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COEFFICIENT BOUNDS FOR SOME CLASSES OF STARLIKE FUNCTIONS

ROGER BARNARD AND JOHN L. LEWIS

Let t be given, $1/4 \leqq t \leqq \infty$, and let $S(t)$ denote the class of normalized starlike univalent functions f in $|z|<1$ satisfying (i) $|f(z)/z| \geqq t$, $|z|<1$, if $1/4 \leqq t \leqq 1$, (ii) $|f(z)/z| \leqq t$, $|z|<1$, if $1<t \leqq \infty$. If $f(z)=z+\sum_{k=2}^{\infty} a_k z^k \in S(t)$ and n is a fixed positive integer, then the authors obtain sharp coefficient bounds for $|a_n|$ when t is sufficiently large or sufficiently near 1/4. In particular a sharp bound is found for $|a_3|$ when $1/4 \leqq t \leqq 1$ and $5 \leqq t \leqq \infty$. Also a sharp bound for $|a_4|$ is found when $1/4 \leqq t \leqq 1$ or $12.259 \leqq t \leqq \infty$.

1. Introduction. Let S denote the class of starlike univalent functions f in $K=\{z:|z|<1\}$ with the normalization, $f(0)=0$, $f'(0)=1$. Given t, $1/4 \leqq t \leqq \infty$, let $S(t)$ denote the subclass of functions $f \in S$ satisfying

$$|f(z)/z| \geqq t,\ z \in K, \text{ if } 1/4 \leqq t \leqq 1, \tag{1.1}$$

$$|f(z)/z| \leqq t,\ z \in K, \text{ if } 1<t \leqq \infty. \tag{1.2}$$

If $1/4<t \leqq 1$, we let $F=F(\cdot,t)$ be defined by

$$zF'(z)/F(z)=[1+2(2b^2-1)z+z^2]^{1/2}/(1-z),\ z \in K, \tag{1.3}$$

where $0 \leqq b<1$ and $t=[(1+b)^{1+b}(1-b)^{1-b}]^{-1}$. The function $F=F(\cdot,t)$ defined by (1.3) is in $S(t)$ for $1/4<t \leqq 1$, as can be shown by a long but straightforward calculation (see Suffridge **[9]**). For fixed t, $1/4<t \leqq 1$, this function maps K onto the complex plane minus a set

$$\{w:|w| \geqq t, \qquad \pi b \leqq \arg w \leqq 2\pi-\pi b\}.$$

If $1<t<\infty$, we let $F=F(\cdot,t) \in S(t)$ be defined by

$$\frac{F(z)}{[1-t^{-1}F(z)]^2}=\frac{z}{(1-z)^2},\ z \in K. \tag{1.4}$$

It is well known (see Nehari [**4**, p. 224, ex. 4]) that the function F maps K onto a domain whose boundary consists of $\{w:|w|=t\}$, and a slit along the negative real axis from $-t$ to $-\lambda$ where $4\lambda t^2=(t+\lambda)^2$. If $t=1/4$ or $t=\infty$, we let

$$F(z,1/4)=F(z,\infty)=z/(1-z)^2,\ z\in K.$$

In [2] the authors proved a subordination theorem for some classes of univalent functions. For $S(t)$ this theorem may be stated as follows:

THEOREM A. *Let t be given, $1/4\leqq t\leqq\infty$. Let $F=F(\cdot,t)$ be as in* (1.3) *and* (1.4). *If $f\in S(t)$, then $\log f(z)/z, z\in K$, is subordinate to $\log F(z)/z$, $z\in K$.*

Theorem A implies for a given t, $1/4\leqq t\leqq\infty$, that $F=F(\cdot,t)$ solves a number of extremal problems in $S(t)$. Some of these problems were pointed out in [2]. There, however, only general prperties of subordination were used. In this note, for certain values of t, we use our specific knowledge of F, together with Theorem A, to obfain coefficient bounds for functions $f\in S(t)$. More specifically, we prove

THEOREM 1. *Let t be given, $1/4\leqq t\leqq\infty$. Let $F(z)=F(z,t)=z+\sum_{k=2}^{\infty}A_k(t)z^k$, $z\in K$, be as in* (1.3) *and* (1.4). *Let $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$, $z\in K$, be in $S(t)$. If n a positive integer is given ($n>2$), then there exist α_n, β_n satisfying $1/4<\alpha_n\leqq 1$, $1\leqq\beta_n<\infty$, with the property that*

$$|a_n|\leqq A_n(t), \tag{1.5}$$

whenever $1/4\leqq t<\alpha_n$ or $\beta_n<t\leqq\infty$. α_n and β_n may be chosen in such a way that equality holds in (1.5) *only if $f(z)=\eta^{-1}F(\eta z)$, $z\in K$, for some η, $|\eta|=1$. In particular*

$$|a_3|\leqq A_3(t)\ \text{if}\ 1/4\leqq t\leqq 1\ \text{or}\ 5<t\leqq\infty, \tag{1.6}$$

$$|a_4|\leqq A_4(t)\ \text{if}\ 1/4\leqq t\leqq 1\ \text{or}\ 12.259\leqq t\leqq\infty. \tag{1.7}$$

Equality holds in (1.6) *and* (1.7) *only if $f(z)=\eta^{-1}F(\eta z)$, $z\in K$, for some η, $|\eta|=1$.*

Let f and t be as in Theorem 1. We note that the inequality $|a_2|\leqq A_2(t)$, $1/4\leqq t\leqq\infty$, is an easy consequence of Theorem A (see [2]). We also note for $1\leqq t\leqq e$ that $|a_3|\leqq 1-t^{-2}$, where equality holds for the function $f\in S(t)$ defined by $f(z)=F(z^2,t^2)^{1/2}$, $z\in K$. This inequality is due to Tammi [10]. The problem of finding a sharp upper bound for $|a_3|$ when $f\in S(t)$, $e<t<5$, is still open. However, Barnard [1] has shown that the function which maximizes $|a_3|$ in $S(t)$ is either F or a function which maps K onto a domain whose boundary consists $\{w:|w|=t\}$ and two radial slits of equal length.

We remark that several authors have considered similar problems in the class $U(t)$ of normalized univalent functions f (i.e., $f(0)=0$,

$f'(0)=1$) bounded above by t, $1<t<\infty$. If $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$, $z\in K$, is in $U(t)$,then Schiffer and Tammi **[6]** showed that $|a_4|\leqq A_4(t)$, for $t\geqq 33\,1/3$. If in addition f has real coefficients, then Singh **[8]** proved that $|a_4|\leqq A_4(t)$ for $t\geqq 11$. Moreover, Schiffer and Tammi **[7]** have proved for each positive integer $n\geqq 2$, that there exists δ_n, $1<\delta_n<\infty$, with the following property: If $f\in U(t)$ and $1<t\leqq\delta_n$, then

$$|a_n|\leqq\frac{2}{n-1}(1-t^{1-n}).$$

Here equality holds for $f(z)=F(z^{n-1},t^{n-1})^{1/(n-1)}$, $z\in K$, which in fact is in $S(t)$. Hence the above inequality is also sharp for functions in $S(t)$ when $1<t\leqq\delta_n$. Finally we remark that Schiffer and Tammi **[6]** have shown that if suffices to take $\delta_4\leqq 34/19$.

2. *Proof of Theorem* 1. Let G, ω, be analytic in K and suppose that

(2.1) $$\omega(0)=0,$$

(2.2) $$|\omega(z)|\leqq 1,\ z\in K.$$

Put $g(z)=G[\omega(z)]$, $z\in K$. Suppose that $G(z)=\sum_{k=1}^{\infty} c_k z^k$, and $g(z)=\sum_{k=1}^{\infty} b_k z^k$. Then Rogosinski **[5**, Thm. VI] proved

THEOREM B. *Let n be a fixed positive integer. If $c_n>0$ and if there exists an analytic function P in K with positive real part satisfying*

$$P(z)=\frac{c_n}{2}+c_{n-1}z+c_{n-2}z^2+\cdots+c_1z^{n-1}+\sum_{k=n}^{\infty} d_k z^k$$

for $z\in K$, then $|b_n|\leqq|c_n|$. Equality can occur only if $g(z)=G(\eta z)$ for some η, $|\eta|=1$, or if $n>1$ and P has the form,

(2.3) $$P(z)=\sum_{i=1}^{J}\lambda_i\left(\frac{1+\varepsilon_i z}{1-\varepsilon_i z}\right),\ z\in K,$$

where $\lambda_i>0$, $|\varepsilon_i|=1$, $1\leqq i\leqq J$, and $J\leqq n-1$.

Furthermore, Carathéodory (see Tsuji **[11**, Ch. 4 §7]) proved

THEOREM C. *The function P in Theorem B exists if and only if the n by n matrix*

$$\begin{pmatrix} c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \\ c_{n-1} & c_n & c_{n-1} & \cdots & c_2 \\ c_{n-2} & c_{n-1} & c_n & \cdots & c_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_1 & c_2 & c_3 & \cdots & c_n \end{pmatrix}$$

is positive semi definite. If P exists, then P has the form (2.3) *only if the above matrix has determinant zero.*

We now use Theorems A, B, and C to prove Theorem 1. Let t be fixed, $1/4 \leqq t \leqq \infty$, and $f \in S(t)$. Then Theorem A implies there exists a function ω satisfying (2.1) and (2.2) for which $f(z)/z = F[\omega(z)]/\omega(z)$, $z \in K$. Hence we may use Theorems B and C with $g(z) = f(z)/z) - 1$, $G(z) = (F(z)/z) - 1$, $z \in K$, and $c_i = A_{i+1}(t)$, $1 \leqq i \leqq n-1$, to prove Theorem 1. To do so we shall want some notation.

Let n and k be fixed positive integers satisfying $2 \leqq k \leqq n$. Let $\delta(k,n,t)$ be the $k-1$ by $k-1$ matrix

$$\delta(k,n,t) = \begin{pmatrix} A_n(t) & A_{n-1}(t) & \cdots & A_{n-k+2}(t) \\ A_{n-1}(t) & A_n(t) & \cdots & A_{n-k+3}(t) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A_{n-k+2}(t) & A_{n-k+3}(t) & \cdots & A_n(t) \end{pmatrix} \tag{2.4}$$

Let $|\delta(k,n,t)|$ denote the determinant of $\delta(k,n,t)$. Then it is well known (see Hohn [3, Thm. 9, 17.3]) that $\delta(n,n,t)$ is positive definite if and only if $|\delta(k,n,t)| > 0$ for $2 \leqq k \leqq n$.

We note that $A_n(\infty) = A_n(1/4) = n$ for $n \geqq 2$. Using this fact we obtain that $|\delta(k,n,\infty)| = |\delta(k,n,1/4)| = (2n+2-k)\ 2^{k-3}$ for $2 \leqq k \leqq n$ and $n > 2$. Since (1.3) and (1.4) imply A_n is continuous as a function of t, $1/4 \leqq t \leqq \infty$, it follows that

$$\lim_{t\to\infty} |\delta(k,n,t)| = |\delta(k,n,\infty)| = \lim_{t\to 1/4} |\delta(k,n,t)| > 0$$

for each positive integer $n > 2$ and $2 \leqq k \leqq n$. From this inequality and our previous remark we see that $\delta(n,n,t)$ is positive definite for sufficiently large t and t near 1/4, say $1/4 \leqq t < \alpha_n$, $\beta_n < t \leqq \infty$. Using Theorems A, B, and C, it follows that (1.5) is true.

To prove (1.6) and (1.7) we make some explicit calculations. The case $t = 1$ is trivial since then $S(t)$ consists only of the identity function. First from (1.4) we find for $x = t^{-1}$, and $1 < t \leqq \infty$, that

$$\begin{aligned} A_2(t) &= 2(1-x), \\ A_3(t) &= (3-5x)(1-x), \\ A_4(t) &= (4+14x^2-16x)(1-x). \end{aligned} \tag{2.5}$$

Second if $1/4 \leqq t < 1$ and $a = 2b^2 - 1$ [b as in (1.3)], then from (1.3) we get

$$\begin{aligned} A_2(t) &= 1+a, \\ A_3(t) &= (1+a)(5+a)/4, \\ A_4(t) &= (1+a)(17+6a+a^2)/12. \end{aligned} \tag{2.6}$$

Here $-1 < a \leqq 1$.

To prove (1.6) it suffices, by the previous argument, to show that $A_2(t) > 0$ and

$$|\delta(3,3,t)| = A_3(t)^2 - A_2(t)^2 > 0$$

for $5 < t < \infty$ or $1/4 \leqq t < 1$. From (2.5) and (2.6) we see that these inequalities are valid for the above values of t. To prove (1.7), we need to show that $\delta(3,4,t) > 0$, $\delta(4,4,t) > 0$, for the stipulated values of t in Theorem 1. To do this we consider two cases. If $1 < t \leqq \infty$, and $x = 1/t$, then from (2.4) and (2.5) we have

$$|\delta(4,4,t)| = (1-x)^3 \begin{vmatrix} 4+14x^2-16x & 3-5x & 2 \\ 3-5x & 4+14x^2-16x & 3-5x \\ 2 & 3-5x & 4+14x^2-16x \end{vmatrix}$$

Adding the second row to the first and third rows we get

$$|\delta(4,4,t)| = (1-x)^5 \begin{vmatrix} 7(1-2x) & 1(1\times 2x) & 5 \\ 3-5x & 4+14x^2-16x & 3-5x \\ 5 & 7(1-2x) & 7(1-2x) \end{vmatrix}$$

Evaluating this determinant we obtain

$$|\delta(4,4,t)| = 4(1-x)^5(1-7x)[3-47x+126x^2-98x^3] > 0$$

for $12.259 \leqq t \leqq \infty$. It is easily checked that $|\delta(3,4,t)| = A_4^2(t) - A_3^2(t) > 0$ for $12.259 \leqq t \leqq \infty$. Hence (1.7) is true for $12.259 \leqq t \leqq \infty$.

If $1/4 \leqq t < 1$, then from (2.4), (2.6), we obtain

$$(12)^3|\delta(4,4,t)| = (1+a)^3 \begin{vmatrix} 17+6a+a^2 & 3(5+a) & 12 \\ 3(5+a) & 17+6a+a^2 & 3(5+a) \\ 12 & 3(5+a) & 17+6a+a^2 \end{vmatrix}$$

Subtracting the second row from the first and third rows, we get

$$(12)^3|\delta(4,4,t)| = (1+a)^5 \begin{vmatrix} a+2 & -a-2 & -3 \\ 3(5+a) & 17+6a+a^2 & 3(5+a) \\ -3 & -a-2 & a+2 \end{vmatrix}$$

Adding six times the first and third rows to the second of this determinant, we find that

$$(12)^3|\delta(4,4,t)| = (1+a)^6 \begin{vmatrix} a+2 & -a-2 & -3 \\ 9 & a-7 & 9 \\ -3 & -a-2 & a+2 \end{vmatrix}$$

Evaluating this determinant we obtain

$$(12)^3|\delta(4,4,t)| = (1+a)^6(a^3+15a^2+93a+215) > 0$$

for $-1 < a \leqq 1$. Hence $|\delta(4,4,t)| > 0$ for $1/4 \leqq t < 1$. It is easily checked that $|\delta(3,4,t)| > 0$ for $1/4 \leqq t < 1$. We conclude that (1.7) is true for $1/4 \leqq t < 1$. The proof of Theorem 1 is now complete.

Finally we remark for $1/4 \leqq t < 1$ that

$$48A_5(t) = (1+a)(74+38a+10a^2-2a^3) < 48A_4(t)$$

for t near 1, $t < 1$. It follows that $|\delta(3,5,t)| < 0$ for t near 1, $t < 1$. Hence our method does not imply for all t, $1/4 \leqq t \leqq 1$, that

$|a_5| \leqq A_5(t)$. However, it is still possible our method implies that α_n in Theorem 1 can be chosen independent of n.

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UNIVERSITY OF KENTUCKY