, we shall consider the extremal problem

$$\max_{\hat{S}^*} \arg[r\hat{f}'(r)/\hat{f}(r)] \tag{5.1}$$

for fixed $r \in (0, 1)$. According to Theorem 4.2 we need to consider only mappings in $\hat{S}_2^*$.

The mappings in S_2^*, which map onto the complement of at most two radial slits, are of the form

$$f(z) = \frac{z}{(1-xz)^{\alpha}(1-yz)^{2-\alpha}} \tag{5.2}$$

where x and y are complex numbers of modulus one and $0 \leqslant \alpha \leqslant 2$. The points on $\partial\mathbb{D}$ that correspond to the tips of the slits can be determined by setting $f'(z) = 0$. They satisfy the quadratic equation

$$xyz^2 + (1-\alpha)(x-y)z - 1 = 0. \tag{5.3}$$

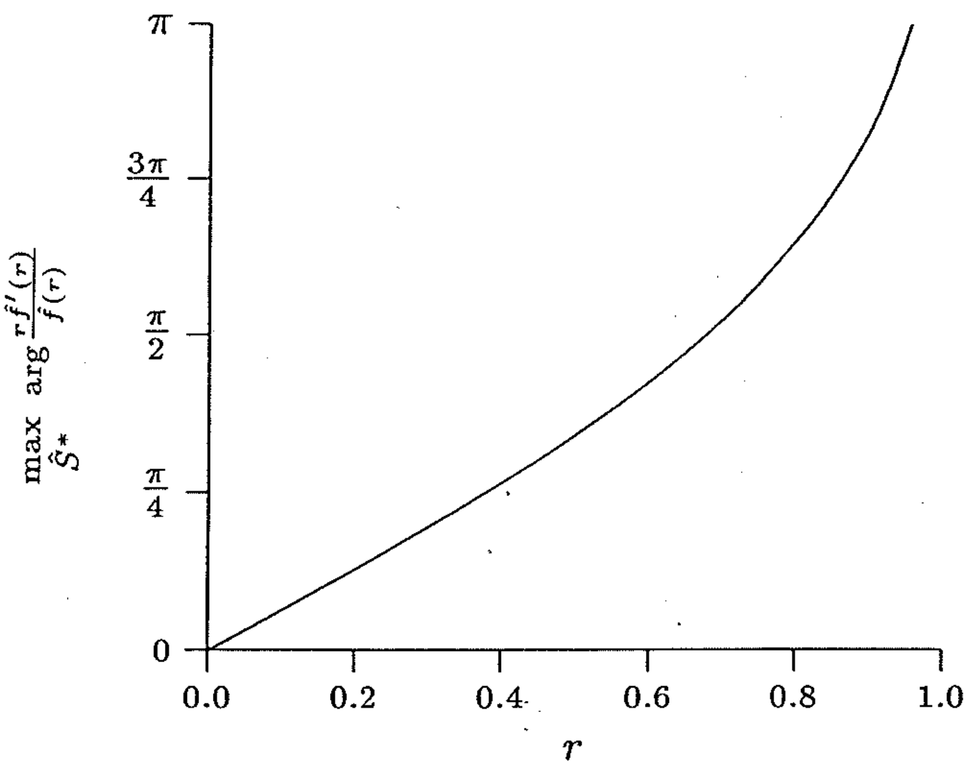

FIGURE 2.

In terms of the function f in (5.2) the functional in (5.1) is

$$\arg[r\hat{f}'(r)/\hat{f}(r)] = \arg\left[\frac{f'(r)}{f(r)[1-f(r)/f(\zeta)]}\right].$$

where ζ satisfies (5.3). The maximum of this expression over all x, y, and α does not seem to admit a closed or even implicit form. It is, however, easily computed numerically. These maxima as a function of r are indicated in Figure 2. The maximum of $\arg[\hat{f}(r)/r]$ over $\hat{S}^*$ as a function of r is also easily computed numerically. Its graph is very similar to, but slightly above, that in Figure 2.

Of special interest in Figure 2 is that value $r=r^*$ at which the curve reaches the height $\pi/2$. That value r^* is approximately 0.6759, which occurs for a mapping (5.2) with $x=\exp(0.2575i)$, $y=\exp(3.5132i)$, and $\alpha=0.4108$. Thus $\text{Re}\{z\hat{f}'(z)/\hat{f}(z)\} \geqslant 0$ is true in the disk $|z|<r^*$ for all functions $\hat{f}\in\hat{S}^*$. In other words, r^* is the sharp value such that $\hat{f}(|z|<r)$ is starlike with respect to the origin for all $r\leqslant r^*$ and all $f\in\hat{S}^*$. The number r^* is called the radius of starlikeness for the family $\hat{S}^*$. It is remarkably close to the radius of starlikeness $\tanh(\pi/4)\approx 0.6558$ for the full class S.

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