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An Inverse Distortion Theorem for Univalent Functions in the Unit Disk

Abstract. We give a sharp estimate for the distance of preimages of points which are close to each other in terms of the distance from the boundary $\partial f(\mathbb{D})$ if f is a univalent function in the unit disk $\mathbb{D}$.

In a recent paper [1] Andrievskii and one of the present authors used the following technical

Lemma 1. *Let f be univalent in the unit disk $\mathbb{D}$ and let $u, v \in \mathbb{D}$ be such that*

$$|f(u) - f(v)| \leq \frac{1}{2} d(f(u), \partial f(\mathbb{D})).$$

Then,

$$|u - v| \leq (e^2 - 1)\,(1 - |u|). \tag{1}$$

Here $d(a, C)$ denotes the distance of a point $a \in \mathbb{C}$ to the curve C. The original proof of this result made use of properties of the hyperbolic distance and certain crude estimates. In the present note we give a sharp version of this result, in a slightly more general setting.

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We introduce the functions G_d, univalent in $\mathbb{D}$ with $G_d(0) = 0$, $G_d'(0) = 1$, and $1/4 \le d \le 1$, which are uniquely defined by their image domains

$$G_d(\mathbb{D}) := \mathbb{C} \setminus \Gamma_d$$

with

$$\Gamma_d := \{x : \ -\infty < x \le -d\} \cup \{w : \ |w| = d, \, |\arg(-w)| \le \alpha\}$$

for some suitable $\alpha = \alpha(d) \le \pi$. An explicit construction of G_d is as follows: let $\frac{1}{4} \le d \le 1$ and write $(s \ge 0)$

$$d = \frac{(1+s)^2}{4}, \quad k(z) = \frac{z}{(1+z)^2}.$$

If $t(z)$ is defined as the solution of

$$k(t(z)) = 4k(s)k(z),$$

then

$$G_d(z) := d\frac{t(z)(1-st(z))}{s-t(z)}, \quad z \in \mathbb{D}.$$

Theorem 1. *Let f be analytic and univalent in $\mathbb{D}$, and for $u, v \in \mathbb{D}$ assume*

$$|f(u) - f(v)| < d(f(u), \partial f(\mathbb{D})).$$

With

$$\tilde{q} := \left| G_{\tilde{d}}^{(-1)} \left(-\frac{|f(u)-f(v)|}{|f'(u)|(1-|u|^2)} \right) \right|, \tag{2}$$

where

$$\tilde{d} := \frac{d(f(u), \partial f(\mathbb{D}))}{|f'(u)|(1-|u|^2)},$$

we then have

$$|u - v| \le \frac{\tilde{q}\,(1-|u|^2)}{1-|u|\,\tilde{q}}. \tag{3}$$

Furthermore, (3) is sharp.

It is well-known that $\frac{1}{4} \leq \tilde{d} \leq 1$ (compare [4; p.22]), so that $\tilde{q}$ is always well defined. Furthermore, we shall see that

$$\tilde{q} \leq \frac{|f(u) - f(v)|}{d(f(u), \partial f(\mathbb{D}))} =: q, \tag{4}$$

with equality only for $\tilde{d} = 1$ or $u = v$. This leads us to the following

Corollary 1. *Let f be analytic and univalent in $\mathbb{D}$ and for $u, v \in \mathbb{D}$, $q \in (0,1)$ assume*

$$|f(u) - f(v)| \leq q\ d(f(u), \partial f(\mathbb{D})). \tag{5}$$

Then

$$|u - v| \leq \frac{q\,(1 - |u|^2)}{1 - |u|q}. \tag{6}$$

If u, q are given, then (6) is sharp for

$$f(z) = \frac{z - u}{1 - \bar{u}z}$$

and

$$v = \begin{cases} \dfrac{u(1-q^2)}{1-q^2|u|^2} - \dfrac{u}{|u|}\dfrac{q(1-|u|^2)}{1-q^2|u|^2}, & u \neq 0, \\ q, & u = 0. \end{cases}$$

Note that (6) implies

$$|u - v| \leq \frac{2q}{1-q}\ d(u, \partial\mathbb{D}),$$

which makes the assumption and the conclusion more similar. This also extends Lemma 1 ($q = 1/2$), with $e^2 - 1$ replaced by 2 (the best possible constant).

For the proof of Theorem 1 we need a distortion theorem for the class $\mathcal{S}_d$ of univalent functions $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$f(0) = 0, \ f'(0) = 1, \ f(\mathbb{D}) \supset \{w : \ |w| < d\}.$$

Obviously $\mathcal{S}_d = \mathcal{S}$ for $0 < d \leq 1/4$ and $\mathcal{S}_d = \emptyset$ for $d > 1$. Furthermore $G_d \in \mathcal{S}_d$ for $1/4 \leq d \leq 1$.

Theorem 2. *Let $\frac{1}{4} \leq d \leq 1$ and $f \in \mathcal{S}_d$. Then, for $z \in \mathbb{D}$, we have*

$$|G_d(-|z|)| \leq |f(z)| \leq |G_d(|z|)|.$$

Theorem 2 is implicit in Baernstein's work [2], but has apparently never been stated explicitly. In fact, a combination of the proof of [2; Theorem 1] and the remark near the bottom of [2; p.163] yields that the convex monotonic integral means of $\log|f|$ for $f \in S_d$ are dominated by those of $\log|G_d|$ for $1/4 \leq d \leq 1$. Theorem 2 then follows by considering the convex functions e^{-px} and e^{px} (letting $p \to \infty$), using the fact that G_d is circularly symmetric and using the fact that the subdomains defined by $G_d(\{z : \ |z| < r\})$ are circularly symmetric as well for $0 < r < 1$ (see Jenkins [3; Theorem 2]).

Proofs

Proof of Theorem 1. Define

$$q := \frac{|f(u) - f(v)|}{d(f(u), \partial f(\mathbb{D}))},$$

so that $0 < q < 1$. We first translate the problem into a normalized form. Fix u in our assumption and define the function F (analytic in $\mathbb{D}$) through

$$F\left(\frac{z - u}{1 - \bar{u}z}\right) = f(z), \quad z \in \mathbb{D}.$$

Then the assumption reads

$$\left|F(0) - F\left(\frac{v - u}{1 - \bar{u}v}\right)\right| \leq q \ d(F(0), \partial F(\mathbb{D})).$$

With $G(w) := \dfrac{F(w) - F(0)}{F'(0)}$, and

$$w := \frac{v-u}{1-\bar{u}v}, \quad d := d(0, \partial G(\mathbb{D})) = \frac{d(F(0), \partial F(\mathbb{D}))}{F'(0)}$$

we arrive at

$$|G(w)| \le q\, d \tag{7}$$

as an equivalent to the assumption.

We also have

$$|u - v| = \left| u - \frac{w+u}{1+\bar{u}w} \right| = \left| \frac{w(1-|u|^2)}{1+\bar{u}w} \right|, \tag{8}$$

which shows that it suffices to determine the w of largest modulus which satisfies (7). The function G, however, is by construction a member of $\mathcal{S}_d$, so Theorem 2 implies

$$|G_d(-|w|)| \le |G(w)|$$

for all w. Therefore, if we choose $|w| > |G_d^{(-1)}(-qd)|$, then condition (7) can never hold. The identity $\tilde{q} = |G_d^{(-1)}(-qd)|$ together with (8) gives now (3). Concerning sharpness we fix $u \in \mathbb{D}, q \in (0,1)$, and $d \in [\frac{1}{4}, 1]$. Then calculate v from the relation

$$\frac{v-u}{1-\bar{u}v} = \frac{u}{|u|} G_d^{(-1)}(-qd).$$

The function

$$f(z) := G_d\left(\frac{|u|}{u} \frac{z-u}{1-\bar{u}z} \right)$$

fulfills

$$|f(u) - f(v)| = \left| G_d\left(\frac{|u|}{u} \frac{v-u}{1-\bar{u}v}\right) \right| = qd = q\, d(f(u), \partial f(\mathbb{D})),$$

and we also have

$$|u - v| = \frac{\tilde{q}(1-|u|^2)}{1-\tilde{q}|u|)}.$$

In the case $u = 0$ we replace $u/|u|$ by 1 to obtain a similar result. ■

Proof of Corollary 1. The relation (4) is equivalent to

$$|G_d^{(-1)}(-qd)| \leq q. \tag{9}$$

To see this we observe that $G_d^{(-1)}(dz)$ is analytic for $z \in \mathbb{D}$ and takes values in $\mathbb{D}$ only. Relation (9) now follows from the Schwarz lemma. The claim concerning sharpness is readily checked. ■

We wish to point out that, although Theorem 1, which relies on Theorem 2, is non-elementary, a very elementary proof for Corollary 1 is available. Starting out from (7), we only need to show that for $G_d \in \mathcal{S}_d$ we have

$$|G_d(w)| \leq qd \;\Rightarrow |w| \leq q. \tag{10}$$

As before the function $G_d^{(-1)}(dz), z \in \mathbb{D}$, is a Schwarz function, and if $h = G_d(w)$ satisfies $|h| \leq qd$, then

$$|w| = |G_d^{(-1)}(h)| = \left|G_d^{(-1)}(d\frac{h}{d})\right| \leq \left|\frac{h}{d}\right| \leq q.$$

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