International Journal of Difference Equations
ISSN 0973-6069, Volume 5, Number 1, pp. 41–54 (2010)
http://campus.mst.edu/ijde

Oscillation Results for Second-Order Delay Dynamic Equations

Raegan Higgins
Texas Tech University
Department of Mathematics & Statistics
Lubbock, TX 79409-1042, U.S.A.
raegan.higgins@ttu.edu

Abstract

In this paper, we consider the second-order linear dynamic equation

$$y^{\Delta\Delta}(t) + q(t)y(\tau(t)) = 0$$

on a time scale $\mathbb{T}$. Our goal is to establish some new oscillation results for this equation. Here we assume that $\tau(t) \leq t$ and $\tau : \mathbb{T} \to \mathbb{T}$. We apply results from the theory of lower and upper solutions for related dynamic equations along with some additional estimates on positive solutions.

AMS Subject Classifications: 34K11, 39A10, 39A99.
Keywords: Oscillation, delay dynamic equations, time scale, Riccati technique.

1 Introduction

In 1988 the theory of time scales was introduced by Stefan Hilger in his Ph.D. Thesis in order to unify continuous and discrete analysis (see [10]). Not only does this theory unify those of differential equations and difference equations, but it also extends these classical situations to cases "in between" – e.g., to the so-called q-difference equations. Moreover, the theory can be applied to other different types of time scales. Since its introduction, many authors have expounded on various aspects of this new theory, and we refer specifically to the paper by Agarwal et al. [1] and the references cited therein. A book on the subject of time scales by Bohner and Peterson [4] summarizes and organizes much of time scale calculus.

In recent years, there has been an increasing interest in studying the oscillation and nonoscillation of solutions of dynamic equations on a time scale (i.e., an arbitrary

Received February 25, 2009; Accepted June 1, 2009
Communicated by Johnny Henderson

nonempty closed subset of the real numbers). This has lead to many attempts to harmonize the oscillation theory for the continuous and the discrete cases, to include them in one comprehensive theory, and to extend the results to more general time scales. We refer the reader to the papers [2, 5, 9, 13, 14], and the references cited therein.

Since we are interested in the oscillatory behavior of solutions near infinity, we assume throughout this paper that our time scale is unbounded above. We assume $t_0 \in \mathbb{T}$ and it is convenient to assume $t_0 > 0$. We define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by

$$[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}.$$

Our main interest is to consider the second-order linear dynamic equation

$$y^{\Delta\Delta}(t) + q(t)y(\tau(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}} \tag{1.1}$$

where $q \in C_{rd}(\mathbb{T}, [0, \infty))$ and where the delay $\tau \in C_{rd}(t_0, \mathbb{T})$ is such that

$$0 < \tau(t) \leq t \quad \text{for all } t \in [t_0, \infty)_{\mathbb{T}} \quad \text{and} \quad \lim_{t\to\infty} \tau(t) = \infty.$$

Let us recall that a solution of (1.1) is a nontrivial real-valued function y satisfying equation (1.1) for $t \geq t_0$. A solution y of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions $y(t)$ of (1.1) which exist on some half-line $[t_y, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|y(t)| : t > T\} > 0$ for any $T \geq t_y$.

We note that (1.1) in its general form includes several types of differential and difference equations with delay arguments. In addition, different equations correspond to the choice of the time scale $\mathbb{T}$. For example, when $\mathbb{T} = \mathbb{R}$, we have $y^{\Delta} = y'$, and so (1.1) becomes the delay differential equation

$$y''(t) + q(t)y(\tau(t)) = 0.$$

In the case $\mathbb{T} = \mathbb{Z}$, $y^{\Delta} = \Delta y$ and (1.1) becomes the second-order delay difference equation

$$\Delta\Delta y(t) + q(t)y(\tau(t)) = 0$$

where Δ denotes the forward difference operator. Finally, when $\mathbb{T} = \left\{q_0^k : k \in \mathbb{N}_0\right\}$ with $q_0 > 1$, (1.1) becomes the second-order delay q_0-difference equation

$$y(q_0^2 t) - (q_0 + 1)y(q_0 t) + q_0 y(t) + q_0(q_0 - 1)^2 t^2 q(t)y(\tau(t)) = 0.$$

In this paper we intend to use the method of upper and lower solutions to obtain oscillation criteria for (1.1) under certain conditions. We also use results about

$$y^{\Delta\Delta}(t) + q(t)y^{\sigma}(t) = 0$$

to obtain results for (1.1). Our results generalize those given in Erbe [6].

2 Preliminary Results

In this section, we state fundamental results needed to prove our main results. We begin with the following lemma.

Lemma 2.1 (See [7, Lemma 1.2]). *Let $y(t)$ be a solution of*

$$y^{\Delta\Delta}(t) + \sum_{i=1}^{n} q_i(t)y(\tau_i(t)) = 0$$

which satisfies

$$y(t) > 0, \quad y^{\Delta}(t) > 0, \quad \textit{and} \quad y^{\Delta\Delta}(t) \leq 0$$

for all $\tau_i(t) \geq T \geq t_0$. Then for each $1 \leq i \leq n$ we have

$$y(\tau_i(t)) \geq \left(\frac{\tau_i(t) - T}{\sigma(t) - T}\right) y^{\sigma}(t), \quad \tau_i(t) > T.$$

In order to prove our main results, we need a method for studying boundary value problems (BVP). Namely we will define functions called upper and lower solutions that, not only imply the existence of a solution of a certain BVP, but also provide bounds on that solution. Consider the second-order equation

$$y^{\Delta\Delta} = f(t, y^{\sigma}) \tag{2.1}$$

where f is continuous on $[a, b]_{\mathbb{T}} \times \mathbb{R}$.

Definition 2.2 (See [4, Definition 6.53]). We say that $\alpha \in C^2_{rd}$ is a *lower solution* of (2.1) on $[a, \sigma^2(b)]_{\mathbb{T}}$ provided

$$\alpha^{\Delta\Delta}(t) \geq f(t, \alpha^{\sigma}(t)) \quad \text{for all} \quad t \in [a, b]_{\mathbb{T}}.$$

Similarly, $\beta \in C^2_{rd}$ is called an *upper solution* of (2.1) on $[a, \sigma^2(b)]_{\mathbb{T}}$ provided

$$\beta^{\Delta\Delta}(t) \leq f(t, \beta^{\sigma}(t)) \quad \text{for all} \quad t \in [a, b]_{\mathbb{T}}.$$

Theorem 2.3 (See [4, Theorem 6.54]). *Let f be continuous on $[a, b]_{\mathbb{T}} \times \mathbb{R}$. Assume that there exist a lower solution α and an upper solution β of* (2.1) *with*

$$\alpha(a) \leq A \leq \beta(a) \quad \textit{and} \quad \alpha(\sigma^2(b)) \leq B \leq \beta(\sigma^2(b))$$

such that

$$\alpha(t) \leq \beta(t) \quad \textit{for all} \quad t \in [a, \sigma^2(b)]_{\mathbb{T}}.$$

Then the BVP

$$y^{\Delta\Delta} = f(t, y^{\sigma}) \textit{ on } [a, b]_{\mathbb{T}}, \quad y(a) = A, \quad y(\sigma^2(b)) = B$$

has a solution y with

$$\alpha(t) \leq y(t) \leq \beta(t) \quad \textit{for all} \quad t \in [a, \sigma^2(b)]_{\mathbb{T}}.$$

We conclude this section with the following generalization of [11, Theorem 7.4].

Theorem 2.4. *Let f be continuous on $[a,b]_{\mathbb{T}} \times \mathbb{R}$. Assume that there exist a lower solution α and an upper solution β of* (2.1) *with $\alpha(t) \le \beta(t)$ for all $t \in [a,\infty)_{\mathbb{T}}$. Then for any $\alpha(a) \le c \le \beta(a)$ the BVP*

$$y^{\Delta\Delta} = f(t, y^{\sigma}), \quad y(a) = c \tag{2.2}$$

has a solution y with

$$\alpha(t) \le y(t) \le \beta(t) \quad \textit{for all} \quad t \in [a,\infty)_{\mathbb{T}}.$$

Proof. It follows from Theorem 2.3 that for each $n \ge 1$ there is a solution $y_n(t)$ of $y^{\Delta\Delta} = f(t, y^{\sigma})$ on $[a,t_n]_{\mathbb{T}}$ with $y_n(a) = c,\ y_n(t_n) = \beta(t_n)$ and $\alpha(t) \le y_n(t) \le \beta(t)$ on $[a,t_n]_{\mathbb{T}}$, where $t_n \to \infty$ as $n \to \infty$. Thus, for any fixed $n \ge 1$, $y_m(t)$ is a solution on $[a,t_n]_{\mathbb{T}}$ satisfying $\alpha(t) \le y_m(t) \le \beta(t)$ for all $m \ge n$. Hence, for $m \ge n$, the sequence $y_m(t)$ is pointwise bounded on $[a,t_n]_{\mathbb{T}}$.

We claim that $\{y_m(t)\}$ is equicontinuous on $[a,t_N]_{\mathbb{T}}$ for any fixed $N \ge 1$. Since f is continuous and $y_m(t) \le \beta(t)$ for all $t \in [a,t_N]_{\mathbb{T}}$, there is constant $K_N > 0$ such that $|y_m^{\Delta\Delta}(t)| = |f(t, y_m^{\sigma}(t))| \le K_N$ for all $t \in [a,t_N]_{\mathbb{T}}$. It follows that

$$\begin{aligned} \left|y_m^{\Delta}(t) - y_m^{\Delta}(a)\right| &= \int_a^t y_m^{\Delta\Delta}(s)\,\Delta s \\ &\le \int_a^t K_N\,\Delta s \\ &= K_N(t-a) \\ &\le K_N(t_N - a) \end{aligned}$$

which gives that

$$|y_m^{\Delta}(t)| \le |y_m^{\Delta}(a)| + |K_N(t_N - a)|.$$

Since $\{y_m(t)\}$ is uniformly bounded on $[a,t_N]_{\mathbb{T}}$ for all $m \ge N$, it follows that $|y_m^{\Delta}(a)| \le L_N$ for some $L_N > 0$ and all $m \ge N$. Consequently,

$$|y_m^{\Delta}(t)| \le L_N + |K_N(t_N - a)| =: M_N,$$

and so,

$$|y_m(t) - y_m(s)| = \left|\int_s^t y_m^{\Delta}\,\Delta s\right| \le M_N|t-s| < \epsilon$$

for all $t, s \in [a,t_N]_{\mathbb{T}}$ provided $|t-s| < \delta = \dfrac{\epsilon}{M_N}$. Hence the claim holds.

So by Ascoli–Arzela and a standard diagonalization argument, $\{y_m(t)\}$ contains a subsequence which converges uniformly on all compact subintervals $[a,t_N]_{\mathbb{T}}$ of $[a,\infty)_{\mathbb{T}}$ to a solution $y(t)$, which is the desired solution of (2.2) that satisfies $\alpha(t) \le y(t) \le \beta(t)$ for all $t \in [a,\infty)_{\mathbb{T}}$. $\square$

3 Main Results

In this section we give four results concerning the oscillatory behavior of

$$y^{\Delta\Delta}(t) + q(t)y(\tau(t)) = 0 \tag{1.1}$$

on the time scale $[t_0, \infty)_{\mathbb{T}}$ where $\sup \mathbb{T} = \infty$ and $q \in C_{rd}\left([0,\infty)_{\mathbb{T}}, [0,\infty)\right)$. These are Theorems 3.1 and 3.9 and Corollaries 3.4 and 3.7.

Theorem 3.1. *Assume that the equation*

$$y^{\Delta\Delta} + \lambda \frac{\tau(t)}{\sigma(t)} q(t) y^{\sigma}(t) = 0 \tag{3.1}$$

is oscillatory on $[t_0, \infty)_{\mathbb{T}}$ *for some* $0 < \lambda < 1$. *Then all solutions of* (1.1) *are oscillatory.*

Proof. Suppose, to the contrary, that (1.1) has an eventually positive solution u. That is, since $\tau(t) \to \infty$ as $t \to \infty$, there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that $u(t) > 0$ and $u(\tau(t)) > 0$ for $t \geq T$. As $q(t) \geq 0$ on $[t_0, \infty)_{\mathbb{T}}$, we have

$$u^{\Delta\Delta}(t) = -q(t)u(\tau(t)) \leq 0 \quad \text{for all} \quad t \geq T, \tag{3.2}$$

and so u^{Δ} decreases to a limit which must be nonnegative. In fact, we must have $u^{\Delta}(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. Indeed, if $u^{\Delta}(t_1) = 0$ for some $t_1 > T$, then $u^{\Delta}(t) \equiv 0$ on $[t_1, \infty)_{\mathbb{T}}$. Consequently, from (1.1) we would have $q(t) \equiv 0$ on $[t_1, \infty)_{\mathbb{T}}$, since $u(\tau(t)) > 0$ on $[T, \infty)_{\mathbb{T}}$, contradicting the fact that (3.1) is oscillatory. So we have

$$u(t) > 0, \quad u^{\Delta}(t) > 0, \quad u^{\Delta\Delta}(t) \leq 0 \quad \text{on} \quad [T, \infty)_{\mathbb{T}}.$$

For any $0 < k < 1$ there is a $T_k \geq T$ such that

$$u(\tau(t)) \geq \left(\frac{\tau(t) - T}{\sigma(t) - T}\right) u^{\sigma}(t) \geq k\frac{\tau(t)}{\sigma(t)} u^{\sigma}(t), \quad t \geq T_k$$

by Lemma 2.1. It follows that

$$u^{\Delta\Delta}(t) + k\frac{\tau(t)}{\sigma(t)} q(t) u^{\sigma}(t) \leq 0, \quad t \geq T_k. \tag{3.3}$$

Let $z(t) = \dfrac{u^{\Delta}(t)}{u(t)}$ and $Q(t) = k\dfrac{\tau(t)}{\sigma(t)} q(t)$. Also, let

$$S[z] = \frac{z^2}{1 + \mu(t) z}.$$

Then

$$1 + \mu(t) z(t) = 1 + \mu(t) \frac{u^{\Delta}(t)}{u(t)} > 0$$

for $t \geq T$ and

$$\begin{aligned} z^{\Delta} + Q + S(z) &= \frac{uu^{\Delta\Delta} - (u^{\Delta})^2}{uu^{\sigma}} + Q + \left(\frac{u^{\Delta}}{u}\right)^2 \frac{1}{1 + \mu\frac{u^{\Delta}}{u}} \\ &= \frac{uu^{\Delta\Delta} - (u^{\Delta})^2}{uu^{\sigma}} + Q + \left(\frac{u^{\Delta}}{u}\right)^2 \frac{u}{u + \mu u^{\Delta}} \\ &= \frac{uu^{\Delta\Delta} - (u^{\Delta})^2}{uu^{\sigma}} + Q + \frac{(u^{\Delta})^2}{uu^{\sigma}} \\ &= \frac{u^{\Delta\Delta}}{u^{\sigma}} + Q \\ &\leq 0 \end{aligned}$$

by (3.2). Hence, by [7, Lemma 1.1], $u^{\Delta\Delta} + Qu^{\sigma} = 0$ is nonoscillatory. Choosing $0 < k < 1$ such that $k > \lambda$, we have $Q(t) > \lambda\frac{\tau(t)}{\sigma(t)}q(t) =: R(t)$. By the Sturm–Picone comparison theorem [8, Lemma 6], we therefore have $u^{\Delta\Delta} + R(t)u^{\sigma}(t) = 0$ is nonoscillatory. This contradiction proves the theorem. □

Before we give the first corollary of Theorem 3.1 we prove the following.

Theorem 3.2. *Assume there is a* $t_* \geq a \in \mathbb{T}$ *and a* $u \in C^1_{rd}[t_*, \infty)$ *such that* $u(t) > 0$ *on* $[t_*, \infty)_{\mathbb{T}}$ *and*

$$\int_{t_*}^{\infty} \{q(t)[u^{\sigma}(t)]^2 - [u^{\Delta}(t)]^2\}\Delta t = \infty.$$

Then the second-order dynamic equation

$$y^{\Delta\Delta}(t) + q(t)y^{\sigma}(t) = 0 \tag{3.4}$$

is oscillatory on $[a, \infty)_{\mathbb{T}}$.

Proof. We prove this theorem by contradiction. So assume (3.4) is nonoscillatory on $[a, \infty)_{\mathbb{T}}$. By [4, Theorem 4.61], y is a solution of (3.4) which is dominant at ∞ such that for $t^* \geq a$, sufficiently large,

$$\int_{t^*}^{\infty} \frac{\Delta t}{y(t)y^{\sigma}(t)} < \infty,$$

and we may assume $y(t) > 0$ on $[t^*, \infty)_{\mathbb{T}}$. Let t_* and u be as in the statement of this theorem. Let T =max$\{t_*, t^*\}$; then let

$$z(t) := \frac{y^{\Delta}(t)}{y(t)}, \quad t \geq T.$$

It follows that

$$\begin{aligned} z^{\Delta}(t) &= \frac{y^{\Delta\Delta}(t)y(t) - (y^{\Delta}(t))^2}{y(t)y^{\sigma}(t)} \\ &= -\frac{q(t)y^{\sigma}(t)}{y^{\sigma}(t)} - \left(\frac{y^{\Delta}(t)}{y(t)}\right)^2 \frac{y(t)}{y^{\sigma}(t)} \\ &= -q(t) - z^2(t)\frac{y(t)}{[y(t)+\mu(t)y^{\Delta}(t)]} \\ &= -q(t) - \frac{z^2(t)}{1+\mu(t)z(t)} \end{aligned}$$

and

$$1+\mu(t)z(t) > 0 \quad \text{for all } t \geq T.$$

Then by [4, Theorem 4.55], we have for $t \geq T$

$$\begin{aligned} &(zu^2)^{\Delta}(t) \\ &= [u^{\Delta}(t)]^2 - q(t)u^2(\sigma(t)) - \left\{\frac{z(t)u(\sigma(t))}{\sqrt{1+\mu(t)z(t)}} - \sqrt{1+\mu(t)z(t)}\,u^{\Delta}(t)\right\}^2 \\ &\leq [u^{\Delta}(t)]^2 - q(t)u^2(\sigma(t)). \end{aligned}$$

Integrating from T to t, we obtain

$$z(t)u^2(t) \leq z(T)u^2(T) - \int_T^t \left\{q(t)u^2(\sigma(t)) - [u^{\Delta}(t)]^2\right\} \Delta t$$

which implies

$$\lim_{t\to\infty} z(t)u^2(t) = -\infty.$$

However, then there is a $T_1 \geq T$ such that for $t \geq T_1$

$$z(t) = \frac{y^{\Delta}(t)}{y(t)} < 0.$$

This implies that $y^{\Delta}(t) < 0$ for $t \geq T_1$, and hence y is decreasing on $[T_1,\infty)_{\mathbb{T}}$. However,

$$\begin{aligned} \int_{T_1}^{\infty} 1\,\Delta s &= y(T_1)y^{\sigma}(T_1)\int_{T_1}^{\infty} \frac{1}{y(T_1)y^{\sigma}(T_1)}\,\Delta s \\ &\leq y(T_1)y^{\sigma}(T_1)\int_{T_1}^{\infty} \frac{1}{y(s)y^{\sigma}(s)}\,\Delta s \\ &< \infty, \end{aligned}$$

which is a contradiction. $\square$

The following example is illustrative.

Example 3.3. If $a > 0$ and

$$\int_a^\infty \sigma^\alpha(t)q(t)\,\Delta t = \infty,$$

where $0 < \alpha < 1$, then $y^{\Delta\Delta} + q(t)y^\sigma = 0$ is oscillatory on $[a, \infty)_{\mathbb{T}}$. We will show that this follows from Theorem 3.2. In the Pötzsche chain rule [4, Theorem 1.90], let $g(t) = t$ and $f(t) = t^{\frac{\alpha}{2}}$, for $0 < \alpha < 1$. Then with $u(t) = (f \circ g)(t) = t^{\frac{\alpha}{2}}$, we have

$$\begin{aligned} u^\Delta(t) = (f \circ g)^\Delta(t) &= \left\{ \int_0^1 \frac{\alpha}{2} \left(t + h\mu(t) \cdot 1\right)^{\frac{\alpha-2}{2}} \,dh \right\} \cdot 1 \\ &= \frac{\alpha}{2} \int_0^1 \left(t + h\mu(t)\right)^{\frac{\alpha-2}{2}} \,dh \\ &\le \frac{\alpha}{2} \int_0^1 t^{\frac{\alpha-2}{2}}\,dh \\ &= \frac{\alpha}{2} t^{\frac{\alpha-2}{2}} \end{aligned}$$

since $\alpha - 2 < 0$. Therefore, it follows that $(u^\Delta(t))^2 \le \dfrac{\alpha^2}{4} t^{\alpha-2}$ for all t. Hence,

$$\int_a^\infty \left\{ q(t)[u^\sigma(t)]^2 - [u^\Delta(t)]^2 \right\} \,\Delta t \ge \int_a^\infty \left\{ q(t)\sigma^\alpha(t) - \frac{\alpha^2}{4} t^{\alpha-2} \right\} \,\Delta t = \infty$$

since $0 < \alpha < 1$ implies

$$\int_a^\infty t^{\alpha-2}\,\Delta t < \infty.$$

Thus $y^{\Delta\Delta} + q(t)y^\sigma = 0$ is oscillatory on $[a, \infty)_{\mathbb{T}}$ by Theorem 3.2.

As a corollary to Theorem 3.1, we have the following.

Corollary 3.4. *All solutions of*

$$y^{\Delta\Delta} + q(t)y(\tau(t)) = 0 \tag{1.1}$$

are oscillatory in case either of the following holds:

(i) $\displaystyle\int^\infty (\sigma(t))^{\alpha-1}\tau(t)q(t)\,\Delta t = \infty$ *for some* $\alpha \in (0, 1)$

(ii) $\displaystyle\liminf_{t\to\infty} t \int_t^\infty \frac{\tau(t)}{\sigma(t)} q(t)\,\Delta t > \frac{1}{4}$ *and* $\mu(t)$ *is bounded*.

Proof. If (i) holds, then for any $\lambda > 0$, $\int^{\infty} \sigma^{\alpha}(t)\lambda\frac{\tau(t)}{\sigma(t)}q(t)\,\Delta t = \infty$. By Example 3.3, $y^{\Delta\Delta} + \lambda\frac{\tau(t)}{\sigma(t)}q(t)y^{\sigma}(t) = 0$ is oscillatory since $0 < \alpha < 1$. Hence, by Theorem 3.1, all solutions of $y^{\Delta\Delta} + q(t)y(\tau(t)) = 0$, equation (1.1), are oscillatory.

Next assume (ii) holds. Then by [12, Theorem 4],

$$y^{\Delta\Delta} + \frac{\tau(t)}{\sigma(t)}q(t)y^{\sigma}(t) = 0$$

is oscillatory. Since $\mu(t)$ is bounded, we have that $y^{\Delta\Delta} + \frac{\tau(t)}{\sigma(t)}q(t)y(\tau(t)) = 0$ is oscillatory by [14, Theorem 2.1] with $f(t) = t$. By the Sturm–Picone comparison theorem [8, Lemma 6], we have (1.1) is oscillatory since $0 < \tau(t) \le t \le \sigma(t)$. □

To prove our second corollary of Theorem 3.1, we will use the method of upper and lower solutions and the following lemma.

Lemma 3.5. *If*

$$\int^{\infty} \tau(t)q(t)\Delta t = \infty, \tag{3.5}$$

then every bounded solution of equation (3.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Suppose that there exists an eventually positive and bounded solution y of (3.1). Then there exists $T \in \mathbb{T}$ such that

$$y(t) > 0, \quad y^{\Delta}(t) > 0, \quad y^{\Delta\Delta}(t) \le 0 \quad \text{for all} \quad t \ge T \ge t_0,$$

and without loss of generality, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$0 < \alpha < y(t) < \beta \quad \text{for all} \quad t \ge T.$$

Let $Y(t) = ty^{\Delta}(t)$. Then

$$\begin{aligned} Y(t) &= Y(T) + \int_T^t Y^{\Delta}(s)\,\Delta s \\ &= Y(T) + \int_T^t \left\{y^{\Delta}(s) + \sigma(s)y^{\Delta\Delta}(s)\right\}\,\Delta s \\ &= Y(T) + \int_T^t \left\{y^{\Delta}(s) - \sigma(s)\lambda\frac{\tau(s)}{\sigma(s)}q(s)y^{\sigma}(s)\right\}\,\Delta s \\ &= Y(T) + y(t) - y(T) - \lambda\int_T^t \tau(s)q(s)y^{\sigma}(s)\,\Delta s \\ &\le Y(T) + \beta - y(T) - \lambda\int_T^t \tau(s)q(s)y^{\sigma}(s)\,\Delta s \\ &\le Y(T) + \beta - y(T) - \lambda\alpha\int_T^t \tau(s)q(s)\,\Delta s \\ &\to -\infty \text{ as } t \to \infty, \end{aligned}$$

i.e., there is a constant $M > 0$ such that

$$y^{\Delta}(t) \leq -\frac{M}{t} \quad \text{for } t \geq \tilde{T}$$

for some $\tilde{T} \geq T$, and this implies that $\lim_{t\to\infty} y(t) = -\infty$ by [3, Example 5.15], contradicting $y(t) > 0$ for all $t \geq T$. Thus every bounded solution of

$$y^{\Delta\Delta} + \lambda\frac{\tau(t)}{\sigma(t)}q(t)y^{\sigma}(t) = 0$$

is oscillatory. □

Example 3.6. Consider the delay dynamic equation

$$y^{\Delta\Delta}(t) + \frac{1}{t\tau(t)}y(\tau(t)) = 0 \quad \text{for } t \geq t_0. \tag{3.6}$$

It follows that

$$\int_{t_0}^{t} \tau(s)q(s)\Delta s = \int_{t_0}^{t} \frac{\Delta s}{s} \to \infty \text{ as } t \to \infty.$$

Therefore, by Lemma 3.5, every bounded solution of (3.6) oscillates on $[t_0, \infty)$.

We can now state and prove our second corollary of Theorem 3.1.

Corollary 3.7. *All bounded solutions of the linear second-order dynamic equation*

$$y^{\Delta\Delta} + q(t)y(\tau(t)) = 0 \tag{1.1}$$

are oscillatory in case (3.5) *holds.*

Proof. Let u be a bounded nonoscillatory solution of (1.1) with $u(t) > 0$ and $u(\tau(t)) > 0$ for $t \geq T$. Since $u^{\Delta\Delta}(t) \leq 0$ for all t, we have $u^{\Delta}(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. As in the proof of Theorem 3.1, for any $0 < k < 1$ there exists a $T_k \geq T$ such that

$$u^{\Delta\Delta}(t) + k\frac{\tau(t)}{\sigma(t)}q(t)u^{\sigma}(t) \leq 0$$

for $t \geq T \geq T_k$. Let $\alpha(t) = u(T)$ and $\beta(t) = u(t)$. Then

$$f(t, \alpha^{\sigma}(t)) = -\lambda\frac{\tau(t)}{\sigma(t)}q(t)u(T) \leq 0 = \alpha^{\Delta\Delta}(t)$$

and

$$f(t, \beta^{\sigma}(t)) = -\lambda\frac{\tau(t)}{\sigma(t)}q(t)u^{\sigma}(t) \geq \beta^{\Delta\Delta}(t) \quad \text{with} \quad k = \lambda.$$

So $\alpha,\ \beta$ are lower and upper solutions, respectively, of

$$y^{\Delta\Delta} + \lambda \frac{\tau(t)}{\sigma(t)} q(t) y^{\sigma}(t) = 0. \tag{3.1}$$

As u is increasing, $\alpha(t) \le \beta(t)$ on $[T_k, \infty)_{\mathbb{T}}$. Then by Theorem 2.4, there is a solution $y(t)$ of (3.1) satisfying $u(T) \le y(t) \le u(t)$ on $[T_k, \infty)_{\mathbb{T}}$. As u is bounded, y is a bounded nonoscillatory solution of (3.1) . This is a contradiction to Lemma 3.5 and proves the theorem. □

In order to prove our last result, which in an extension of [2, Theorem 4.4], we need the following lemma.

Lemma 3.8. *Let $y(t)$ be a positive solution of* (1.1) *defined on $[T, \infty)_{\mathbb{T}}$ for some $T > 0$ that satisfies $y^{\Delta}(t) > 0$ and $y^{\Delta\Delta}(t) \le 0$ on $[T, \infty)_{\mathbb{T}}$. If* (3.5) *holds, then there exists a $T_1 \ge T$ such that*

$$\frac{y(t)}{y^{\Delta}(t)} \ge t \quad \textit{and} \quad \frac{y(t)}{t} \textit{ is decreasing} \tag{3.7}$$

on $[T_1, \infty)_{\mathbb{T}}$.

Proof. Let $y(t)$ be as in the statement of the lemma and assume (3.5) holds. Also let $Y(t) := y(t) - ty^{\Delta}(t)$. Then $Y^{\Delta}(t) = -\sigma(t) y^{\Delta\Delta}(t) \ge 0$ for $t \in [T, \infty)_{\mathbb{T}}$. This implies that $Y(t)$ is increasing on $[T, \infty)_{\mathbb{T}}$.

We claim there is a $T_1 \in [T, \infty)$ such that $Y(t) \ge 0$ on $[T_1, \infty)_{\mathbb{T}}$. If not, then $Y(t) < 0$ on $[T_1, \infty)_{\mathbb{T}}$. Therefore

$$\left(\frac{y(t)}{t}\right)^{\Delta} = \frac{ty^{\Delta}(t) - y(t)}{t\sigma(t)} = -\frac{Y(t)}{t\sigma(t)} > 0, \quad t \in [T_1, \infty)_{\mathbb{T}}$$

which implies that $\dfrac{y(t)}{t}$ is increasing on $[T_1, \infty)_{\mathbb{T}}$. Choose $T_2 \in [T_1, \infty)_{\mathbb{T}}$ such that $\tau(t) \ge \tau(T_2)$ for $t \ge T_2$. Then

$$\frac{y(\tau(t))}{\tau(t)} \ge \frac{y(\tau(T_2))}{\tau(T_2)} =: D > 0,$$

which gives $y(\tau(t)) \ge D\tau(t)$ for $t \ge T_2$. Now by integrating both sides of (1.1) from T_2 to t, we obtain

$$y^{\Delta}(t) - y^{\Delta}(T_2) + \int_{T_2}^{t} q(s) y(\tau(s)) \Delta s = 0.$$

This implies that

$$y^{\Delta}(T_2) > D \int_{T_2}^{t} q(s) \tau(s) \Delta s,$$

which contradicts (3.5). Hence there is a $T_1 \in [T, \infty)_{\mathbb{T}}$ such that $Y(t) \geq 0$ on $[T_1, \infty)_{\mathbb{T}}$, and the first part of (3.7) holds. Moreover,

$$\left(\frac{y(t)}{t}\right)^{\Delta} = \frac{ty^{\Delta}(t) - y(t)}{t\sigma(t)} = -\frac{Y(t)}{t\sigma(t)} \leq 0, \quad t \in [T_1, \infty)_{\mathbb{T}},$$

and we have that $\dfrac{y(t)}{t}$ is decreasing on $[T_1, \infty)_{\mathbb{T}}$. This completes the proof of the lemma. □

Theorem 3.9. *Assume* (3.5) *holds. If*

$$\lim_{t\to\infty} \left(t \int_t^{\infty} q(s) \frac{\tau(s)}{\sigma(s)} \, \Delta s \right) = \infty, \tag{3.8}$$

then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. To the contrary, suppose y is a nonoscillatory solution of (1.1). Then there exists $T \in \mathbb{T}$ such that

$$y(t) > 0, \quad y^{\Delta}(t) > 0, \quad \text{and} \quad y^{\Delta\Delta}(t) \leq 0 \quad \text{for all} \quad t \geq T \geq t_0.$$

It follows that for $s \geq t \geq T$ we have

$$\int_t^s q(u) y(\tau(u)) \, \Delta u = -\int_t^s y^{\Delta\Delta}(u) \, \Delta u = y^{\Delta}(t) - y^{\Delta}(s) \leq y^{\Delta}(t)$$

and hence

$$\int_t^{\infty} q(u) y(\tau(u)) \, \Delta u \leq y^{\Delta}(t).$$

From the above inequality and Lemma 3.8, it follows that for sufficiently large $t \in \mathbb{T}$

$$\begin{aligned} y(t) &\geq ty^{\Delta}(t) \\ &\geq t \int_t^{\infty} q(u) y(\tau(u)) \, \Delta u \\ &\geq t \int_t^{\infty} q(u) \frac{\tau(u)}{u} y(u) \, \Delta u \\ &\geq y(t) \left(t \int_t^{\infty} q(u) \frac{\tau(u)}{u} \, \Delta u \right) \\ &\geq y(t) \left(t \int_t^{\infty} q(u) \frac{\tau(u)}{\sigma(u)} \, \Delta u \right) \end{aligned}$$

and so

$$1 \geq t \int_t^{\infty} q(u) \frac{\tau(u)}{\sigma(u)} \, \Delta u,$$

a contradiction to (3.8). This completes the proof. □

Example 3.10. Let $h > 0$ and $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$. In this case (1.1) becomes

$$y(t + 2h) - 2y(t + h) + y(t) + h^2 q(t)y(\tau(t)) = 0. \tag{3.9}$$

Assume that

$$\lim_{b\to\infty} \sum_{k=t_0/h}^{b/h-1} h\tau(hk)q(hk) = \infty.$$

Then every bounded solution of (3.9) oscillates on $[t_0, \infty)_{\mathbb{T}}$. Additionally, if,

$$\lim_{n\to\infty} \left\{ h^2 n \lim_{b\to\infty} \sum_{k=n}^{b/h-1} q(hk)\frac{\tau(hk)}{h+k} = \infty \right\},$$

then every solution of (3.9) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

4 Conclusion and Future Directions

In this paper we have obtained sufficient conditions for the oscillatory behavior of

$$y^{\Delta\Delta}(t) + q(t)y(\tau(t)) = 0.$$

This was done by comparing nonoscillatory solutions of the delay dynamic equation with the solutions of a corresponding linear dynamic equation and then using known properties of the linear equation to obtain a desired contradiction.

Possibilities for further exploration include changing the leading term to $(p(t)y^{\Delta})^{\Delta}$ where $p(t) > 0$ on the time scale interval $[t_0, \infty)_{\mathbb{T}}$ and $\int^{\infty} \frac{\Delta t}{p(t)} = \infty$, and replacing the delay $\tau(t)$ with the advance $\xi : \mathbb{T} \to \mathbb{T}$ where $\sigma(t) \leq \xi(t)$ and $\lim_{t\to\infty} \xi(t) = \infty$.

References

[1] Ravi Agarwal, Martin Bohner, Donal O'Regan, and Allan Peterson. Dynamic equations on time scales: a survey. *J. Comput. Appl. Math.*, 141(1-2):1–26, 2002. Dynamic equations on time scales.

[2] Ravi Agarwal, Martin Bohner, and Samir H. Saker. Oscillation of second order delay dynamic equations. *Can. Appl. Math. Q.*, 13(1):1–17, 2005.

[3] Martin Bohner and Gusein Sh. Guseinov. Improper integrals on time scales. *Dynam. Systems Appl.*, 12(1-2):45–65, 2003. Special issue: dynamic equations on time scales.

[4] Martin Bohner and Allan Peterson. *Dynamic Equations on Time Scales: An Introduction with Applications*. Birkhäuser Boston Inc., Boston, MA, 2001.

[5] Martin Bohner and Samir H. Saker. Oscillation of second order nonlinear dynamic equations on time scales. *Rocky Mountain J. Math.*, 34(4):1239–1254, 2004.

[6] Lynn Erbe. Oscillation criteria for second order nonlinear delay equations. *Canad. Math. Bull.*, 16:49–56, 1973.

[7] Lynn Erbe and Allan Peterson. Some oscillation results for second order linear delay dynamic equations. volume 53 of *Adv. Stud. Pure Math.*, pages 237–245. Math. Soc. Japan, Tokyo, 2009.

[8] Lynn Erbe, Allan Peterson, and Pavel Řehák. Comparison theorems for linear dynamic equations on time scales. *J. Math. Anal. Appl.*, 275(1):418–438, 2002.

[9] Lynn Erbe, Allan Peterson, and Samir H. Saker. Oscillation criteria for second-order nonlinear dynamic equations on time scales. *J. London Math. Soc. (2)*, 67(3):701–714, 2003.

[10] Stefan Hilger. Analysis on measure chains—a unified approach to continuous and discrete calculus. *Results Math.*, 18(1-2):18–56, 1990.

[11] Lloyd K. Jackson. Subfunctions and second-order ordinary differential inequalities. *Advances in Math.*, 2:307–363 (1968), 1968.

[12] Pavel Řehák. How the constants in Hille-Nehari theorems depend on time scales. *Adv. Difference Equ.*, pages Art. ID 64534, 15, 2006.

[13] Samir H. Saker. Oscillation of nonlinear dynamic equations on time scales. *Appl. Math. Comput.*, 148(1):81–91, 2004.

[14] B. G. Zhang and Zhu Shanliang. Oscillation of second-order nonlinear delay dynamic equations on time scales. *Comput. Math. Appl.*, 49(4):599–609, 2005.