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On applications of hypergeometric functions

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Dedicated to Professor Haakon Waadeland on the occasion of his 70th birthday

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Abstract

This is a survey article on the author's involvement over the years with hypergeometric functions. We discuss our counter-example to one of M. Robertson's conjectures, our results on the omitted values problems, Brannan's conjecture on the coefficients of a certain power series, generalizations of Ramanujan's asymptotic formulas for complete elliptic integrals and Muir's 1883 approximation to the arc length of an ellipse involving an inequality for some ${}_3F_2$'s. © 1999 Elsevier Science B.V. All rights reserved.

This is a survey article on the author's involvement over the years with hypergeometric functions. Although much of my early work has been in the field of geometric function theory, hypergeometric functions have frequently reoccurred in many interesting ways. It started very early. In 1968 my masters thesis was on "Univalent solutions to hypergeometric differential equations", where Robertson's univalence criterion in [36] was applied to solutions of the differential equation, $w''+p(z)w=0$. Then, in 1972 while on a postdoctoral appointment at the University of Kentucky, I heard Richard Askey give one of his inspiring talks on why everyone should know all about hypergeometric functions. It left an impression.

In a very early paper of the author [7], one of Robertson's conjectures was investigated. To introduce some notation, let $(a)_n$ be Pochhammer's symbol for the generalized factorial: $(a)_n = a(a+1)\dots(a+n-1)$ and define

$${}_2F_1(a,b;c:z):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z_n.$$

Let $D_r=\{z\colon |z|<r\}$ with $D=D_1$. Let S be the class of functions, f, analytic on D, normalized by $f(z)=z+\cdots+b_nz^n+\cdots$ and S^* be those functions in S that map D onto a domain starlike with

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respect to the origin. In [37,38] Robertson had conjectured that

$$|2b_n - b_2 b_{n-1}| \leqslant 2 \tag{1}$$

for any f in S^* and had shown that this is true for $n = 1, 2$ and 3. He had shown in [38] that if (1) held in S^* then his conjecture that

$$|nb_n - mb_n| \leqslant |n^2 - m^2| \tag{2}$$

for all functions in the more general class of close to convex functions would be valid. Conjecture (2) was later verified in 1979 by Leung in [26]. However, in 1975 in [7] the function defined by

$$f_\alpha(z) = \sum_{k=0}^{\infty} \frac{(2-\alpha)_k}{k!}\, {}_2F_1(-k, \alpha; 1-\alpha-k : 1) z^{k+1} = \sum_{k=1}^{\infty} b_n(\alpha) z^n$$

was considered.

It follows from the geometric properties of $f(D)$ that f_α is in S^* for α in $[0, 2)$. It was noted that for $T_n(\alpha) = 2b_n(\alpha) - b_2(\alpha)b_{n-1}(\alpha)$, $T_5(0) = T_5(1) = T_5(2) = 2$, while $\partial T_5(\alpha)/\partial\alpha > 0$ at $\alpha = 0$. Thus (1) does not hold for some $\alpha > 0$ for $n = 5$. In fact, for $\alpha \neq 1$, and $0 < \alpha < 2$, it was shown that

$$2 < T_5(\alpha) \leqslant T_5(1 - \sqrt{2}/2) = 7/3.$$

In the 1980s, the author investigated, along with coauthors J.L. Lewis and K. Pearce, the omitted values problem first posed [23] in 1949 by Goodman, restated by MacGregor in his survey article [29] in 1972, then reposed in a more general setting by Brannan, [5] in 1977. It also appeared in Bernardi's survey article [16] and has appeared in several open problem sets since then including [9,20,31].

For a function f in S let $A(f)$ denote the Lebesgue measure of the set $D \backslash f(D)$ and let $L(f, r)$ denote the Lebesgue measure of the set $\{D \backslash f(D)\} \cap \{w\colon |w| = r\}$ for some fixed r, $0 < r < 1$. Two explicit problems posed by Goodman and Brannan were to determine

$$A = \sup_{f \in S} A(f)$$

and

$$L(r) = \sup_{f \in S} L(f, r).$$

Goodman had shown in [23] that $0.22\pi < A < 0.50\pi$ and with Reich in [24] improved the upper bound to 0.38π. In a series of papers we gave a geometric characterization in [8,28] of an extremal function f_0 for A and gave in [12] the currently best known lower bound, constructively, of $0.24\pi \leqslant A$. The upper bound is conceptually harder. Indeed it appears difficult to use our geometric description of $f_0(D)$ to calculate A directly. However, we used an indirect proof in [11] to obtain the best known upper bound of $A < 0.31\pi$.

Open Problem. Show that f_0 is unique and determine A explicitly.

The corresponding problem for starlike functions of determining

$$A^* = \sup_{f \in S^*} A(f)$$

was completely solved by Lewis in [28]. The uniquely defined extremal function f_1 in S^* gives $A^* = A(f_1) \approx 0.235\pi$. The corresponding problem for starlike functions of determining

$$L^*(r) = \sup_{f \in S^*} L(f,r)$$

was solved by Lewandowski in [27] and Stankiewicz in [39].

However, to this date, the interesting corresponding problems for the class S^c of convex functions in S remain open.

For the class S^c of functions in S whose images are convex domains the corresponding problem of determining

$$A^c(r) = \sup_{f \in S^c} A(f,r)$$

and

$$L^c(r) = \sup_{f \in S^c} L(f,r)$$

presents some interesting difficulties. One particular difficulty is that the basic tool of circular symmetrization used in the solution to each of the previous determinations is no longer useful. The example of starting with the convex domain bounded by a square shows that convexity is not always preserved under circular symmetrization. However, Steiner symmetrization (see [25,40]), can still be used in certain cases such as sectors. Another difficulty is the introduction of distinctly different extremal domains for different ranges of r. Since every function in S^c covers a disk of radius $\frac{1}{2}$ (see [22]) r needs only to be considered in the interval $(\frac{1}{2},1)$. Waniurski has obtained some partial results in [42]. He defined r_1 and r_2 to be the unique solutions to certain transcendental equations where $r_1 \approx 0.594$ and $r_2 \approx 0.673$. If $F_{\pi/2}$ is the map of D onto the half plane $\{w\colon \operatorname{Re} w > -\frac{1}{2}\}$ and F_α maps D onto the sector

$$\left\{w\colon \left|\arg\left(w + \frac{\pi}{4\alpha}\right)\right| < \alpha\right\}$$

whose vertex, $v = -\pi/4\alpha$, is located inside D, then

$$A^c(r) = A(F_{\pi/2}, r) \quad \text{for } 1/2 < r < r_1,$$
$$L^c(r) = L(F_{\pi/2}, r) \quad \text{for } 1/2 < r < r_1,$$

and

$$L^c(r) = L(F_\alpha, r) \quad \text{for } r_1 < r < r_2.$$

The author announced in his survey talk at the 1985 *Symposium on the Occasion of the Proof of the Bierberbach Conjecture* the following conjecture:

Conjecture. *The external domains in determining* $A^c(r)$ and $L^c(r)$ *will be half-planes, symmetric sectors and domains bounded by singles arcs of* $|w| = r$ *along with tangent lines to the endpoints of these arcs, the different domains depending on different ranges* r *in* $(\frac{1}{2},1)$.

This conjecture was also made independently by Waniurski at the end of his paper [42] in 1987.

Determining explicit values for $A^c(r)$ and $L^c(r)$ would involve computing the function that maps D onto the convex domain bounded by an arc of $\{w\colon |w| = r\}$ along with the two tangent lines at the endpoints of this arc. The function defining this map involves the quotient of two hypergeometric functions (cf. [33]). In particular, an extensive verification shows that the function g is given by a renormalization of

$$g(z) = \frac{z_2F_1((2\alpha-1)/4,(2\alpha+3)/4;1+\alpha:z)}{{}_2F_1((2\alpha+1)/-4,(3-2\alpha)/4;1-\alpha:z)}.$$

A difficulty arises when determining the explicit preimage of the center of the circle so that g can be renormalized to the mapping function f in S taking D onto a domain whose boundary circle is centered at the origin.

While spending a semester as a Research Scholar at the University of Kentucky in 1984, the author began working in approximation theory with L. Reichel who had just finished his degree under G. Dahlquist. We were using a differential equation model for difference equations developed by Dahlquist. We were investigating the problem of obtaining exact estimates for the discrete norm approximation of continuous functions by Gram polynomials in terms of a sup norm. We had reduced the problem to the verification of an inequality involving Kummer's hypergeometric function, ${}_1F_1$. From considerable numerical evidence we made the following conjecture.

Conjecture. ${}_1F_1((1-\alpha)/2;1:x) \leqslant {}_1F_1(\frac{1}{2};1:x),\ \alpha,x>0.$

This conjecture was also announced at the survey talk on open problems and conjectures in complex analysis and special function theory given at the *Symposium on the Proof of the Bieberbach Conjecture* in 1985. It was discussed at several open problems sessions at conferences and appeared in my survey paper "Open Problems and Conjectures in Complex Analysis" in [9]. While working with one of my former Ph.D students K. Richards, we combined Bailey's reduction formula and Hankel's integral formula for ${}_1F_1$'s with properties of Bessel functions to prove in [10] that the more general inequality,

$$\left|{}_1F_1\left(\frac{1-\alpha}{2};c:x\right)\right| \leqslant F\left(\frac{1}{2};c:x\right)$$

holds for all $\alpha\geqslant 0$, $c\geqslant\frac{1}{2}$ and $x\geqslant 0$. By using the asymptotic behaviour of Whittaker's version of Kummer's function we showed that the constants were sharp.

An innocent looking, but not so trivial, conjecture was made by Brannan in 1973 in [19] on the coefficients of a specific power series. The problem originated in the Brannan et al. paper [21] (later completed by Aharonov and Friedland in [1]) solving the coefficient problem for functions of bounded boundary rotation. Consider the coefficients in the expansion

$$\frac{(1-xz)^\alpha}{(1-z)^\beta} = \sum_{n=0}^{\infty} A_n^{(\alpha,\beta)}(x)z^n, \quad |x|=1,\ \alpha>0,\ \beta>0. \tag{3}$$

Brannan posed the problem as to when

$$|A_n^{(\alpha,\beta)}(x)| \leqslant A_n^{(\alpha,\beta)}(1). \tag{4}$$

He gave a short elegant proof that (4) held if $\beta=1$ and $\alpha\geqslant 1$. However, he showed the surprising result that for $\beta=1, 0<\alpha<1$, (4) did *not* hold in general for the even coefficients. He showed that for $x=\mathrm{e}^{\mathrm{i}\theta}$, (4) held for odd coefficients in a sufficiently small neighborhood of $\theta=0$. He also noted that for $0<\alpha<1$, $|A_3^{(\alpha,\alpha)}(x)| \leqslant A_3^{(\alpha,\alpha)}(1)$.

By using the expansion

$$A_n^{(\alpha,\beta)}(x) = \frac{(\beta)_n F(-n, -\alpha; 1-\beta-n : -x)}{n!}$$

and the properties of the hypergeometric function, we had shown the following:

1. $|A_n^{(\alpha,\beta)}(x)| < A_n^{(\alpha,\beta)}(1)$, for $\alpha \leqslant \beta$, $\beta+\alpha \geqslant 1$ and $|x| = 1$, $x \neq 1$ and $n = 1,2,3,\ldots$.
2. $|A_{2n+1}^{(\alpha,1)}(x)| \leqslant A_{2n+1}^{(\alpha,1)}(1)$, $n = 1,2,3,\ldots$ for $0 < \alpha < \alpha + \varepsilon$, $1-\delta < \alpha < 1$ for ε and δ sufficiently small and positive.
3. $|A_3^{(\alpha,\beta)}(x)| \leqslant A_3^{(\alpha,\beta)}(1)$, $0 < \alpha < \beta < 1$.

In [32], Moak has shown that (4) holds for $\alpha \geqslant 1$, $\beta \geqslant 1$. Milcetich, in [30], has shown that (4) holds for $n=5, \beta=1$ and $2 < \alpha < n$ but does *not* hold for noninteger α's less than $n-1$, β near zero, for odd $n \geqslant 3$. Along with our current Ph.D student W. Wheeler, using matrix theory and computer capabilities, we verified in [15] that (4) holds for $n = 7$, $0 < \alpha < 1$, $\beta = 1$.

The following conjecture on the coefficients in (3) is still open.

Conjecture. $|A_{2n+1}^{(\alpha,1)}(x)| \leqslant A_{2n+1}^{(\alpha,1)}(1)$ for $x = e^{i\alpha}$, $0 < \alpha < 1$ and $n \geqslant 4$.

Another interesting set of problems arose from M. Vuorinen's investigation of Gauss and Ramanujan's results on the complete elliptic integrals of the first and second kinds:

$$K(x) = \pi/2F\left(\frac{1}{2}, \frac{1}{2}; 1 : x^2\right) \quad \text{and} \quad E(x) = \pi/2F\left(-\frac{1}{2}, \frac{1}{2}; 1 : x^2\right). \tag{5}$$

Consider what are called the zero-balanced hypergeometric functions, $F(a,b,a+b,x)$ where $a,b > 0$. Gauss had shown that for the Beta function, $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, that

$$F(a,b;a+b:x) \sim \frac{1}{B(a,b)} \log\frac{1}{1-x} \quad \text{as } x \to 1,$$

while Ramanujan [17] refined this to

$$B(a,b)F(a,b;a+b:x) + \log(1-x) = R + O((1-x)\log(1-x))$$

with $R = 2\psi(1) - \psi(a) - \psi(b)$ where $\psi(x) = \Gamma'(x)/\Gamma(x)$ and $\psi(1) = -\gamma = -0.5772\ldots$. Over a number of years we were able to considerably refine these in [3] to the following result.

Theorem A. (1) *For* $a,b \in (0,\infty)$ *the function*

$$f(x) \equiv \frac{1 - F(a,b;a+b:x)}{\log(1-x)}$$

is strictly increasing for x *in* $(0,1)$ *and maps* $(0,1)$ *on to* $(ab/(a+b), 1/B)$, *where* $B = B(a,b)$.

(2) *For* $a,b \in (0,\infty)$ *the function*

$$g(x) \equiv BF(a,b;a+b:x) + \log(1-x)$$

is strictly decreasing on $(0,1)$ *and maps* $(0,1)$ *onto* (R,B), *where*

$$R = -\psi(a) - \psi(b) - 2\gamma.$$

Theorem B. *For* $a, b \in (0, \infty)$, *let*

$$f(x) = xF(a, b; a+b : x)/\log(1/(1-x))$$

on (0,1) *and let* B, R *be defined as in Theorem A.*

1. *If* $a, b \in (0,1)$, *then* f *is decreasing with range* $(1/B, 1)$.
2. *If* $a, b \in (1, \infty)$, *then* f *is increasing with range* $(1, 1/B)$.
3. *If* $a, b \in (0,1)$ *the function* $g_1(x) \equiv BF(a, b; a+b : x) + (1/x)\log(1-x)$ *is increasing on* (0,1) *and maps* (0,1) *onto* $(B-1, R)$.
4. *If* $a, b \in (1, \infty)$, *then* g_1 *is decreasing on* (0,1) *and maps* (0,1) *onto* $(R, B-1)$.

While studying relationships between the arithmetic-geometric means Borwein and Borwein [18] had proved that

$$F\left(\frac{1}{2}, \frac{1}{2}; 1 : 1 - x^c\right) < F\left(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 1 : 1 - x^d\right) \tag{6}$$

for all $x \in (0,1)$, with $c = 2, d = 3, \delta = \frac{1}{6}$.

We obtained the following generalization.

Theorem C. *For* $c, d \in (0, \infty)$ *with* $4c < \pi d$, *inequality* (6) *holds for all* $x \in (0,1)$ *and for al* $\delta \in (0, \delta_0)$, *where* $\delta_0 = ((d\pi - 4c)/(4\pi d))^{1/2}$.

Theorems A and B have recently appeared in the text [4]. An improvement and generalizatio of Theorem C has appeared in a brief survey by Ponnusamy [34]. More recently, the best possibl value of $\delta_0 = \delta_0(c, d)$ for which the truth of the general inequality.

$$F(a, b; a+b : \ 1 - x^c) < F(a - \delta, b + \delta; a + b + 1 : 1 - x^d) : a, b, c, d > 0, d > c,$$

for all $x \in (0,1)$ and all $\delta \in (0, \delta_0)$, has been obtained in [2] which in particular, settles Conjectur 4.10 (1) of [3]. This result gives a precise form of Theorem C. In Investigating properties o $E(x)$ in (5) Vuorinen [41] considered the fact that the arc length of an ellipse with semiaxes o length 1 and b, where $b < 1$ can be expressed as $L(1, b) = 2\pi F(\frac{1}{2}, -\frac{1}{2}; 1 : 1 - b^2)$. From antiquity various approximations for the arc length of an ellipse have been suggested. A relatively simpl approximation, first suggested by Muir in 1883 and again in Ramanujan's notebooks, is given by

$$L(1, b) \approx 2\pi[(1 + b^{3/2})/2]^{2/3}$$

(see Berndt's *Ramanujan's Notebooks* Vol III; [17]). A computer examination of this approximatio led Vuorinen to ask in his survey on open problems [41], which has been discussed at severa international conferences, if the function defined by

$$G(r) = F\left(\frac{1}{2}, -\frac{1}{2}; 1 : r\right) - [(1 + (1-r)^{3/4})/2]^{2/3}$$

is positive for r in $(0,1)$.

Using computer algebra and Sturm sequence arguments we initially showed [14] that, if $G(r) = \sum_{r=4}^{\infty} a_n r^n$, then the function $G(r)/r^4$ is an increasing map of $[0,1]$ onto $[\frac{1}{16384}, 2/\pi - \frac{1}{2^{2/3}}] \approx [0.00006$ 0.00667]. This shows the surprising accuracy of the original Muir–Ramanujan approximation. Com puter experiments suggested however, that a much stronger result held. Recently, in [13], Richards

Pearce and the author proved that $a_n > 0$ for $n \geqslant 4$. The proof is fairly involved using recurrence relations, a transformation formula for ${}_3F_2$'s of Thomae, Gauss's and Whipple's contiguous relations for ${}_2F_1$'s and ${}_3F_2$-type, respectively, and generating functions for certain generalized hypergeometric functions, ${}_3F_2$'s. A critical lemma in our proof is the general result that ${}_3F_2(-n, a, b; a+b+1, 1+\varepsilon-n : 1) > 0$ for all $n > 0$ whenever $1 > \varepsilon > ab/(a+b+1) > 0$, which we proved using an idea suggested in an early paper of Askey et al. [6].

For further reading

[35]

References

[1] D. Aharonov, S. Friedland, On an inequality connected with the coefficient conjecture for functions of bounded boundary rotation, Ann. Acad. Sci. Fenn. Ser. A.I. 524 (1972) 14.

[2] G.D. Anderson, S.-L. Qiu, M. Vuorinen, Precise estimates for differences of the Gaussian hypergeometric functions, J. Math. Anal. Appl. 215 (1997) 212–234.

[3] G.D. Anderson, R.W. Barnard, K.C. Richards, M.K. Vamanamurthy, M. Vuorinen, Inequalities for zero-balanced hypergeometric functions, Trans. Amer. Math. Soc. 347 (1995) 1713–1723.

[4] G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen, Conformal Invarients, Inequalities, and Quasiconformal Mappings, Wiley, New York, 1997.

[5] J.M. Anderson, K.F. Barth, D.A. Brannan, Research problems in complex analysis, Bull. London Math. Soc. 9 (1977) 129–162.

[6] R. Askey, G. Gasper, M. Ismail, A positive sum from summability theory, J. Approx. Theory 13 (1975) 413–420.

[7] R.W. Barnard, On a coefficient inequality for starlike functions, Proc. Amer. Math. Soc. 47 (1975) 429–430.

[8] R.W. Barnard, The omitted area problem for univalent functions, Contemp. Math. 38 (1985) 53–60.

[9] R.W. Barnard, Open problems and conjectures in complex analysis, Computational Methods and Functions Theory, Lecture Notes in Mathematics and Function Theory, vol. 1435, Springer Berlin, 1990, pp. 1–25.

[10] R.W. Barnard, G. Dahlquist, K. Pearce, L. Reichel, K.C. Richards, Gram polynomials and the Kummer function, J. Approx. Theory 94 (1998) 128–143.

[11] R.W. Barnard, J.L. Lewis, On the omitted area problem, Mich. Math. J. 34 (1987) 13–22.

[12] R.W. Barnard, K. Pearce, Rounding corners of gearlike domains and the omitted area problem, J. Comput. Appl. Math. 14 (1986) 217–226.

[13] R.W. Barnard, K. Pearce, K. Richards, An inequality involving the generalized hypergeometric function and the arc length of an ellipse, preprint.

[14] R.W. Barnard, K. Pearce, L. Schovanec, On the Muir–Ramanujan approximation to the arc length of an ellipse, SIAM J. Math. Anal., to appear.

[15] R.W. Barnard, K. Pearce, W. Wheeler, On a coefficient conjecture of Brannan, Complex Variables Theory Appl. 33 (1997) 51–61.

[16] S.D. Bernardi, A survey of the development of the theory of schlicht functions, Duke Math. J. 19 (1952) 263–287.

[17] B.C. Berndt, Ramanujan's Notebooks, Part III, Springer, New York, 1991.

[18] J. Borwein, P. Borwein, Inequalities for compound mean iterations with logarithmic asymptotes, J. Math. Anal. Appl. 177 (1993) 572–582.

[19] D.A. Brannan, On coefficient problems for certain power series, in: Clunie, Hayman (Eds.), Symposium on Complex Analysis, Cantebury, London Math. Soc. Lecture Note Ser. 12 (1973) 17–27.

[20] D.A. Brannan, J.G. Clunie (Eds.), Aspects of Contemporary Complex Analysis (Durham, 1979), Academic Press, London, 1980.

[21] D.A. Brannan, J.G. Clunie, W.E. Kirwan, On the coefficient problem for functions of bounded boundary rotation, Ann. Acad. Sci. Fenn. Ser. A.I. Math. 3 (523) (1973) 1–18.
[22] P.L. Duren, Univalent Functions, vol. 259, Springer, New York, 1980.
[23] A.W. Goodman, Note on regions omitted by univalent functions, Bull. Amer. Math. Soc. 55 (1949) 363–369.
[24] A.W. Goodman, E. Reich, On the regions omitted by univalent functions II, Can. J. Math. 7 (1955) 83–88.
[25] W.K. Hayman, Multivalent Functions, Cambridge Tracts in Math. and Math. Phys., Cambridge University Press, Cambridge, 1958.
[26] Y.J. Leung, Robertson's conjecture on the coefficients of close-to-convex functions, Proc. Amer. Math. Soc. 76 (1979) 89–94.
[27] Z. Lewandowski, On circular symmetrization of starshaped domains, Ann. Univ. Mariae Curie–Sklodowska, Sect A 17 (1963) 35–38.
[28] J.L. Lewis, On the minimum area problem, Indiana Univ. Math. J. 34 (1985) 631–661.
[29] T.H. MacGregor, Geometric problems in complex analysis, Amer. Math. Mon. 79 (1972) 447–468.
[30] J.G. Milcetich, On a coefficient conjecture of Brannan, J. Math. Anal. Appl. 139 (1989) 515–522.
[31] S. Miller (Ed.), Complex analysis, Brockport, NY, 1976, Lecture Notes in Pure and Appl. Math., vol. 36, Dekker, New York, 1978.
[32] D. Moak, An application of hypergeometric functions to a problem in functions theory, Int. J. Math. Math. Sci. 7 (3) (1984) 503–506.
[33] Z. Nehari, Conformal Mapping, McGraw-Hill, New York, 1952.
[34] S. Ponnusamy, Role of special functions in function theory, in: K. Srinivasa Rao, R. Jagannthan, G. Vanden Berghe, J. Van der Jeugy (Eds.), Proceedings of Special Functions and Differential Equations, Allied Publishers, 1998, in press.
[35] S. Ponnusamy, M. Vuorinen, Asymptotic expansion and inequalities for hypergeometric functions, Mathematika 44 (1997) 278–301.
[36] M.S. Robertson, Schlicht solutions of $w'' + pw = 0$, Trans. Amer. Math. Soc. 76 (1954) 254–274.
[37] M.S. Robertson, A generalization of the Bieberbach coefficient problem for univalent functions, Mich. Math. J. 13 (1966) 185–192.
[38] M.S. Robertson, Quasi-subordination and coefficient inequalities, Bull. Amer. Math. Soc. 76 (1970) 1–9.
[39] J. Stankiewiez, On a family of starlike functions. Ann. Univ. Mariae Curie–Sklodowska, Sect A. 22–24 (1968–70) 175–181.
[40] F.A. Valentine, Convex Sets, McGraw-Hill, New York, 1964.
[41] M. Vuorinen, Hypergeometric functions in geometric function theory, in: K. Srinivasa Rao, R. Jagannthan, G. Van der Jeugy, (Eds.), Proceedings of Speicial Functions and Differential Equations, Allied Publishers, 1998, in press.
[42] J. Waniurski, On values omitted by convex univalent mappings, Complex Variables Theory and Appl. 8 (1987) 173–180.

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[2] H. Brunner, A survey of recent advances in the numerical treatment of Volterra integral and integro-differential equations, *J. Comput. Appl. Math.* **8** (1982) 213–229.
[3] P.R. Graves-Morris, The numerical calculation of Padé approximants, in: L. Wuytack, Ed., *Padé Approximation and its Application* (Springer, Berlin, 1979) 231–245.

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