Möbius Transformations of Convex Mappings II

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For the class of functions referred to in the title, this article finds the Koebe disk, the radius of convexity, and sharp estimates for the coefficient functional \(|a_3 + a_2^2|\) for \(r\) in a certain interval.

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1. INTRODUCTION

Let \(S\) denote the class of analytic univalent functions \(f\) defined in the unit disk \(U = \{z: |z| < 1\}\) and normalized so that \(f(0) = f'(0) - 1 = 0\). The convex subclass \(K\) consists of those functions \(f \in S\) such that \(f(U)\) is a convex set.

If \(f \in S\) and \(w \notin f(U)\), then the function

\[
\hat{f} = f/(1 - f/w)
\]

(1.1)

belongs again to \(S\). The transformation \(f \to \hat{f}\) is a familiar one in the
study of univalent functions. If $F$ is a subset of $S$, let

$$
\hat{F} = \{ \hat{f} : f \in F \text{ and } w \in \mathbb{C} \setminus f(U)\}.
$$

Since $w = \infty$ belongs to $\mathbb{C}^* = \mathbb{C} \cup \{ \infty \}$, it follows that $F \subset \hat{F} \subset S$. Other obvious properties are that $\hat{F} = \hat{F}$ and $\hat{S} = S$. It is an interesting question to ask which properties of $F$ are inherited by $\hat{F}$. In this article we shall consider the class $\hat{K}$. The class $\hat{K}$ is compact, and simple examples show that $\hat{K} \neq K$.

In an earlier paper [1] we applied a variational procedure to a class of extremal problems for $\hat{K}$. If $\lambda : \hat{K} \to \mathbb{R}$ is a continuous functional that satisfies certain admissibility criteria, we showed that the problem

$$
\max_{\hat{K}} \lambda
$$

has a relatively elementary extremal function $\hat{f}$. More specifically, we showed that $\hat{f}$ either is a half-plane mapping

$$
\hat{f}(z) = z/(1 - e^{i\alpha}z)
$$

or is generated through (1.1) by a parallel strip mapping $f \in K$.

The class of functionals considered contained the second-coefficient functional $\hat{\lambda}(\hat{f}) = \Re a_2$ and the functionals $\lambda(\hat{f}) = \Re \Phi(\log|\hat{f}(z)/z|)$ where $\Phi$ is entire and $z \in U$ is fixed. The latter functionals include the problems of maximum and minimum modulus ($\Phi(w) = \pm w$). Therefore, for such problems it is necessary to test the functional only over Möbius transformations (1.2) and over functions $\hat{f}$ generated through (1.1) by strip mappings $f \in K$. We remark that the extremal strip domains $f(U)$ need not be symmetric about the origin. This adds an interesting and nontrivial character to the problems. In particular, in [1] an explicit determination was made of such an extremal function for the second-coefficient functional, and the sharp bound $|a_2| \leq 1.327 \ldots$ was obtained for $\hat{K}$.

In this paper similar techniques are used to find the Koebe disk for $\hat{K}$, that is, the largest domain centered at the origin always covered by $\hat{f}(U)$ for $\hat{f} \in \hat{K}$, the radius of convexity for the class $\hat{K}$, and sharp bounds for the functional

$$
\lambda_1(\hat{f}) = |ta_3 + a_2^2|
$$

for a certain range of $t$. A corollary is the solution of the third-coefficient problem for the inverses of functions in $K$. 

2. PRELIMINARIES

Since $\hat{K}$ is compact, a continuous functional

$$\hat{\lambda} : \hat{K} \to \mathbb{R}$$

will assume its maximum at some function $f \in \hat{K}$. We call $\hat{\lambda}$ admissible if it has the following properties. (i) At an extremal function $f$ it has an expansion of the form

$$\hat{\lambda}(f^*) = \hat{\lambda}(f) + \varepsilon \int_{|\xi| = 1} \sigma(\xi) \, d\psi(\xi) + o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0 \quad (2.1)$$

under Julia variations

$$f^*(z) = f(z) + \varepsilon \int_{|\xi| = 1} \left\{zf'(\xi)[(\xi + z)/(\xi - z)] - f(z)\right\} \, d\psi(\xi) + o(\varepsilon). \quad (2.2)$$

within $\hat{K}$, where $d\psi(\xi) = \left[\varphi(f(\xi))/|f'(\xi)|\right] d\xi/(i\xi)$ and $\varphi$ is a piecewise smooth, real-valued function. We require the function $\sigma$ to be continuous and to vanish at no more than two points of the circle $|\xi| = 1$. (ii) In addition, we require that there is a constant $c_f \neq 0$ such that

$$\hat{\lambda}(f/(1 - f/w)) = \hat{\lambda}(f) + \text{Re}\{c_f/w\} + o(1/w) \quad (2.3)$$

as $w \to \infty$ in $\mathbb{C} \setminus f(U)$.

By choosing appropriate variations we proved the following theorem in [1, Theorem 8.2].

**Theorem 1** If $\hat{\lambda}$ is an admissible functional, then $\hat{\lambda}$ assumes its maximum over $\hat{K}$ either at a half-plane mapping (1.2) or at a mapping generated through (1.1) by a parallel strip mapping $f \in K$ with $w$ a finite point of $\partial f(U)$.

The parallel strip mappings in $K$ are rotations $e^{-i\alpha} f_x(e^{i\alpha}z)$, $\alpha \in \mathbb{R}$, of the vertical strip mappings

$$f_x(z) = \left[1/(2i \sin x)\right] \log[(1 + e^{i\alpha}z)/(1 + e^{-i\alpha}z)], \quad 0 < x < \pi. \quad (2.4)$$

3. KOEBE DISK AND RADIUS OF CONVEXITY

It is an interesting problem to compare properties of the transformed class $\hat{K}$ with those of $K$. Of course, each $f \in K$ covers the disk $|w| < 1/2$. The following theorem determines the size of this Koebe disk for the class $\hat{K}$. 
Theorem 2 If \( f \in \tilde{K} \), then \( f(U) \) covers the disk \( |w| < \pi/8 \). Furthermore, \( \bigcap_{U \in \tilde{K}} f(U) = \{ w : |w| < \pi/8 \} \).

Proof We begin by showing first that if \( f \in \tilde{K} \), then

\[
|f(z)/z| \geq f_{\pi/2}(1)/[1 - f_{\pi/2}(1)/f_{\pi/2}(-1)] = \pi/8. \tag{3.1}
\]

The minimum modulus principle will then imply that the disk \( |w| < \pi/8 \) is contained in \( f(U) \). By considering rotations of \( \tilde{K} \), it follows that this disk is the largest set covered by all functions in \( \tilde{K} \).

In [1] we observed that for fixed \( z \in U \) the functional \( \lambda(f) = -\log |f(z)/z| \) is admissible. Thus, to obtain its maximum value, it is sufficient by Theorem 1 to examine the functions (1.2) and transforms of rotations of (2.4). A comparison with the functions \( \tilde{K} \) eliminates (1.2). Therefore

\[
|z/f(z)| \leq \max_{x \neq x'} |1/[e^{-ix} f_x(e^{ix})] - y/[e^{-ix} f_x(e^{iy})]| 
\]

\[
\leq \max_{y \neq x'} |1/f_x(e^{iy}) - 1/f_x(e^{iy})|
\]

by the maximum principle. For a fixed vertical strip, \( |1/f_x(e^{iy}) - 1/f_x(e^{iy})| \)

will be a maximum for \( y = 0 \) and \( \beta = \pi \). That is,

\[
|1/f_x(e^{iy}) - 1/f_x(e^{iy})| \leq |1/f_x(1) - 1/f_x(-1)| = 2\pi |\sin x|/[x(\pi - x)].
\]

Note that \( g(x) = (\sin x)/[x(\pi - x)] \) is symmetric about \( x = \pi/2 \). In addition, \( g'(x) = h(x)/[x(\pi - x)]^2 \) where \( h(x) = x(\pi - x) \cos x - (\pi - x^2) \sin x \). Since \( h'(x) = (\pi - \sqrt{\pi^2 - 8})/2 \), the function \( h \) increases from \( h(0) = 0 \) to \( h((\pi - \sqrt{\pi^2 - 8})/2) \) and then decreases to \( h(\pi/2) = 0 \). That is, the only critical point of \( g \) in \( (0, \pi) \) is at \( x = \pi/2 \). It provides the maximum of \( g \) since \( g''(\pi/2) < 0 \). Consequently, \( |z/f(z)| \leq 2\pi g(\pi/2) = 8/\pi \) and (3.1) is proved.

To obtain the radius of convexity for the class \( \tilde{K} \), we apply the Marty transformation and use the sharp bound for the second coefficient in \( \tilde{K} \).

Theorem 3 If \( f \in \tilde{K} \), then \( f(|z| < r) \) is convex for \( r \leq r_0 \approx 0.4547 \) where

\[ r_0 = A_2 - \sqrt{A_2^2 - 1}, \quad A_2 = (2/x_0) \sin x_0 - \cos x_0 \approx 1.3270, \quad x_0 \approx 2.0816 \]

is the unique solution of the equation \( \cot x = (1/x) - (x/2) \) in the interval \( (0, \pi) \). This result is sharp.
Proof. For any function \( f \in \hat{K} \) and any \( \zeta \in U \) the function
\[
F_z(z) = \left[ f((z + \zeta)/(1 + \bar{\zeta}z)) - f(\zeta) \right]/[f'(\zeta)(1 - |\zeta|^2)] = z + a_2(\zeta)z^2 + \cdots
\]
belongs again to \( \hat{K} \) since it is a Möbius transform of some function in \( K \). Therefore the coefficient
\[
\hat{a}_2(\zeta) = (1/2)(1 - |\zeta|^2)f''(\zeta)/f'(\zeta) - \bar{\zeta}
\]
has the bound \( A_2 \) in the statement of the theorem, by what was proved in [1, Theorem 9.1]. Thus we have
\[
|1 + \zeta f''(\zeta)/f'(\zeta) - (1 + |\zeta|^2)/(1 - |\zeta|^2)| \leq 2|\zeta|A_2/(1 - |\zeta|^2)
\]
and
\[
\text{Re}\{1 + \zeta f''(\zeta)/f'(\zeta)\} \geq (1 - 2|\zeta|A_2 + |\zeta|^2)/(1 - |\zeta|^2). \tag{3.2}
\]
The latter will be positive for \(|\zeta| < r_0 = A_2 - \sqrt{A_2^2 - 1}\).

If \( \zeta = -r_0 \) and \( f \in \hat{K} \) is chosen so that \( F_z(z) = f_{x_n}(z)/(1 - f_{x_n}(z)/f_{x_n}(1)) \), where \( f_{x_n} \) is defined by (2.4), then \( a_2(\zeta) = A_2 \) and both sides of (3.2) become zero. Therefore \( r_0 \) is the sharp radius of convexity for \( \hat{K} \).

4. THE FUNCTIONAL \( \lambda_t(f) = \|a_3 + a_2^2\| \)

In this section we shall apply Theorem 1 to give a sharp estimate for the functional (1.3) for \( t \) in a certain interval.

For \( t = -1 \) a sharp bound for the functional is already known. In fact, if \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \) belongs to \( K \) and \( \hat{f}(z) = z + \hat{a}_2z^2 + \hat{a}_3z^3 + \cdots \) is the transform (1.1), then \( \hat{a}_3 - \hat{a}_2^2 = a_3 - a_2^2 \). That is, for \( t = -1 \) the functional is invariant under our Möbius transformations. J. A. Hummel [2] proved that
\[
|a_3 - a_2^2| \leq 1/3 \tag{4.1}
\]
is a sharp estimate in the class \( K \), and so this sharp estimate persists for the class \( \hat{K} \).

The initial coefficients of the strip mapping (2.4) are \( a_2(x) = -\cos x \) and \( a_3(x) = (4/3)\cos^2 x - 1/3 \), and the initial coefficients of its transform \( \hat{f} \) are
\[
\hat{a}_2(x) = a_2(x) + 1/w \quad \text{and} \quad \hat{a}_3(x) = a_3(x) + 2a_2(x)/w + 1/w^2.
\]
In this case

\[ |\hat{a}_3(x) + \hat{a}_2(x)^2| = |-(t/3)\sin^2 x + (1 + t)(-\cos x + 1/w)^2|, \]

Points \( w \) on the boundary of the strip \( f(x) \) satisfy \( \Re w = f_x(1) = x/(2\sin x) \) or \( \Re w = f_x(-1) = (x - \pi)/(2\sin x) \). In these cases \( 1/w \) is of the form \( (1 + e^{i\theta})(\sin x)/x \) or \( (1 + e^{i\theta})(\sin x)/(x - \pi) \) for some real \( \theta \). If

\[ h(x, \theta, t) = -(t/3)\sin^2 x + (1 + t)[ -\cos x + (1 + e^{i\theta})(\sin x)/x]^2, \quad (4.2) \]

then \( t\hat{a}_3(x) + \hat{a}_2(x)^2 = h(x, \theta, t) \) or \( t\hat{a}_3(x) + \hat{a}_2(x)^2 = h(\pi - x, \theta, t) \). In particular,

\[ m(t) = \max_{0 \leq \theta \leq \pi} \max_{0 \leq \theta \leq \pi} |h(x, \theta, t)| \quad (4.3) \]

is a lower bound for the maximum of our functional for each \( t \). The following theorem shows that \( m(t) \) actually is the maximum of our functional for a certain range of \( t \).

**Theorem 4** If \( f(z) = z + a_2z^2 + a_3z^3 + \ldots \) belongs to \( \hat{K} \) and if \( -0.7652 \leq t \leq 1.682 \), then the estimate

\[ |ta_3 + a_2^2| \leq M(t) \]

holds where \( M(t) \) is the unique maximum of the function \( -(t/3)\sin^2 x + (1 + t)[2(\sin x)/x - \cos x]^2 \) on the interval \( \pi/2 < x < 3\pi/4 \), and this estimate is sharp.

The values of \( M(t) \) are easily obtained numerically: for example,

\[
\begin{align*}
M(-1.7) &\approx 0.7356 & M(0.1) &\approx 1.9120 & M(0.9) &\approx 3.1321 \\
M(-1.6) &\approx 0.8763 & M(1.1) &\approx 2.0636 & M(1.0) &\approx 3.2854 \\
M(-1.5) &\approx 1.0197 & M(1.2) &\approx 2.2155 & M(1.1) &\approx 3.4388 \\
M(-1.4) &\approx 1.1652 & M(1.4) &\approx 2.3677 & M(1.2) &\approx 3.5923 \\
M(-1.3) &\approx 1.3125 & M(1.5) &\approx 2.5202 & M(1.3) &\approx 3.7458 \\
M(-1.2) &\approx 1.4610 & M(1.6) &\approx 2.6729 & M(1.4) &\approx 3.8995 \\
M(-1.1) &\approx 1.6106 & M(1.7) &\approx 2.8259 & M(1.5) &\approx 4.0531 \\
M(0.0) &\approx 1.7610 & M(1.8) &\approx 2.9789 & M(1.6) &\approx 4.2069.
\end{align*}
\]

*An application* We postpone the proof of Theorem 4 for a moment and consider an application of Theorem 4. Let \( g \) be the inverse of a function \( f \in \hat{K} \), that is, \( g = f^{-1} \). Then

\[ g(w) = w + a_2w^2 + a_3w^3 + \ldots \quad (4.4) \]
in some neighborhood of \( w = 0 \). In view of Theorem 2 the series expansion (4.4) is valid at least for \( |w| < \pi / 8 \). In terms of the coefficients of \( f \), we have \( \alpha_2 = -a_2 \) and \( \alpha_3 = -a_3 + 2a_2^2 \). Therefore the initial coefficients of the inverse function have the sharp bounds \( |\alpha_2| \leq A_2 \) where \( A_2 \) is defined in Theorem 3 and

\[
|\alpha_3| \leq 2M(-1/2) \approx 2.0393.
\]

**Proof of Theorem 4** Since \( \bar{K} \) is preserved under the rotations \( f(z) \rightarrow e^{-4i\theta}e^{i\theta}z \), it is sufficient to consider the functional

\[
A_r(f) = \text{Re}\{ta_3 + a_2^2\}.
\]

(4.5)

In order to verify the admissibility condition (i), we shall use the formulas

\[
a_2^\pm = a_2 + e \int_{|z|=1} (a_2 + 2\bar{\zeta}) d\psi(\zeta) + o(\varepsilon)
\]

\[
a_3^\pm = a_3 + 2e \int_{|z|=1} (a_3 + 2a_2\bar{\zeta} + \bar{\zeta}^2) d\psi(\zeta) + o(\varepsilon)
\]

as \( \varepsilon \to 0 \) under the variations (2.2). Thus the functional (4.5) has the expansion (2.1) with

\[
\sigma(\zeta) = 2 \text{Re}\{t(a_3 + 2a_2\bar{\zeta} + \bar{\zeta}^2) + a_2(a_2 + 2\bar{\zeta})\}.
\]

We need to show that \( \sigma \) has at most two zeros on \( |\zeta| = 1 \). It is equivalent to show that the polynomial

\[
\zeta^2 \sigma(\zeta) = t\zeta^4 + 2(1 + t)a_2\zeta^3 + 2 \text{Re}\{t(a_3 + a_2^2)\zeta^2 + 2(1 + t)a_2\bar{\zeta} + t\}
\]

has at most two zeros on \( |\zeta| = 1 \). If \( t = 0 \), this is obvious. Assume therefore that \( t \neq 0 \). If more than two zeros were on \( |\zeta| = 1 \), then all four would be there since the product of the zeros equals one. In this case

\[
[(2/t)(1 + t)a_2]\zeta^2 - (4/t) \text{Re}\{t(a_3 + a_2^2)\} \leq 4
\]

(4.6)

since \( A^2 - 2B \) is the sum of the squares of the zeros of the polynomial \( \zeta^4 + A\zeta^3 + B\zeta^2 + C\zeta + D \). Substitute \( (1 + t)a_2 = (ta_3 + a_2^2) - t(a_2 - a_2^2) \) into (4.6); then

\[
(4/t^2) \text{Re}\{ta_3 + a_2^2\} - (4/t)(1 + t) \text{Re}\{a_3 - a_2^2\} \leq 4.
\]

Using the estimate (4.1), we can simplify this to

\[
\text{Re}\{ta_3 + a_2^2\} \leq t^2 + |t(1 + t)|/3.
\]

(4.7)
Due to the extreme nature of $\text{Re}\{ta_3 + a_2^2\}$, it would follow that $h(x, 0, t) \leq t^2 + |t(1 + t)|/3$ for every choice of $x$. If we can choose $x$ to violate this inequality, then the admissibility condition (i) will be satisfied. From a computer-assisted search, two good choices for $x$ appear to be $x = 1.7422$ and $x = 2.2297$. Then the inequality $h(1.7422, 0, t) > t^2 + |t(1 + t)|/3$ is satisfied at least for $-0.7652 \leq t < 0$, and the inequality $h(2.2297, 0, t) > t^2 + |t(1 + t)|/3$ is satisfied at least for $0 < t \leq 1.682$. Therefore (i) is satisfied for $-0.7652 \leq t \leq 1.682$, and this is assumed in the hypothesis.

In order to verify the admissibility condition (ii), we compute

$$\Lambda_t(f/(1 - f/w)) - \Lambda_t(f) = 2(t + 1) \text{Re}\{a_2/w\} + a(1/w)$$

as $w \to \infty$.

Since $t \neq -1$, the coefficient $c_f = 2(t + 1)a_2$ could be zero only if $a_2 = 0$. If this were the case for an extremal function, then (4.1) would imply that $\text{Re}\{ta_3 + a_2^2\} \leq |t|/3$. However, this inequality is violated whenever (4.7) is violated since $|t|/3 \leq t^2 + |t(1 + t)|/3$. Thus (ii) is satisfied.

Now Theorem 1 applies; that is, $|ta_3 + a_2^2|$ will be a maximum either at a half-plane mapping or at a mapping generated through (1.1) by a parallel strip mapping. Since this functional is invariant under rotations, it is sufficient to consider the strip mappings (2.4). Consequently, the maximum value of $|ta_3 + a_2^2|$ is $m(t)$, defined in (4.3), where the half-plane mappings correspond to $x = 0, \pi$ and the strip mappings to $0 < x < \pi$. The remainder of this proof concerns a more specific description of $m(t)$.

For $t \geq -1$, we may estimate

$$|h(x, \theta, t)| \leq (|t|/3) \sin^2 x + (1 + t)|((\sin x)/x - \cos x + e^{\theta}(\sin x)/x|^2$$

$$\leq (|t|/3) \sin^2 x + (1 + t)[2(\sin x)/x - \cos x]^2$$

since $(\sin x)/x - \cos x \geq 0$. At least for $0 \leq x \leq \pi/4$ this is a sum of increasing functions, and so $|h(x, \theta, t)| \leq |t|/6 + (1/2)(1 + t)(8/\pi - 1)^2$ whenever $0 \leq x \leq \pi/4$. One easily verifies that

$$|t|/6 + (1/2)(1 + t)(8/\pi - 1)^2 \leq -t/6 + (1/2)(1 + t)[8/(3\pi) + 1]^2 = h(3\pi/4, 0, t).$$

Therefore, for fixed $t$ the maximum of $|h(x, \theta, t)|$ occurs when $\pi/4 \leq x \leq \pi$.

As a function of $\theta$, the function $|h(x, \theta, t)|$ is of the form $(1 + t)a|b + 2c e^{\theta} + e^{2\theta}|$ where $a = [((\sin x)/x)]^2$ and $c = 1 - x \cot x$ are nonnegative, $b = -tx^2/[3(1 + t)] + c^2$ is real, and $-0.7652 \leq t \leq 1.682.$
It can be written as

\[(1 + t)u\sqrt{4c^2 + (1 - h)^2 + 4c(1 + h)\cos \theta + 4b \cos^2 \theta}.
\]

We wish to show that the maximum occurs for \(\theta = 0\). This is obvious if \(-.7652 \leq t \leq 0\); assume therefore that \(0 < t \leq 1.682\). It is sufficient to show that \(4c(1 + h)(\cos \theta - 1) + 4b(\cos^2 \theta - 1) \leq 0\) or that \(c(1 + b) + 2b \geq 0\). After multiplying by \((1 + t)/x^2\), we note that the latter inequality becomes

\[(c/x^2)\{(1 + c)^2 + t[(1 + c)^2 - x^2/3]\} - 2t/3 \geq 0.
\]

Since

\[c = x^2/3 + \sum_{k=2}^{\infty} \frac{|B_{2k}|(2x)^{2k}/(2k)!}{},\]

where the \(B_{2k}\) are Bernoulli numbers, the function

\[(c/x^2)\{(1 + c)^2 + t[(1 + c)^2 - x^2/3]\} - 2t/3
\]

is increasing at least for \(\pi/4 \leq x < \pi\). At \(x = \pi/4\) one verifies directly that this expression is positive. As a result, for each fixed \(t\) the maximum value of \(|h(x, 0, t)|\) occurs for \(\pi/4 \leq x \leq \pi\) and \(\theta = 0\).

Using the notation of the previous paragraph, we observe that \(h(x, 0, t) = a\{(1 + c)^2 + t[(1 + c)^2 - x^2/3]\}\) is positive for \(\pi/4 \leq x \leq \pi\), and so \(|h(x, 0, t)| = h(x, 0, t)\) over this interval. Next, we shall show that the maximum value of \(h(x, 0, t)\) over \(\pi/4 \leq x \leq \pi\) occurs in the smaller interval \(\pi/2 < x < 3\pi/4\).

The derivative \(H = (\partial h/\partial x)(x, 0, t)\) may be written as \(H = 2xa\{-t(1 - c)/3 + (1 + t)(1 + c)(1 - 2c/x^2)\}\) where \(a = (\sin^2 x)/x^2\) and \(c = 1 - x \cot x\) as before. For \(\pi/4 \leq x \leq \pi/2\), we have \(1 - \pi/4 \leq c \leq 1\) and \(c/x^2 \leq 4/\pi^2\). On this interval \(H\) is clearly positive of \(-.7652 \leq t \leq 0\). If \(t > 0\), then \(-t(1 - c)/3 + (1 + t)(1 + c)(1 - 2c/x^2) \geq -t\pi/12 + (1 + t)(2 - \pi/4)(1 - 8/\pi^2)\), which is positive for \(t \leq 1.682\). Therefore \(h(x, 0, t)\) does not assume a maximum in \(\pi/4 \leq x \leq \pi/2\).

The derivative \(H\) may also be written as \(H = 4(1 + t)G/x^3\) where \(G = (x/12)[18 - (3 + 4t)x^2/(1 + t)]\sin 2x - (x^2 - 1)\cos 2x - 1\). We shall show that \(H\), or equivalently \(G\), is negative for \(3\pi/4 \leq x \leq \pi\).

Since \(G\) is a monotone function of \(t\), it is sufficient to show that \(G\) is negative for \(t = -.7652\) and \(t = 1.682\). First, if \(t = -.7652\), then the first
two terms in $G$ are at most zero and the third term is negative. Second, if $t = 1.682$, then $\frac{\partial G}{\partial x} = (1466x^2/1341 - 1/2)\sin 2x - (2432x^2/4023 - 1)x\cos 2x$ and both terms give negative contributions on $3\pi/4 < x < \pi$; in addition, $G$ is negative at $x = 3\pi/4$. Therefore $h(x, 0, t)$ does not assume a maximum in $3\pi/4 < x < \pi$.

With the notation of the previous paragraph, it is easy to show that $(1 + t) \frac{\partial^2 G}{\partial x^2} = \frac{1}{2} + (3 + 4t)x^2/3] x\sin 2x + x^2\cos 2x$ is negative for $\pi/2 < x < 3\pi/4$. Since $G$ is concave downward, positive when $x = \pi/2$, and negative when $x = 3\pi/4$, it follows that $G$, and hence $H$, has at most one zero on this interval. We conclude that $h(x, 0, t)$ has a unique maximum on the interval $\pi/2 < x < 3\pi/4$ for each fixed $t$.

In summary, the maximum $m(t)$ occurs as the unique maximum of the function $h(x, 0, t)$ on the interval $\pi/2 < x < 3\pi/4$, and it is the maximum of our functional for the given range of $t$.

References
