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Möbius Transformations of Convex Mappings II

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Dedicated to Helmut Grunsky on the occasion of his 80th birthday

For the class of functions referred to in the title, this article finds the Koebe disk, the radius of convexity, and sharp estimates for the coefficient functional $|ta_3 + a_2^2|$ for t in a certain interval.

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1. INTRODUCTION

Let S denote the class of analytic univalent functions f defined in the unit disk $U = \{z: |z| < 1\}$ and normalized so that $f(0) = f'(0) - 1 = 0$. The convex subclass K consists of those functions $f \in S$ such that $f(U)$ is a convex set.

If $f \in S$ and $w \notin f(U)$, then the function

$$\hat{f} = f/(1 - f/w) \tag{1.1}$$

belongs again to S. The transformation $f \to \hat{f}$ is a familiar one in the study of univalent functions. If F is a subset of S, let

$$\hat{F} = \{\hat{f}: f \in F \text{ and } w \in \mathbb{C}^* \setminus f(U)\}.$$

Since $w = \infty$ belongs to $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, it follows that $F \subset \hat{F} \subset S$. Other obvious properties are that $\hat{\hat{F}} = \hat{F}$ and $\hat{S} = S$. It is an interesting question to ask which properties of F are inherited by $\hat{F}$. In this article we shall consider the class $\hat{K}$. The class $\hat{K}$ is compact, and simple examples show that $\hat{K} \neq K$.

In an earlier paper [1] we applied a variational procedure to a class of extremal problems for $\hat{K}$. If $\lambda: \hat{K} \to \mathbb{R}$ is a continuous functional that satisfies certain admissibility criteria, we showed that the problem

$$\max_{\hat{K}} \lambda$$

has a relatively elementary extremal function $\hat{f}$. More specifically, we showed that $\hat{f}$ either is a half-plane mapping

$$\hat{f}(z) = z/(1 - e^{i\alpha} z) \tag{1.2}$$

or is generated through (1.1) by a parallel strip mapping $f \in K$.

The class of functionals considered contained the second-coefficient functional $\lambda(\hat{f}) = \operatorname{Re} a_2$ and the functionals $\lambda(\hat{f}) = \operatorname{Re} \Phi(\log[\hat{f}(z)/z])$ where Φ is entire and $z \in U$ is fixed. The latter functionals include the problems of maximum and minimum modulus ($\Phi(w) = \pm w$). Therefore, for such problems it is necessary to test the functional only over Möbius transformations (1.2) and over functions $\hat{f}$ generated through (1.1) by strip mappings $f \in K$. We remark that the extremal strip domains $f(U)$ need not be symmetric about the origin. This adds an interesting and nontrivial character to the problems. In particular, in [1] an explicit determination was made of such an extremal function for the second-coefficient functional, and the sharp bound $|a_2| \leqslant 1.327\ldots$ was obtained for $\hat{K}$.

In this paper similar techniques are used to find the Koebe disk for $\hat{K}$, that is, the largest domain centered at the origin always covered by $\hat{f}(U)$ for $\hat{f} \in \hat{K}$, the radius of convexity for the class $\hat{K}$, and sharp bounds for the functional

$$\lambda_t(\hat{f}) = |t a_3 + a_2^2| \tag{1.3}$$

for a certain range of t. A corollary is the solution of the third-coefficient problem for the inverses of functions in K.

2. PRELIMINARIES

Since $\hat{K}$ is compact, a continuous functional

$$\lambda : \hat{K} \to \mathbb{R}$$

will assume its maximum at some function $f \in \hat{K}$. We call λ *admissible* if it has the following properties. (i) At an extremal function f it has an expansion of the form

$$\lambda(f^*) = \lambda(f) + \varepsilon \int_{|\zeta|=1} \sigma(\zeta)\, d\psi(\zeta) + o(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0 \tag{2.1}$$

under Julia variations

$$f^*(z) = f(z) + \varepsilon \int_{|\zeta|=1} \{zf'(z)[(\zeta + z)/(\zeta - z)] - f(z)\}\, d\psi(\zeta) + o(\varepsilon). \tag{2.2}$$

within $\hat{K}$, where $d\psi(\zeta) = [\varphi(f(\zeta))/|f'(\zeta)|]\, d\zeta/(i\zeta)$ and φ is a piecewise smooth, real-valued function. We require the function σ to be continuous and to vanish at no more than two points of the circle $|\zeta| = 1$. (ii) In addition, we require that there is a constant $c_f \neq 0$ such that

$$\lambda(f/(1 - f/w)) = \lambda(f) + \operatorname{Re}\{c_f/w\} + o(1/w) \tag{2.3}$$

as $w \to \infty$ in $\mathbb{C} \backslash f(U)$.

By choosing appropriate variations we proved the following theorem in [1, Theorem 8.2].

THEOREM 1 *If λ is an admissible functional, then λ assumes its maximum over $\hat{K}$ either at a half-plane mapping* (1.2) *or at a mapping generated through* (1.1) *by a parallel strip mapping $f \in K$ with w a finite point of $\partial f(U)$.*

The parallel strip mappings in K are rotations $e^{-i\alpha}f_x(e^{i\alpha}z)$, $\alpha \in \mathbb{R}$, of the vertical strip mappings

$$f_x(z) = [1/(2i \sin x)] \log[(1 + e^{ix}z)/(1 + e^{-ix}z)], \qquad 0 < x < \pi. \tag{2.4}$$

3. KOEBE DISK AND RADIUS OF CONVEXITY

It is an interesting problem to compare properties of the transformed class $\hat{K}$ with those of K. Of course, each $f \in K$ covers the disk $|w| < 1/2$. The following theorem determines the size of this Koebe disk for the class $\hat{K}$.

THEOREM 2 *If $f \in \hat{K}$, then $f(U)$ covers the disk $|w| < \pi/8$. Furthermore, $\bigcap_{\hat{K}} f(U) = \{w: |w| < \pi/8\}$.*

Proof We begin by showing first that if $f \in \hat{K}$, then

$$|f(z)/z| \geqslant f_{\pi/2}(1)/[1 - f_{\pi/2}(1)/f_{\pi/2}(-1)] = \pi/8. \tag{3.1}$$

The minimum modulus principle will then imply that the disk $|w| < \pi/8$ is contained in $f(U)$. By considering rotations of $\hat{f}_{\pi/2}$ it follows that this disk is the largest set covered by all functions in $\hat{K}$.

In [1] we observed that for fixed $z \in U$ the functional $\lambda(f) = -\log|f(z)/z|$ is admissible. Thus, to obtain its maximum value, it is sufficient by Theorem 1 to examine the functions (1.2) and transforms of rotations of (2.4). A comparison with the functions $\hat{f}_{\pi/2}$ eliminates (1.2). Therefore

$$|z/f(z)| \leqslant \max_{\alpha,\beta,x} |z/[e^{-i\alpha}f_x(e^{i\alpha}z)] - z/[e^{-i\alpha}f_x(e^{i\beta})]|$$

$$\leqslant \max_{\beta,\gamma,x} |1/f_x(e^{i\gamma}) - 1/f_x(e^{i\beta})|$$

by the maximum principle. For a fixed vertical strip, $|1/f_x(e^{i\gamma}) - 1/f_x(e^{i\beta})|$ will be a maximum for $\gamma = 0$ and $\beta = \pi$. That is,

$$|1/f_x(e^{i\gamma}) - 1/f_x(e^{i\beta})| \leqslant |1/f_x(1) - 1/f_x(-1)| = 2\pi(\sin x)/[x(\pi - x)].$$

Note that $g(x) = (\sin x)/[x(\pi - x)]$ is symmetric about $x = \pi/2$. In addition, $g'(x) = h(x)/[x(\pi - x)]^2$ where $h(x) = x(\pi - x)\cos x - (\pi - 2x)\sin x$. Since $h'(x) = (2 - \pi x + x^2)\sin x$, the function h increases from $h(0) = 0$ to $h((\pi - \sqrt{\pi^2 - 8})/2)$ and then decreases to $h(\pi/2) = 0$. That is, the only critical point of g in $(0, \pi)$ is at $x = \pi/2$. It provides the maximum of g since $g''(\pi/2) < 0$. Consequently, $|z/f(z)| \leqslant 2\pi g(\pi/2) = 8/\pi$ and (3.1) is proved. ∎

To obtain the radius of convexity for the class $\hat{K}$, we apply the Marty transformation and use the sharp bound for the second coefficient in $\hat{K}$.

THEOREM 3 *If $f \in \hat{K}$, then $f(|z| < r)$ is convex for $r \leqslant r_0 \approx .4547$ where $r_0 = A_2 - \sqrt{A_2^2 - 1}$, $A_2 = (2/x_0)\sin x_0 - \cos x_0 \approx 1.3270$, and $x_0 \approx 2.0816$ is the unique solution of the equation $\cot x = (1/x) - (x/2)$ in the interval $(0, \pi)$. This result is sharp.*

Proof For any function $f \in \hat{K}$ *and any* $\zeta \in U$ the function

$$F_\zeta(z) = [f((z+\zeta)/(1+\bar{\zeta}z)) - f(\zeta)]/[f'(\zeta)(1-|\zeta|^2)] = z + a_2(\zeta)z^2 + \cdots$$

belongs again to $\hat{K}$ since it is a Möbius transform of some function in K. Therefore the coefficient

$$a_2(\zeta) = (1/2)(1-|\zeta|^2)f''(\zeta)/f'(\zeta) - \bar{\zeta}$$

has the bound A_2 in the statement of the theorem, by what was proved in [1, Theorem 9.1]. Thus we have

$$|1 + \zeta f''(\zeta)/f'(\zeta) - (1+|\zeta|^2)/(1-|\zeta|^2)| \leqslant 2|\zeta|A_2/(1-|\zeta|^2)$$

and

$$\mathrm{Re}\{1 + \zeta f''(\zeta)/f'(\zeta)\} \geqslant (1 - 2|\zeta|A_2 + |\zeta|^2)/(1-|\zeta|^2). \qquad (3.2)$$

The latter will be positive for $|\zeta| < r_0 = A_2 - \sqrt{A_2^2 - 1}$.

If $\zeta = -r_0$ and $f \in \hat{K}$ is chosen so that $F_\zeta(z) = f_{x_0}(z)/(1 - f_{x_0}(z)/f_{x_0}(1))$, where f_{x_0} is defined by (2.4), then $a_2(\zeta) = A_2$ and both sides of (3.2) become zero. Therefore r_0 is the sharp radius of convexity for $\hat{K}$.

4. THE FUNCTIONAL $\lambda_t(f) = |ta_3 + a_2^2|$

In this section we shall apply Theorem 1 to give a sharp estimate for the functional (1.3) for t in a certain interval.

For $t = -1$ a sharp bound for the functional is already known. In fact, if $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belongs to K and $\hat{f}(z) = z + \hat{a}_2z^2 + \hat{a}_3z^3 + \cdots$ is the transform (1.1), then $\hat{a}_3 - \hat{a}_2^2 = a_3 - a_2^2$. That is, for $t = -1$ the functional is invariant under our Möbius transformations. J. A. Hummel [2] proved that

$$|a_3 - a_2^2| \leqslant 1/3 \qquad (4.1)$$

is a sharp estimate in the class K, and so this sharp estimate persists for the class $\hat{K}$.

The initial coefficients of the strip mapping (2.4) are $a_2(x) = -\cos x$ and $a_3(x) = (4/3)\cos^2 x - 1/3$, and the initial coefficients of its transform $\hat{f}_x$ are

$$\hat{a}_2(x) = a_2(x) + 1/w \qquad \text{and} \qquad \hat{a}_3(x) = a_3(x) + 2a_2(x)/w + 1/w^2.$$

In this case

$$|t\hat{a}_3(x) + \hat{a}_2(x)^2| = |-(t/3)\sin^2 x + (1+t)(-\cos x + 1/w)^2|.$$

Points w on the boundary of the strip $f_x(U)$ satisfy $\operatorname{Re} w = f_x(1) = x/(2\sin x)$ or $\operatorname{Re} w = f_x(-1) = (x-\pi)/(2\sin x)$. In these cases $1/w$ is of the form $(1+e^{i\theta})(\sin x)/x$ or $(1+e^{i\theta})(\sin x)/(x-\pi)$ for some real θ. If

$$h(x,\theta,t) = -(t/3)\sin^2 x + (1+t)[-\cos x + (1+e^{i\theta})(\sin x)/x]^2, \tag{4.2}$$

then $t\hat{a}_3(x) + \hat{a}_2(x)^2 = h(x,\theta,t)$ or $t\hat{a}_3(x) + \hat{a}_2(x)^2 = h(\pi - x,\theta,t)$. In particular,

$$m(t) = \max_{\substack{0 \leq x \leq \pi \\ 0 \leq \theta < 2\pi}} |h(x,\theta,t)| \tag{4.3}$$

is a lower bound for the maximum of our functional for each t. The following theorem shows that $m(t)$ actually is the maximum of our functional for a certain range of t.

THEOREM 4 *If* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *belongs to* $\hat{K}$ *and if* $-.7652 \leq t \leq 1.682$, *then the estimate*

$$|ta_3 + a_2^2| \leq M(t)$$

holds where $M(t)$ *is the unique maximum of the function* $-(t/3)\sin^2 x + (1+t)[2(\sin x)/x - \cos x]^2$ *on the interval* $\pi/2 < x < 3\pi/4$, *and this estimate is sharp.*

The values of $M(t)$ are easily obtained numerically; for example,

$M(-.7) \approx 0.7356$	$M(.1) \approx 1.9120$	$M(.9) \approx 3.1321$
$M(-.6) \approx 0.8763$	$M(.2) \approx 2.0636$	$M(1.0) \approx 3.2854$
$M(-.5) \approx 1.0197$	$M(.3) \approx 2.2155$	$M(1.1) \approx 3.4388$
$M(-.4) \approx 1.1652$	$M(.4) \approx 2.3677$	$M(1.2) \approx 3.5923$
$M(-.3) \approx 1.3125$	$M(.5) \approx 2.5202$	$M(1.3) \approx 3.7458$
$M(-.2) \approx 1.4610$	$M(.6) \approx 2.6729$	$M(1.4) \approx 3.8995$
$M(-.1) \approx 1.6106$	$M(.7) \approx 2.8259$	$M(1.5) \approx 4.0531$
$M(0) \approx 1.7610$	$M(.8) \approx 2.9789$	$M(1.6) \approx 4.2069.$

An application We postpone the proof of Theorem 4 for a moment and consider an application of Theorem 4. Let g be the inverse of a function $f \in \hat{K}$, that is, $g = f^{-1}$. Then

$$g(w) = w + \alpha_2 w^2 + \alpha_3 w^3 + \cdots \tag{4.4}$$

in some neighborhood of $w = 0$. In view of Theorem 2 the series expansion (4.4) is valid at least for $|w| < \pi/8$. In terms of the coefficients of f, we have $\alpha_2 = -a_2$ and $\alpha_3 = -a_3 + 2a_2^2$. Therefore the initial coefficients of the inverse function have the sharp bounds $|\alpha_2| \leqslant A_2$ where A_2 is defined in Theorem 3 and

$$|\alpha_3| \leqslant 2M(-1/2) \approx 2.0393.$$

Proof of Theorem 4 Since $\hat{K}$ is preserved under the rotations $f(z) \to e^{-i\theta} f(e^{i\theta} z)$, it is sufficient to consider the functional

$$\Lambda_t(f) = \mathrm{Re}\{ta_3 + a_2^2\}. \tag{4.5}$$

In order to verify the admissibility condition (i), we shall use the formulas

$$a_2^* = a_2 + \varepsilon \int_{|\zeta|=1} (a_2 + 2\bar{\zeta})\, d\psi(\zeta) + o(\varepsilon)$$

$$a_3^* = a_3 + 2\varepsilon \int_{|\zeta|=1} (a_3 + 2a_2\bar{\zeta} + \bar{\zeta}^2)\, d\psi(\zeta) + o(\varepsilon)$$

as $\varepsilon \to 0$ under the variations (2.2). Thus the functional (4.5) has the expansion (2.1) with

$$\sigma(\zeta) = 2\,\mathrm{Re}\{t(a_3 + 2a_2\bar{\zeta} + \bar{\zeta}^2) + a_2(a_2 + 2\bar{\zeta})\}.$$

We need to show that σ has at most two zeros on $|\zeta| = 1$. It is equivalent to show that the polynomial

$$\zeta^2\sigma(\zeta) = t\zeta^4 + 2(1+t)\overline{a_2}\zeta^3 + 2\,\mathrm{Re}\{ta_3 + a_2^2\}\zeta^2 + 2(1+t)a_2\zeta + t$$

has at most two zeros on $|\zeta| = 1$. If $t = 0$, this is obvious. Assume therefore that $t \neq 0$. If more than two zeros were on $|\zeta| = 1$, then all four would be there since the product of the zeros equals one. In this case

$$|[(2/t)(1+t)\overline{a_2}]^2 - (4/t)\,\mathrm{Re}\{ta_3 + a_2^2\}| \leqslant 4 \tag{4.6}$$

since $A^2 - 2B$ is the sum of the squares of the zeros of the polynomial $\zeta^4 + A\zeta^3 + B\zeta^2 + C\zeta + D$. Substitute $(1+t)a_2^2 = (ta_3 + a_2^2) - t(a_3 - a_2^2)$ into (4.6); then

$$(4/t^2)\,\mathrm{Re}\{ta_3 + a_2^2\} - (4/t)(1+t)\,\mathrm{Re}\{a_3 - a_2^2\} \leqslant 4.$$

Using the estimate (4.1), we can simplify this to

$$\mathrm{Re}\{ta_3 + a_2^2\} \leqslant t^2 + |t(1+t)|/3. \tag{4.7}$$

Due to the extreme nature of $\operatorname{Re}\{ta_3 + a_2^2\}$, it would follow that $h(x, 0, t) \leqslant t^2 + |t(1+t)|/3$ for every choice of x. If we can choose x to violate this inequality, then the admissibility condition (i) will be satisfied. From a computer-assisted search, two good choices for x appear to be $x = 1.7422$ and $x = 2.2297$. Then the inequality $h(1.7422, 0, t) > t^2 + |t(1+t)|/3$ is satisfied at least for $-.7652 \leqslant t < 0$, and the inequality $h(2.2297, 0, t) > t^2 + |t(1+t)|/3$ is satisfied at least for $0 < t \leqslant 1.682$. Therefore (i) is satisfied for $-.7652 \leqslant t \leqslant 1.682$, and this is assumed in the hypothesis.

In order to verify the admissibility condition (ii), we compute

$$\Lambda_t(f/(1 - f/w)) - \Lambda_t(f) = 2(t+1)\operatorname{Re}\{a_2/w\} + o(1/w) \qquad \text{as} \quad w \to \infty.$$

Since $t \neq -1$, the coefficient $c_f = 2(t+1)a_2$ could be zero only if $a_2 = 0$. If this were the case for an extremal function, then (4.1) would imply that $\operatorname{Re}\{ta_3 + a_2^2\} \leqslant |t|/3$. However, this inequality is violated whenever (4.7) is violated since $|t|/3 \leqslant t^2 + |t(1+t)|/3$. Thus (ii) is satisfied.

Now Theorem 1 applies; that is, $|ta_3 + a_2^2|$ will be a maximum either at a half-plane mapping or at a mapping generated through (1.1) by a parallel strip mapping. Since this functional is invariant under rotations, it is sufficient to consider the strip mappings (2.4). Consequently, the maximum value of $|ta_3 + a_2^2|$ is $m(t)$, defined in (4.3), where the half-plane mappings correspond to $x = 0, \pi$ and the strip mappings to $0 < x < \pi$. The remainder of this proof concerns a more specific description of $m(t)$.

For $t \geqslant -1$, we may estimate

$$\begin{aligned}|h(x, \theta, t)| &\leqslant (|t|/3)\sin^2 x + (1+t)|(\sin x)/x - \cos x + e^{i\theta}(\sin x)/x|^2 \\ &\leqslant (|t|/3)\sin^2 x + (1+t)[2(\sin x)/x - \cos x]^2\end{aligned}$$

since $(\sin x)/x - \cos x \geqslant 0$. At least for $0 \leqslant x \leqslant \pi/4$ this is a sum of increasing functions, and so $|h(x, \theta, t)| \leqslant |t|/6 + (1/2)(1+t)(8/\pi - 1)^2$ whenever $0 \leqslant x \leqslant \pi/4$. One easily verifies that $|t|/6 + (1/2)(1+t)(8/\pi - 1)^2 \leqslant -t/6 + (1/2)(1+t)[8/(3\pi) + 1]^2 = h(3\pi/4, 0, t)$. Therefore, for fixed t the maximum of $|h(x, \theta, t)|$ occurs when $\pi/4 \leqslant x \leqslant \pi$.

As a function of θ, the function $|h(x, \theta, t)|$ is of the form $(1+t)a|b + 2ce^{i\theta} + e^{2i\theta}|$ where $a = [(\sin x)/x]^2$ and $c = 1 - x\cot x$ are nonnegative, $b = -tx^2/[3(1+t)] + c^2$ is real, and $-.7652 \leqslant t \leqslant 1.682$.

It can be written as

$$(1+t)a\sqrt{4c^2+(1-b)^2+4c(1+b)\cos\theta+4b\cos^2\theta}\,.$$

We wish to show that the maximum occurs for $\theta = 0$. This is obvious if $-.7652 \leqslant t \leqslant 0$; assume therefore that $0 < t \leqslant 1.682$. It is sufficient to show that $4c(1+b)(\cos\theta - 1) + 4b(\cos^2\theta - 1) \leqslant 0$ or that $c(1+b) + 2b \geqslant 0$. After multiplying by $(1+t)/x^2$, we note that the latter inequality becomes

$$(c/x^2)\{(1+c)^2 + t[(1+c)^2 - x^2/3]\} - 2t/3 \geqslant 0.$$

Since

$$c = x^2/3 + \sum_{k=2}^{\infty} |B_{2k}|(2x)^{2k}/(2k)!,$$

where the B_{2k} are Bernoulli numbers, the function

$$(c/x^2)\{(1+c)^2 + t[(1+c)^2 - x^2/3]\} - 2t/3$$

is increasing at least for $\pi/4 \leqslant x < \pi$. At $x = \pi/4$ one verifies directly that this expression is positive. As a result, for each fixed t the maximum value of $|h(x,\theta,t)|$ occurs for $\pi/4 \leqslant x \leqslant \pi$ and $\theta = 0$.

Using the notation of the previous paragraph, we observe that $h(x,0,t) = a\{(1+c)^2 + t[(1+c)^2 - x^2/3]\}$ is positive for $\pi/4 \leqslant x \leqslant \pi$, and so $|h(x,0,t)| = h(x,0,t)$ over this interval. Next, we shall show that the maximum value of $h(x,0,t)$ over $\pi/4 \leqslant x \leqslant \pi$ occurs in the smaller interval $\pi/2 < x < 3\pi/4$.

The derivative $H = (\partial h/\partial x)(x,0,t)$ may be written as $H = 2xa[-t(1-c)/3 + (1+t)(1+c)(1-2c/x^2)]$ where $a = (\sin^2 x)/x^2$ and $c = 1 - x\cot x$ as before. For $\pi/4 \leqslant x \leqslant \pi/2$, we have $1 - \pi/4 \leqslant c \leqslant 1$ and $c/x^2 \leqslant 4/\pi^2$. On this interval H is clearly positive of $-.7652 \leqslant t \leqslant 0$. If $t > 0$, then $-t(1-c)/3 + (1+t)(1+c)(1-2c/x^2) \geqslant -t\pi/12 + (1+t)(2-\pi/4)(1-8/\pi^2)$, which is positive for $t \leqslant 1.682$. Therefore $h(x,0,t)$ does not assume a maximum in $\pi/4 \leqslant x \leqslant \pi/2$.

The derivative H may also be written as $H = 4(1+t)G/x^3$ where $G = (x/12)[18 - (3+4t)x^2/(1+t)]\sin 2x - (x^2-1)\cos 2x - 1$. We shall show that H, or equivalently G, is negative for $3\pi/4 \leqslant x \leqslant \pi$. Since G is a monotone function of t, it is sufficient to show that G is negative for $t = -.7652$ and $t = 1.682$. First, if $t = -.7652$, then the first two terms in G are at most zero and the third term is negative. Second, if $t = 1.682$, then $\partial G/\partial x = (1466x^2/1341 - 1/2)\sin 2x - (2432x^2/4023 - 1)x\cos 2x$ and both terms give negative contributions on $3\pi/4 < x < \pi$; in addition, G is negative at $x = 3\pi/4$. Therefore $h(x, 0, t)$ does not assume a maximum in $3\pi/4 \leqslant x \leqslant \pi$.

With the notation of the previous paragraph, it is easy to show that $(1 + t)\,\partial^2 G/\partial x^2 = [1/2 + (3 + 4t)x^2/3]x \sin 2x + x^2 \cos 2x$ is negative for $\pi/2 < x < 3\pi/4$. Since G is concave downward, positive when $x = \pi/2$, and negative when $x = 3\pi/4$, it follows that G, and hence H, has at most one zero on this interval. We conclude that $h(x, 0, t)$ has a unique maximum on the interval $\pi/2 < x < 3\pi/4$ for each fixed t.

In summary, the maximum $m(t)$ occurs as the unique maximum of the function $h(x, 0, t)$ on the interval $\pi/2 < x < 3\pi/4$, and it is the maximum of our functional for the given range of t. ■

References

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