Complex Variables, 1984, Vol. 3, pp. 55–69
0278-1077/84/0301-0055 $18.50/0

Möbius Transformations of Convex Mappings*

ROGER W. BARNARD

Texas Tech University, Lubbock, Texas 79409

and

GLENN SCHOBER**

Indiana University, Bloomington, Indiana 47405

(*Received March 29, 1983*)

Extremal problems are considered for the class of functions in S that are Möbius transformations of convex mappings. A variational method based on Julia's formula is used to describe extremal functions for a certain class of problems. These admissible problems are shown to have extremal functions which are either half-plane mappings or Möbius transformations of strip mappings. An explicit solution is given for the second-coefficient problem.

AMS (MOS): 30A32

1. INTRODUCTION

Let S denote the familiar class of normalized analytic univalent functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the unit disk $U = \{z : |z| < 1\}$. The convex subclass K consists of those functions $f \in S$ such that $f(U)$ is a convex set.

*Dedicated to Malcolm Robertson on the occasion of his eightieth birthday.

**Supported in part by a grant from the National Science Foundation.

If $f \in S$ and $w \notin f(U)$, then the function

$$\hat{f} = f/(1 - f/w) \tag{1.1}$$

belongs again to S. The transformation $f \to \hat{f}$ is important in the study of univalent functions. It is useful in the proofs of both elementary and not so elementary properties of S.

If F is a subset of S, let

$$\hat{F} = \{\hat{f} : f \in F, w \in \mathbb{C}^* \setminus f(U)\}.$$

Here $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. Since we admit $w = \infty$, it is clear that $F \subset \hat{F} \subset S$, and since the composition of normalized Möbius transformations is again a normalized Möbius transformation, it follows that $\hat{\hat{F}} = \hat{F}$.

If F is compact in the topology of locally uniform convergence, then so is $\hat{F}$. If F is rotationally invariant, that is, $f_\alpha(z) = e^{-i\alpha}f(e^{i\alpha}z)$ belongs to F whenever f does, then $\hat{F}$ is also rotationally invariant. It is an interesting question to ask which properties of F are inherited by $\hat{F}$. Since $\hat{S} = S$, this question is trivial for S.

In this article we shall consider the class $\hat{K}$. Simple examples show that $\hat{K}$ is strictly larger than K. Since the coefficients of functions in K are uniformly bounded (by one), J. Clunie and T. Sheil-Small asked whether the coefficients of functions in $\hat{K}$ have a uniform bound. The affirmative solution of this problem was given recently by R. R. Hall [3]. The question of the best uniform bound remains open as well as the individual coefficient problems for $\hat{K}$.

The purpose of this article is to apply a variational procedure to a class of extremal problems for $\hat{K}$. If $\lambda : \hat{K} \to \mathbb{R}$ is a continuous functional that satisfies certain admissibility criteria, we shall show that the problem

$$\max_{\hat{K}} \lambda$$

has a relatively elementary extremal function $\hat{f}$. More specifically, in Section 8 we shall show that $\hat{f}$ either is a half-plane mapping $\hat{f}(z) = z/(1 - e^{i\alpha}z)$ or is generated through (1.1) by a parallel strip mapping $f \in K$.

The class of functionals considered contains the second-coefficient functional $\lambda(\hat{f}) = \operatorname{Re} a_2$ and the functionals $\lambda(\hat{f}) = \operatorname{Re}\Phi(\log \hat{f}(z)/z)$ where Φ is entire and z is fixed. The latter functionals include the problems of maximum and minimum modulus ($\Phi(w) = \pm w$). Therefore for such problems it is necessary to test the functional only over Möbius transformations $\hat{f}(z) = z/(1 - e^{i\alpha}z)$ and over functions $\hat{f}$ generated by strip mappings $f \in K$. In general, the extremal strip domains $f(U)$ need not be symmetric about the origin. This adds a nontrivial and interesting character to the problems. Finally, in Section 9 we explicitly determine such an extremal function for the second-coefficient problem and obtain the sharp bound $|a_2| \leqslant 1.327\ldots$ in $\hat{K}$.

2. JULIA VARIATIONAL FORMULA

In this section we formulate the Julia variational formula and later show how it can be used to produce variations within a given class of functions. The basic idea occurs in [6], and it is expanded in [2] and [1]. For our purposes this method appears to be superior to other methods since it will permit us to preserve easily certain geometric properties and since the family $\hat{K}$ does not seem to admit a useful structural formula.

Let f belong to S and map U onto a domain Ω whose boundary is a piecewise analytic arc Γ. We denote also by f the extension of f to Γ. Let $n(w)$ be the unit exterior normal to Ω at $w \in \Gamma$, and let $\phi(w)$ be a real-valued, continuous, piecewise continuously differentiable function on Γ which vanishes at points of nonanalyticity of Γ. If $\epsilon > 0$ is sufficiently small, then $w^* = w + \epsilon\phi(w)n(w)$ maps Γ homeomorphically onto an arc Γ^*, which is the boundary of a domain Ω^*. It follows from Julia's work [4] that the Riemann mapping function g^* of U onto Ω^*, with $g^*(0) = 0$ and $g^{*\prime}(0) > 0$, is given by

$$g^*(z) = f(z) + \frac{\epsilon z f'(z)}{2\pi i} \int_\Gamma \frac{\zeta + z}{\zeta - z} \frac{\phi(w)n(w)}{[\zeta f'(\zeta)]^2}\, dw + o(\epsilon) \tag{2.1}$$

as $\epsilon \to 0$. Here $w = f(\zeta)$ and the $o(\epsilon)$ term is uniform for z in compact subsets of U. In our development we shall see that this formula remains valid for certain variations of curvilinear polygons.

It will be convenient to denote

$$d\psi = \frac{\phi(w)n(w)}{i[\zeta f'(\zeta)]^2}\, dw = \frac{\phi(w)}{|f'(\zeta)|}\, d\theta$$

where $\zeta = e^{i\theta}$. Then it is apparent that $d\psi$ is real and that formula (2.1) becomes

$$g^*(z) = f(z) + \frac{\epsilon z f'(z)}{2\pi} \int_\Gamma \frac{\zeta + z}{\zeta - z}\, d\psi + o(\epsilon).$$

Since

$$g^{*\prime}(0) = 1 + \frac{\epsilon}{2\pi} \int_\Gamma d\psi + o(\epsilon),$$

the function $f^*(z) = g^*(z)/g^{*\prime}(0)$ belongs again to S and has the asymptotic form

$$f^*(z) = f(z) + \frac{\epsilon}{2\pi}\left[z f'(z) \int_\Gamma \frac{\zeta + z}{\zeta - z}\, d\psi - f(z) \int_\Gamma d\psi \right] + o(\epsilon). \quad (2.2)$$

3. POLYGONS

For $n = 1, 2, 3, \ldots$, let $K_n = \{ f \in K : f(U)$ is a polygon with at most n sides$\}$. We admit unbounded polygons so that K_1 consists of half-plane mappings and K_2 contains, in addition, wedge and parallel strip mappings. By discretely approximating the measures in the representation

$$f'(z) = \exp\left\{ -2 \int_{|\eta|=1} \log(1 - \eta z)\, d\mu \right\}, \qquad \int_{|\eta|=1} d\mu = 1,$$

for functions in K, it is easy to see that $\bigcup_{n=1}^{\infty} K_n$ is dense in K. Furthermore, each set K_n is compact.

Functions $\hat{f}$ in $\hat{K}_n$ map U onto curvilinear polygons with at most n sides and with interior angles at most π. Furthermore, if $\hat{f} = f/(1 - f/w)$, then the sides of $\partial \hat{f}(U)$ all lie on circles or lines through the point $w_1 = -w$ (see Figure 1). In fact, these two properties characterize functions in $\hat{K}_n$. That is, if $g \in S$ and $\partial g(U)$ is a curvilinear n-gon with interior angles at most π and if the sides of $\partial g(U)$ all lie on circles or lines through a point $-w \notin g(U)$, then $f = g/(1 + g/w)$ belongs to K_n and so $\hat{f} = g$ belongs to $\hat{K}_n$.

Finally, straightforward arguments show that $\bigcup_{n=1}^{\infty} \hat{K}_n$ is dense in $\hat{K}$ and that each $\hat{K}_n$ is compact.

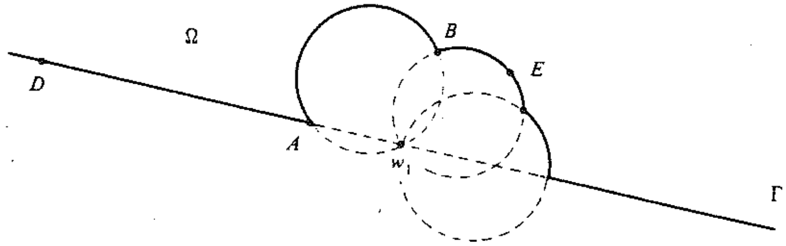

Figure 1

4. ADMISSIBLE FUNCTIONALS

We shall consider continuous functionals

$$\lambda : \hat{K} \to \mathbb{R}.$$

Since $\bigcup_{n=1}^{\infty} \hat{K}_n$ is dense in $\hat{K}$, the maximum of λ over $\hat{K}_n$ converges to the maximum of λ over $\hat{K}$ as $n \to \infty$. In our development even more will be true. We shall show that for admissible functionals the maximum of λ over $\hat{K}_n$, $n \geqslant 2$, occurs for a function in $\hat{K}_2$. It follows that the same function provides the maximum of λ over the entire family $\hat{K}$.

Since we shall be concerned principally with the family $\hat{K}$ for the next several sections, it will be convenient to drop the ˆ in reference to functions in $\hat{K}$ and $\hat{K}_n$.

The functional λ will be called *admissible* if at an extremal function f it has an expansion

$$\lambda(f^*) = \lambda(f) + \frac{\epsilon}{2\pi} \int_\Gamma \sigma(\zeta)\, d\psi + o(\epsilon) \qquad \text{as} \quad \epsilon \to 0 \tag{4.1}$$

under variations of the form (2.2) and if the function σ is continuous and vanishes at no more than two points of $|\zeta| = 1$. In addition, we shall require that there is a constant $c_f \neq 0$ such that

$$\lambda(f/(1 - f/w)) = \lambda(f) + \operatorname{Re}\left\{ \frac{c_f}{w} \right\} + o\left(\frac{1}{w} \right) \tag{4.2}$$

as $w \to \infty$ in $\mathbb{C} \setminus f(U)$.

The requirement (4.2) has an immediate consequence. If f is an extremal function for the problem

$$\max_{\hat{K}_n} \lambda,$$

then (4.2) implies

$$\operatorname{Re}\left\{\frac{c_f}{w}\right\} + o\left(\frac{1}{w}\right) \leq 0$$

as $w \to \infty$ in $\mathbb{C}\setminus f(U)$. This means that the values omitted by f lie asymptotically in a half-plane. Thus the curvilinear polygon $f(U)$ cannot be bounded, and so the edge(s) to ∞ must be straight. Furthermore, if two edges meet at ∞, their exterior angle cannot be more than π. Since the interior angle is *a priori* not more than π, it follows that there is exactly one edge of $f(U)$ through ∞ (see Figure 1). Thus, if f is not a half-plane mapping, then $f = g/(1 - g/w)$ where $g \in K_n$ and w is a finite nonvertex point of $\partial g(U)$.

5. EXAMPLES

Consider the functional $\lambda(f) = \operatorname{Re} a_2$. Under the variations (2.2) the second coefficient satisfies

$$a_2^* = a_2 + \frac{\epsilon}{2\pi}\int_\Gamma \left(a_2 + 2\bar{\zeta}\right) d\psi + o(\epsilon),$$

and the function

$$\sigma(\zeta) = \operatorname{Re}\{a_2 + 2\zeta\}$$

has at most two zeros on $|\zeta| = 1$. In addition,

$$\lambda(f/(1 - f/w)) = \operatorname{Re} a_2 + \operatorname{Re}\left\{\frac{1}{w}\right\} + o\left(\frac{1}{w}\right) \qquad \text{as} \quad w \to \infty$$

so that (4.2) is satisfied. Therefore $\lambda(f) = \operatorname{Re} a_2$ is admissible.

Next let Φ be a nonconstant entire function, and let $z \in U\setminus\{0\}$ be fixed. Consider the functional $\lambda(f) = \operatorname{Re}\{\Phi(\log f(z)/z)\}$. Under the variations (2.2) we have

$$\Phi\left(\log\frac{f^*(z)}{z}\right) = \Phi\left(\log\frac{f(z)}{z}\right) + \frac{\epsilon}{2\pi}\Phi'\left(\log\frac{f(z)}{z}\right)\int_\Gamma\left[\frac{zf'(z)}{f(z)}\,\frac{\zeta + z}{\zeta - z} - 1\right] d\psi + o(\epsilon).$$

Since $(\zeta + z)/(\zeta - z)$ carries the circle $|\zeta| = 1$ onto a circle, in order to show that the function

$$\sigma(\zeta) = \mathrm{Re}\left\{\Phi'\left(\log\frac{f(z)}{z}\right)\left[\frac{zf'(z)}{f(z)}\,\frac{\zeta + z}{\zeta - z} - 1\right]\right\}$$

can vanish at most twice, it is sufficient to show that $\Phi'(\log(f(z)/z))$ and $(zf'(z)/f(z))$ are different from zero. The latter is obviously different from zero, and W. E. Kirwan [5] has given an argument which shows that $\Phi'(\log f(z)/z)$ is not zero if f is extremal and the family (here $\hat{K}_n$) is rotationally invariant. In addition,

$$\lambda(f/(1 - f/w))$$

$$= \mathrm{Re}\left\{\Phi\left(\log\frac{f(z)}{z}\right)\right\} - \mathrm{Re}\left\{\frac{f(z)}{w}\Phi'\left(\log\frac{f(z)}{z}\right)\right\} + o\left(\frac{1}{w}\right)$$

where $c_f = -f(z)\Phi'(\log(f(z)/z))$ is not zero if f is extremal. Thus $\lambda(f) = \mathrm{Re}\{\Phi(\log(f(z)/z))\}$ is admissible.

6. VARIATIONS FOR CIRCULAR ARCS

In this section we shall construct variations within $\hat{K}_n$ by moving certain circular arcs. The purpose is to produce variations $w^* = w + \epsilon\phi(w)n(w)$ so that ϕ is positive on one part of the arc and negative on another part.

Let $f \in \hat{K}_n \setminus \hat{K}_1$ and $\Omega = f(U)$, and assume that $\Gamma = \partial\Omega$ is unbounded. Suppose that D, A, B, E are points of Γ, in that order. Assume that A and B are consecutive vertices so that AB is a circular arc and DA is a straight (see Figure 1).

Fix a point P on the arc AB. We shall construct variations of two similar types:

$$\phi < 0 \text{ on the open arc } AP \text{ and } \phi > 0 \text{ on the open arc } PB; \tag{6.1}$$

$$\phi > 0 \text{ on the open arc } AP \text{ and } \phi < 0 \text{ on the open arc } PB. \tag{6.2}$$

Off the arc AB the function ϕ will be zero. If P is the endpoint A, then ϕ will be positive on the open arc AB in case (6.1) and negative in case (6.2). Similarly, if P is the endpoint B, then ϕ will be negative in case (6.1) and positive in case (6.2).

Let w_1 be a point through which continuations of the arcs of Γ must pass (i.e., the point $-w$ in §3). Under certain circumstances it is possible that $w_1 = A$ or $w_1 = B$. In these cases we shall restrict $P \neq A$ or $P \neq B$, respectively. Thus P is always different from w_1.

Note that the arc AB lies on a circle C which belongs to the family of circles through the points P and w_1. The variations (6.1) and (6.2) will be obtained essentially by displacing the arc AB to arcs of neighboring circles through P and w_1.

To define (6.1) let C^* be a circle through P and w_1 which has the open arc AP in its interior and the open arc PB in its exterior. Then C^* meets the segment AD at a point A^* so that the open arc A^*P lies in Ω. Similarly, if C^* is sufficiently close to C, then C^* will intersect the circle or line on which BE lies at a point B^* near to (or at) B and the open arc PB^* will be exterior to Ω. The varied domain Ω^* will be obtained by replacing the curvilinear arc $DABE$ by the curvilinear arc DA^*B^*E (see Figure 2). However, in order to derive a variational formula of the form (2.2), an additional construction will be used in certain situations. Its purpose is to make ϕ vanish continuously at the endpoints of AB.

Denote the interior angles at A and B by α and β, respectively. If $P \neq A$ and $\alpha \geqslant \pi/2$, let A_1^* be the end point on the arc A^*P such that the interior angle between the segment AA_1^* and the arc AP is $\pi/2 - \delta$ for a sufficiently small fixed $\delta > 0$. If $P \neq B$ and $\beta \leqslant \pi/2$, let B_1^* be the point on the arc B^*P so that the angle between the segment BB_1^* and the arc BP is $\pi/2 - \eta$ for a sufficiently small fixed $\eta > 0$.

Now to each point w on AB we associate a point w^* on the ray through w from the center of C. The point $w^* = w^*(w)$ is the intersection of this ray with the segment AA_1^* and arc A_1^*P if A_1^* is defined or with the segment AA^* and arc A^*P if A_1^* is not defined, and it is the intersection with the arc PB_1^* and segment B_1^*B if B_1^* is

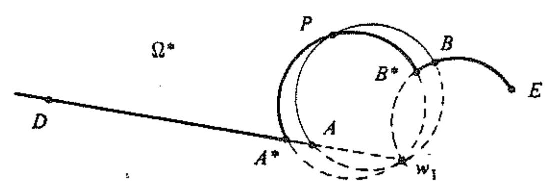

Figure 2

defined or with the arcs PB^* and B^*B if B_1^* is not defined. On the rest of Γ define $w^*(w) = w$.

If C^* is sufficiently close to C, then the normal displacement is

$$\epsilon\phi(w) = \begin{cases} -|w^*(w) - w| & \text{on the arc } AP \\ |w^*(w) - w| & \text{on the arc } PB \\ 0 & \text{on the rest of } \Gamma. \end{cases}$$

The resulting $\phi(w)$ is continuous, piecewise continuously differentiable, and vanishes at the vertices A and B. Thus Julia's variational formula (2.1) applies to the mapping function g_0^* onto the varied domain. However, the function $g_0^*(z)/g_0^{*\prime}(0)$ does not necessarily belong to $\hat{K}_n$ because of the possible adjustments near the endpoints in order to obtain Julia's formula. Nevertheless, if g^* denotes the mapping onto the desired domain Ω^* obtained by replacing the curvilinear arc $DABE$ by the curvilinear arc DA^*B^*E, then the argument used in [2, pp. 348–356] shows that $g^* - g_0^* = o(\epsilon)$ as $\epsilon \to 0$ (see also [7]). Consequently, $f^*(z) = g^*(z)/g^{*\prime}(0)$ does belong to $\hat{K}_n$ and admits an asymptotic development of the form (2.2) where ϕ satisfies (6.1).

To define (6.2) a very similar construction is used. The circle C^* passes again through w_1 and P, but PB^* lies in Ω while A^*P does not. Proceeding as before, we obtain variations f^* within $\hat{K}_n$ that have the asymptotic form (2.2) where ϕ satisfies (6.2).

7. AN ADDITIONAL VARIATION

If λ is an admissible functional and if f provides the maximum of λ over all functions in $\hat{K}_n$, then based on the variations of Section 6 we shall show in Section 8 that f belongs to $\hat{K}_2$, that is, $\Omega = f(U)$ has at most two sides. In order to obtain further properties of f we shall make use of an additional variation.

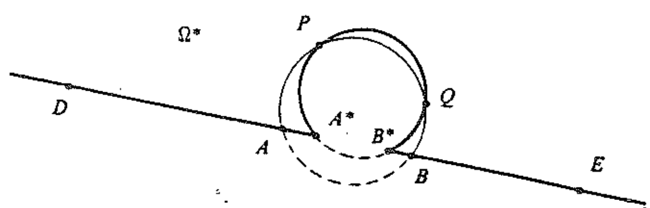

Figure 3

For that purpose assume f belongs to $\hat{K}_2 \backslash \hat{K}_1$. Suppose that D, A, B, E are consecutive points of $\Gamma = \partial\Omega$ such that A and B are vertices of a circular arc and such that DA and BE are colinear segments. In addition, assume that $A \neq B$ so that the equal interior angles at A and B are positive.

Let P and Q be distinct points on the arc AB, different from A and B. We shall make use of variations of the following types:

$$\phi > 0 \text{ on } PQ \text{ and } \phi < 0 \text{ on both } AP \text{ and } QB; \tag{7.1}$$

$$\phi < 0 \text{ on } PQ \text{ and } \phi > 0 \text{ on both } AP \text{ and } QB. \tag{7.2}$$

Off the arc AB the function ϕ will be zero.

The constructions are very similar to those in Section 6, with P and Q in place of P and w_1. The arc AB is replaced by an arc A^*B^* of a circle through P and Q. The development of the asymptotic form (2.1) for the mapping g^* onto the varied domain obtained by replacing $DABE$ by DA^*B^*E is identical to the previous section, and we omit it. It is clear that the various circles through P and Q produce variations of both types (7.1) and (7.2).

Since the point w_1 (i.e., the point $-w$ in §3) does not remain on the circle C^* containing the arc A^*B^*, as it did in the variations of the previous section, we must still show that the variations $f^*(z) = g^*(z)/g^{*\prime}(0)$ belong to $\hat{K}_2$. However, in this case note that f^* is the transform of a function in K_2 which maps U onto a wedge domain.

8. GENERAL THEOREMS

We are now in a position to describe extremal functions for the class of admissible functionals λ defined in Section 4. First, we shall consider the problem

$$\max_{\hat{K}_n} \lambda. \tag{8.1}$$

Since λ is continuous and $\hat{K}_n$ is compact, an extremal function exists. The following theorem describes its properties.

THEOREM 8.1 *Suppose $n \geqslant 2$ and λ is an admissible functional which assumes its maximum over $\hat{K}_n$ at $\hat{f} = f/(1 - f/w)$, $f \in K_n$. Then either $\hat{f}$ is a half-plane mapping or else f belongs to K_2, f maps U onto an infinite strip domain, and w is a finite point of $\partial f(U)$.*

As a consequence of Theorem 8.1, there is a common solution to the problem (8.1) for all $n \geqslant 2$. Since λ is continuous and $\bigcup_{n=1}^{\infty} \hat{K}_n$ is dense in $\hat{K}$, it follows that this same function solves the problem

$$\max_{\hat{K}} \lambda.$$

Notice, however, that this limiting procedure does not prevent the possibility of additional extremal functions in $\hat{K} \setminus \bigcup_{n=1}^{\infty} \hat{K}_n$. Before proving Theorem 8.1 we record this consequence.

THEOREM 8.2 *If λ is an admissible functional, then λ assumes its maximum over $\hat{K}$ at a function $\hat{f} = f/(1 - f/w)$ where $f \in K_2$. Furthermore, either $\hat{f}$ is a half-plane mapping or else f is a strip mapping and w is a finite point of $\partial f(U)$.*

By specializing Theorem 8.2 to one of the admissible functionals in Section 5 we have the following corollary. Application to the second-coefficient functional will be the subject of Section 9.

COROLLARY *Let Φ be a nonconstant entire function, and let $z \in U \setminus \{0\}$ be fixed. Then the functional*

$$\lambda(g) = \operatorname{Re}\left\{\Phi\left(\log \frac{g(z)}{z}\right)\right\}$$

assumes its maximum over $\hat{K}$ at a function $\hat{f} = f/(1 - f/w)$ where $f \in K_2$. Furthermore, either $\hat{f}$ is a half-plane mapping or else f is a strip mapping and w is a finite point of $\partial f(U)$.

Proof of Theorem 8.1 Suppose that an admissible functional λ assumes its maximum over $\hat{K}_n$ at $\hat{f} = f/(1 - f/w)$ where $f \in K_n$. We know from Section 4 that the boundary $\Gamma = \partial \hat{f}(U)$ contains ∞ as an interior point of one straight edge and that either $\hat{f}$ is a Möbius transformation or w is a finite nonvertex point of $\partial f(U)$. Assume for the purpose of contradiction that $\hat{f}$ is neither a half-plane mapping nor generated by a strip mapping f.

As in Section 6, let D, A, B, E be points of Γ such that AB is a circular arc and DA is straight. Necessarily A is different from B, for otherwise, AB would be a full circle and $\hat{f}$ would be generated by a strip mapping f. Denote by γ_{AB} the open arc of $|\zeta| = 1$ that $\hat{f}$ carries onto the open arc AB. Since the functional λ is admissible, there are three possibilities for the function σ in the expansion

$$\lambda(\hat{f}^*) = \lambda(\hat{f}) + \frac{\epsilon}{2\pi}\int_0^{2\pi}\sigma(\zeta)\frac{\phi(\hat{f}(\zeta))}{|\hat{f}(\zeta)|}\,d\theta + o(\epsilon), \qquad \zeta = e^{i\theta}: \quad (4.1')$$

(i) σ does not vanish on γ_{AB};
(ii) σ vanishes exactly once on γ_{AB};
(iii) σ vanishes exactly twice on γ_{AB}.

If it is possible to make variations $\hat{f}^*$ within $\hat{K}_n$ so that $\phi \circ \hat{f}$ has the same sign as σ, then (4.1′) shows that $\hat{f}$ cannot be extremal. One of the variations (6.1) or (6.2) will have this property by choosing P to be an end-point of AB in case (i) and by choosing P to correspond to the zero of σ in case (ii). Therefore we are left only with the alternative (iii), in which σ vanishes twice on γ_{AB}.

If Γ were to contain more than one circular arc, then a second choice of points D, A, B, E would be possible (i.e., coming from ∞ in the opposite direction). However, by repeating the argument of the previous paragraph we could conclude that σ has altogether at least four zeros, in contradiction to the admissibility criteria. Thus Γ has only one circular arc AB, the points A and B are distinct, and σ vanishes twice on γ_{AB}.

Now let P and Q be the points of the arc AB that correspond to the two zeros of σ on γ_{AB}. Then one of the variations (7.1) or (7.2) will have the same sign as σ on γ_{AB}. In this final case formula (4.1′) shows that $\hat{f}$ cannot be extremal. Since no alternative remains, the theorem is proved.

9. THE SECOND-COEFFICIENT PROBLEM

In this section we shall apply Theorem 8.2 to give a sharp estimate for the second coefficient of functions in $\hat{K}$. Surprisingly, the answer is not an obvious one.

THEOREM 9.1 *If $\hat{f}(z) = z + a_2 z^2 + \ldots$ belongs to $\hat{K}$, then*

$$|a_2| \leqslant \frac{2}{x_0}\sin x_0 - \cos x_0 \approx 1.3270$$

where $x_0 \approx 2.0816$ *is the unique solution of the equation*

$$\cot x = \frac{1}{x} - \frac{1}{2}x \tag{9.1}$$

in the interval $(0, \pi)$. *Equality occurs for the functions* $e^{-i\alpha}\hat{f}(e^{i\alpha}z)$, $\alpha \in \mathbb{R}$, *where* $\hat{f}(z) = f(z)/[1 - f(z)/f(1)]$ *and* f *is the vertical strip mapping defined by*

$$f(z) = \frac{1}{2i\sin x_0}\log\frac{1 + e^{ix_0}z}{1 + e^{-ix_0}z}. \tag{9.2}$$

Proof Since the family $\hat{K}$ is rotationally invariant, the maxima of $\operatorname{Re} a_2$ and $|a_2|$ are the same. Thus by Theorem 8.2 we need to consider only half-plane mappings $\hat{f}(z) = z/(1 - e^{i\alpha}z)$, whose second coefficients have modulus one, and the transforms of strip mappings. Therefore consider

$$\hat{f} = f/(1 - f/w) \tag{9.3}$$

where f is a strip mapping and w is a finite boundary point. Since $|a_2|$ is invariant under rotations, we may rotate $\hat{f}$ and f so that f is a vertical strip mapping. Hence it is sufficient to assume that f has the form (9.2) with x_0 replaced by $x \in (0, \pi)$ and to determine x and $w \in \partial f(U)$ so that the modulus of the second coefficient of (9.3) is as large as possible. We shall see that a maximum exists and is larger than one. Therefore this maximum is a sharp bound for $|a_2|$.

The second coefficient of (9.2) is $-\cos x$, and so the second coefficient of (9.3) is

$$a_2 = -\cos x + 1/w. \tag{9.4}$$

Since the points $1/w$ vary on a circle that is symmetric with respect to the real axis, it follows that the modulus of (9.4) can be a maximum only when $1/w$ is real. The point w must be finite; hence

$$w = f(1) = \frac{x}{2\sin x} > 0 \quad \text{or} \quad w = f(-1) = \frac{x - \pi}{2\sin x} < 0.$$

In these cases (9.4) becomes

$$a_2 = h(x) \quad \text{or} \quad a_2 = -h(\pi - x)$$

where

$$h(x) = -\cos x + \frac{2}{x}\sin x \tag{9.5}$$

and $h(0) = 1$. Thus the problem is reduced to finding the extreme values of the function h for $0 \leqslant x \leqslant \pi$.

Since $h(0) = h(\pi) = 1$ and $h(\pi/2) = 4/\pi > 1$, the maximum of h is larger than one, and it occurs at a point $x_0 \in (0, \pi)$ where

$$h'(x) = \frac{2\sin x}{x}\left[\cot x - \frac{1}{x} + \frac{1}{2}x\right] \tag{9.6}$$

vanishes. We shall show that there is only one such point in $(0, \pi)$. It follows then that $1 \leqslant h(x) \leqslant h(x_0)$, that $h(x_0)$ provides the maximum of $|a_2|$, and that there is an extremal function of the indicated form.

It is clear from (9.6) that $h'(x)$ can vanish in $(0, \pi)$ only if equation (9.1) is satisfied. By inserting series into (9.5) one arrives at the expansion

$$h'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n-1}}{(2n+1)(2n-2)!}.$$

Since $(x^{2n-1}/(2n+1)(2n-2)!)$ is a decreasing function of n as long as $0 < x < \sqrt{10/3}$, it follows that $h'(x) > 0$ for $0 < x < \sqrt{10/3} \approx 1.825$. Thus equation (9.1) can be satisfied only in the smaller interval $[\sqrt{10/3}, \pi)$. Finally, by checking derivatives, observe that $x + \cot x$ is decreasing and $x + (1/x - (1/2)x)$ is increasing on the interval $[\sqrt{10/3}, \pi)$. Therefore equation (9.1) has precisely one solution in $(0, \pi)$, and the proof is complete.

References

[1] Barnard, R. W., On bounded univalent functions whose ranges contain a fixed disk, *Trans. Amer. Math. Soc.* **225** (1977), 123–144.

[2] Barnard, R. W. and Lewis, J. L., Subordination theorems for some classes of starlike functions, *Pacific J. Math.* **56** (1975), 333–366.

[3] Hall, R. R., On a conjecture of Clunie and Sheil-Small. *Bull. London Math. Soc.* **12** (1980), 25–28.

[4] Julia, G., Sur une équation aux dérivées fonctionelles liée à la représentation conforme. *Ann. Sci. École Norm. Sup.* **39** (1922), 1–28.

[5] Kirwan, W. E., A note on extremal problems for certain classes of analytic functions. *Proc. Amer. Math. Soc.* **17** (1966), 1028–1030.

[6] Krzyż, J., Distortion theorems for bounded convex functions. II. *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **14** (1960), 7–18.

[7] Warschawski, S. E., On Hadamard's variation formula for Green's function. *J. Math. Mech.* **9** (1960), 497–512.