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ASYMPTOTIC BEHAVIOR OF SECOND-ORDER NONLINEAR DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we consider the second-order nonlinear dynamic equation

$$\Big(p(t)y^{\Delta}(t)\Big)^{\Delta} + f(t, y^{\sigma})g(p(t)y^{\Delta}) = 0,$$

on a time scale $\mathbb{T}$. Our goal is to establish necessary and sufficient conditions for the existence of certain types of solutions of this dynamic equation. We apply results from the theory of lower and upper solutions for related dynamic equations and use several results from calculus.

1. **Introduction.**

We are concerned with the asymptotic behavior of the solutions of the following second-order nonlinear dynamic equation:

$$\big(p(t)y^{\Delta}\big)^{\Delta} + f(t, y^{\sigma})g(p(t)y^{\Delta}) = 0, \tag{1}$$

where $\sup \mathbb{T} = \infty$. We shall assume the following conditions hold:
(A_0) $f, f_y : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are continuous in y and rd-continuous in t and $g : \mathbb{R} \to \mathbb{R}$ is continuous.
(A_1) $f(t, 0) = 0,\ t \in [0, \infty)_{\mathbb{T}}$.
(A_2) $f_y(t, y) \geq 0$ and is nondecreasing in y for $t \in [0, \infty)_{\mathbb{T}}$ and $y \geq 0$.
(A_3) $g(v) > 0$ for all $v \geq 0$.
(A_4) $p : (0, \infty)_{\mathbb{T}} \to (0, \infty)_{\mathbb{T}}$ is right-dense continuous and delta-differentiable, has a bounded derivative, and satisfies $\displaystyle\int^{\infty} \frac{\Delta t}{p(t)} = \infty$.

We shall study (1) by considering the equation

$$\big(p(t)y^{\Delta}\big)^{\Delta} + f_y(t, \alpha)y = 0, \tag{2}$$

where α is some real constant depending on the solutions of (1). In this paper, we intend to use the method of upper and lower solutions to obtain oscillation criteria for (1) under certain conditions. Our results generalize those given in Erbe [12].

As a way to unify a discussion of many types of problems for equations in the continuous and discrete cases, the theory of time scales was introduced by Stefan Hilger [17]. Not only does this unify the theories of differential equations and difference equations, but it also extends these classical situations to cases "in between"–

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e.g. to the so-called $q-$difference equations. Moreover, the theory can be applied to other different types of time scales. Since its introduction, many authors have expounded on various aspects of this new theory. A book on the subject of time scales by Bohner and Peterson [8] summarizes and organizes much of time scale calculus.

In recent years, there has been an increasing interest in studying the asymptotic behavior solutions of dynamic equations on a time scale (i.e., a closed subset of the real lin $\mathbb{R}$). This has lead to many attempts to harmonize the oscillation theory for the continuous and the discrete cases, to include them in one comprehensive theory, and to extend the results to more general time scales. We refer the reader to the papers [1], [2], [4], [6], [10], [11], [13], [15], [16], and the references cited therein. To illustrate some of the results, we mention the work of Erbe, Peterson, and Saker [15] who considered the nonlinear dynamic equation

$$\big(p(t)y^{\Delta}(t)\big)^{\Delta}+q(t)(f\circ x^{\sigma})=0, \quad t\in\mathbb{T}, \tag{3}$$

where p, q are positive real-valued right-dense continuous functions and $f:\mathbb{R}\to\mathbb{R}$ is continuous and satisfies

$$yf(y)>0 \text{ and } |f(y)|\geq K|y| \text{ for } y\neq 0 \text{ for some } K>0.$$

By means of generalized Riccati transformation techniques and generalized exponential functions, the authors give some oscillation criteria for the above equation.

In Bohner and Saker [9], the authors considered (3) and used Riccati techniques to give some sufficient conditions for oscillation when $\int^{\infty}\frac{\Delta t}{p(t)}\Delta t$ converges or diverges. They obtained sufficient conditions which guarantee that every solution oscillates or converges to zero.

2. **Preliminary results.** In this section, we state fundamental results needed to prove our main results. We begin by introducing the auxiliary function

$$P(t,a)=\int_a^t\frac{1}{p(s)}\Delta s \quad \text{for} \quad a\in\mathbb{T} \text{ with } a<t,$$

and with the following theorem which is a generalization of [20, Theorem 3].

Theorem 2.1. *Let $f(t,y)$ be a continuous function of the variables $t\geq t_0$ and $|y(t)|<\infty$. Assume that for all $t>0$ and $y\neq 0$, $yf(t,y)>0$, and for each fixed t, $f(t,y)$ is nondecreasing in y for $y>0$. Then a necessary condition for*

$$\big(p(t)y^{\Delta}\big)^{\Delta}+f(t,y^{\sigma})=0, \quad t\geq t_0>0, \tag{4}$$

to have a bounded nonoscillatory solution is that

$$\int^{\infty}P(t,a)f(t,c)\Delta t<\infty, \tag{5}$$

for some constant $c>0$.

Proof. Suppose y is a bounded eventually positive solution of (4). So there exists $T\in[t_0,\infty)_{\mathbb{T}}$ such that $y(t)>0$ for $t\geq T$. As $f(t,y)>0$ for all $y>0$, $\big(p(t)y^{\Delta}\big)^{\Delta}$ is eventually negative. So $p(t)y^{\Delta}(t)$ is decreasing and tends to a limit L that is positive, zero, negative, or $-\infty$. If $L<0$ or if $L=-\infty$, $p(t)y(t)$ would be eventually negative. Hence $\lim\limits_{t\to\infty}p(t)y^{\Delta}(t)=L$ with $0\leq L<\infty$.

Integrating (4) from s to T_1, we obtain

$$p(T_1)y^\Delta(T_1) - p(s)y^\Delta(s) + \int_s^{T_1} f(r, y^\sigma(r))\Delta r = 0.$$

It follows that

$$y^\Delta(s) \geq \frac{1}{p(s)} \int_s^\infty f(r, y^\sigma(r))\Delta r.$$

Integrating again for $T < t_1 < t$, we obtain

$$y(t) - y(t_1) \geq \int_{t_1}^t \frac{1}{p(s)} \int_s^\infty f(r, y^\sigma(r))\Delta r \Delta s.$$

If we let

$$I_1(t) := \int_{t_1}^t \frac{1}{p(s)} \int_s^\infty f(r, y^\sigma(r))\Delta r \Delta s$$

and

$$I_2(t) := \int_{t_1}^t P(r, t_1) f(r, y^\sigma(r))\Delta r + \int_t^\infty P(t, t_1) f(r, y^\sigma(r))\Delta r,$$

we obtain $I_1(t) \geq I_2(t)$. Consequently, for $t \geq t_1 \geq T$, we have

$$\begin{aligned} y(t) - y(t_1) &= \int_{t_1}^t y^\Delta(s)\Delta s \\ &\geq I_1(t) \\ &\geq I_2(t) \\ &\geq \int_{t_1}^t P(r, t_1) f(r, y^\sigma(r))\Delta r, \end{aligned}$$

and so

$$y(t) > \int_{t_1}^t P(r, t_1) f(r, y^\sigma(r))\Delta r.$$

Since $y(t) \leq M$ for some $M > 0$ and $\int_{t_1}^t P(r, t_1) f(r, y^\sigma(r))\Delta r$ is an increasing function of t, we have

$$\int_{t_1}^\infty P(r, t_1) f(r, y^\sigma(r))\Delta r < \infty.$$

By the monotonicity of f, we have

$$\int^\infty P(r, t_1) f\left(r, y(t_1)\right)\Delta r < \infty.$$

By letting $c = y(t_1)$, we obtain the desired result. This completes the proof. □

In order to prove our main results, we need a method of studying separated boundary value problems (SBVPs). Namely, we will define functions called upper and lower solutions that not only imply the existence of a solution of a SBVP but also provide bounds on the location of the solution. Consider the SBVP

$$-\left(p(t)x^\Delta\right)^\Delta + q(t)x^\sigma = f(t, x^\sigma), \quad t \in [a, b]^{\kappa^2} \tag{6}$$

$$x(a) = A, \qquad x(b) = B, \tag{7}$$

where the functions $f \in C([a,b]^{\kappa^2} \times \mathbb{R}, \mathbb{R})$ and $p, q \in C_{rd}([a,b]^{\kappa^2})$ are such that $p(t) > 0$ and $q(t) \geq 0$ on $[a,b]^{\kappa^2}$. We define the set

$$\mathbb{D}_1 := \{x \in \mathbb{X} : x^\Delta \text{ is continuous and } px^\Delta \text{ is delta differentiable on } [a,b]^\kappa \text{ and } (px^\Delta)^\Delta \text{ is rd-continuous on } [a,b]^{\kappa^2}\},$$

where the Banach space $\mathbb{X} = C([a,b])$ is equipped with the norm $\|\cdot\|$ defined by

$$\|x\| := \max_{t \in [a,b]_{\mathbb{T}}} |x(t)| \quad \text{for all} \quad x \in \mathbb{X}.$$

A function x is called a solution of the equation $-\left(p(t)y^\Delta\right)^\Delta + q(t)y^\sigma = 0$ on $[a,b]^{\kappa^2}$ if $x \in \mathbb{D}_1$ and the equation $-\left(p(t)x^\Delta\right)^\Delta + q(t)x^\sigma = 0$ holds for all $t \in [a,b]^{\kappa^2}$. Next we define for any $u, v \in \mathbb{D}_1$ the sector $[u,v]_1$ by

$$[u,v]_1 := \{w \in \mathbb{D}_1 : u \leq w \leq v\}.$$

Definition 2.2. [7, Definition 6.1] We call $\alpha \in \mathbb{D}_1$ a lower solution of the SBVP (6)-(7) on $[a,b]$ provided

$$-\left(p\alpha^\Delta\right)^\Delta(t) + q(t)\alpha^\sigma(t) \leq f(t, \alpha^\sigma(t)) \quad \text{for all} \quad t \in [a,b]^{\kappa^2}$$

and

$$\alpha(a) \leq A, \quad \alpha(b) \leq B.$$

Similarly, $\beta \in \mathbb{D}_1$ is called an upper solution of the SBVP (6)-(7) on $[a,b]$ provided

$$-\left(p\beta^\Delta\right)^\Delta(t) + q(t)\beta^\sigma(t) \geq f(t, \beta^\sigma(t)) \quad \text{for all} \quad t \in [a,b]^{\kappa^2}$$

and

$$\beta(a) \geq A, \quad \beta(b) \geq B.$$

Theorem 2.3. [7, Theorem 6.5] *Assume that there exists a lower solution α and an upper solution β of the SBVP (6)-(7) such that*

$$\alpha(t) \leq \beta(t) \quad \textit{for all} \quad t \in [a,b].$$

Then the SBVP (6)-(7) has a solution $x \in [\alpha, \beta]_1$ on $[a,b]$.

The following is an extension of the previous theorem to $[a, \infty)_{\mathbb{T}}$.

Theorem 2.4. *Assume that there exists a lower solution α and an upper solution β of (6) with $\alpha(t) \leq \beta(t)$ for all $t \in [a,\infty)_{\mathbb{T}}$. Then*

$$-\left(p(t)x^\Delta\right)^\Delta + q(t)x^\sigma = f(t, x^\sigma) \tag{8}$$

has a solution x with $x(a) = A$ and $x \in [\alpha, \beta]_1$ on $[a, \infty)_{\mathbb{T}}$.

Proof. It follows from Theorem 2.3 that for each $n \geq 1$ there is a solution $x_n(t)$ of (6) on $[a, t_n]_{\mathbb{T}}$ with

$$x_n(a) = A, \quad x_n(t_n) = \beta(t_n), \quad \text{and} \quad \alpha(t) \leq x_n(t) \leq \beta(t)$$

on $[a, t_n]_{\mathbb{T}}$, where $\{t_n\}$ is such that $\lim_{n\to\infty} t_n = \infty$. Thus, for any fixed $n \in \mathbb{N}$, $x_m(t)$ is a solution on $[a,t_n]_{\mathbb{T}}$ satisfying $\alpha(t) \leq x_m(t) \leq \beta(t)$ for all $m \geq n$. Hence, for $m \geq n$, the sequence $x_m(t)$ is pointwise bounded on $[a,t_n]_{\mathbb{T}}$.

We claim that $\{x_m(t)\}$ is equicontinuous on $[a, t_N]_{\mathbb{T}}$ for any fixed $N \geq 1$. As q is right-dense continuous, it is regulated. It follows that q is bounded on $[a, t_N]_{\mathbb{T}}$, and so there exists a constant $Q_N > 0$ such that $|q(t)| \leq Q_N$ for all $t \in [a, t_N]_{\mathbb{T}}$.

Furthermore, since f is continuous and $x_m(t) \le \beta(t)$ for all $t \in [a, t_N]_{\mathbb{T}}$, there is a constant $K_N > 0$ such that

$$\begin{aligned} \left|[p(t)x_m^\Delta(t)]^\Delta\right| &= |q(t)x_m^\sigma(t) - f(t, x_m^\sigma(t))| \\ &\le q(t)|x_m^\sigma(t)| + |f(t, x_m^\sigma(t))| \\ &\le Q_N|x_m^\sigma(t)| + K_N \\ &\le Q_N\|\beta\| + K_N \\ &=: M_N \end{aligned}$$

for all $t \in [a, t_N]_{\mathbb{T}}$. It follows that

$$\begin{aligned} \left|p(t)x_m^\Delta(t) - p(a)x_m^\Delta(a)\right| &= \int_a^t [p(s)x_m^\Delta(s)]^\Delta \,\Delta s \\ &\le M_N(t-a) \\ &\le M_N(t_N - a), \end{aligned}$$

which gives

$$p(t)|x_m^\Delta(t)| = |p(t)x_m^\Delta(t)| \le |p(a)x_m^\Delta(a)| + |M_N(t_N - a)|.$$

Since $\{p(t)x_m(t)\}$ is uniformly bounded on $[a, t_N]_{\mathbb{T}}$ for all $m \ge N$, it follows that $|p(a)x_m^\Delta(a)| \le L_N$ for some $L_N > 0$ and all $m \ge N$. Consequently,

$$\left|p(t)x_m^\Delta(t)\right| \le L_N + |M_N(t_N - a)| := C_N,$$

and, immediately, we have

$$|x_m^\Delta(t)| \le \frac{C_N}{p(t)} \le C_N P_N \quad \text{for all} \quad t \in [a, t_N]_{\mathbb{T}}$$

since $1/p(t)$ is continuous on the compact interval $[a, t_N]_{\mathbb{T}}$. Consequently,

$$|x_m(t) - x_m(s)| = \left|\int_s^t x_m^\Delta(u)\,\Delta u\right| \le C_N P_N|t - s| < \epsilon$$

for all $t, s \in [a, t_N]_{\mathbb{T}}$ provided $|t - s| < \delta = \dfrac{\epsilon}{C_N P_N}$. Hence the claim holds.

The Ascoli-Arzela theorem along with a diagonalization argument gives $\{x_m(t)\}$ contains a subsequence which converges uniformly on all compact subintervals $[a, t_N]_{\mathbb{T}}$ of $[a, \infty)_{\mathbb{T}}$ to a solution $x(t)$, which is the desired solution of (8) that satisfies $x \in [\alpha, \beta]_1$ on $[a, \infty)_{\mathbb{T}}$. □

3. **Main results.** In this section, we establish necessary and sufficient conditions for the existence of certain types of solutions of (1).

Theorem 3.1. *Assume (A_0)-(A_4) hold and let $\alpha_0 > 0$. Furthermore, assume $P(\sigma(t), a)/P(t, a)$ is bounded. Then the following statements are equivalent:*

(i) For each $0 < \alpha < \alpha_0$ there is a solution $u_\alpha(t)$ of (1) satisfying $\lim_{t\to\infty} u_\alpha(t) = \alpha$.

(ii) $\displaystyle\int^\infty P(\sigma(t), a) f_y(t, \alpha)\Delta t < \infty$ for all $0 < \alpha < \alpha_0$.

Proof. Assume $\displaystyle\int^\infty P(\sigma(t), a) f_y(t, \alpha_1)\Delta t = \infty$ for some $0 < \alpha_1 < \alpha_0$ and let $\alpha_1 < \beta < \alpha_0$. Let u_β be the corresponding solution of (1) with $\lim_{t\to\infty} u_\beta(t) = \beta$.

Without loss of generality, assume $u_\beta(t) > 0$ on $[0, \infty)_{\mathbb{T}}$. Choose $\delta > 0$ such that $\alpha_1 + \delta < \beta$ and let $T \geq 0$ be such that $u_\beta^\sigma(t) \geq \alpha_1 + \delta$ for all $t \geq T$. Then for $t \geq T$

$$\big(p(t)u_\beta^\Delta\big)^\Delta = -f(t, u_\beta^\sigma)g(p(t)u_\beta^\Delta) \leq 0,$$

and so $p(t)u_\beta^\Delta(t)$ is nonincreasing for $t \geq T$.

We claim that $p(t)u_\beta^\Delta(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. If not, there is a $t_1 \geq T$ such that $p(t_1)u_\beta^\Delta(t_1) \leq 0$. Then

$$p(t)u_\beta^\Delta(t) \leq p(t_1)u_\beta^\Delta(t_1), \quad t \in [t_1, \infty)_{\mathbb{T}},$$

and therefore

$$u_\beta^\Delta(t) \leq \frac{p(t_1)u_\beta^\Delta(t_1)}{p(t)}, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Integrating, we obtain

$$u_\beta(t) - u_\beta(t_1) = \int_{t_1}^t u_\beta^\Delta(s)\Delta s \leq p(t_1)u_\beta^\Delta(t_1) \int_{t_1}^t \frac{\Delta s}{p(s)} \to -\infty$$

as $t \to \infty$, which implies that $u_\beta(t)$ is eventually negative. This contradiction proves the claim.

Furthermore, $p(t)u_\beta^\Delta(t)$ decreases to 0 on $[T, \infty)_{\mathbb{T}}$. Indeed, let $L := \lim_{t\to\infty} p(t)u_\beta^\Delta(t)$ and assume $L > 0$. Then $p(t)u_\beta^\Delta(t) > L - \epsilon$ for all large t and for some $\epsilon > 0$. It follows that

$$u_\beta^\Delta(t) > \frac{L-\epsilon}{p(t)} \quad \text{for all large } t,$$

and so

$$u_\beta(t) - u_\beta(T) > \int_T^t \frac{L-\epsilon}{p(t)}\Delta s \quad \text{for all large } t.$$

Consequently, $u_\beta(t) \to \infty$ as $t \to \infty$, which is a contradiction.

By applying the Mean Value Theorem, we obtain

$$\frac{f(t, u_\beta^\sigma(t)) - f(t, \alpha_1)}{u_\beta^\sigma(t) - \alpha_1} = f_y(t, \eta(t)) \quad \text{for some } \eta(t) \in (\alpha_1, u_\beta^\sigma(t)).$$

Now by the monotonicity of f_y (condition (A$_2$)), we have

$$\begin{aligned}
f_y(t, \alpha_1) &\leq f_y(t, \eta(t)) \\
&= \frac{f(t, u_\beta^\sigma(t)) - f(t, \alpha_1)}{u_\beta^\sigma(t) - \alpha_1} \\
&\leq \frac{u_\beta^\sigma(t)}{u_\beta^\sigma(t) - \alpha_1} \frac{f(t, u_\beta^\sigma(t))}{u_\beta^\sigma(t)} \\
&\leq \frac{\beta}{\delta} \frac{f(t, u_\beta^\sigma(t))}{u_\beta^\sigma(t)}
\end{aligned}$$

for $t \geq T$ as $u_\beta(t) \leq \beta$ on $[T,\infty)_{\mathbb{T}}$. Since $\lim_{t\to\infty} p(t)u_\beta^\Delta(t) = 0$, there exists $T_1 \geq T$ such that $g(p(t)u_\beta^\Delta(t)) \geq \dfrac{g(0)}{2} > 0$ for all $t \geq T_1$. Hence, for $t \geq T_1$, we have

$$\begin{aligned}\left(p(t)u_\beta^\Delta(t)\right)^\Delta &= -f(t,u_\beta^\sigma(t))g(p(t)u_\beta^\Delta(t))\\ &\leq -\frac{f_y(t,\alpha_1)}{\beta}\delta u_\beta^\sigma(t)\frac{g(0)}{2}\\ &= -kf_y(t,\alpha_1)u_\beta^\sigma(t),\end{aligned}$$

where $k = \dfrac{\delta g(0)}{2\beta}$. Also, $\left(p(t)\alpha_1^\Delta\right)^\Delta = 0 \geq -kf_y(t,\alpha_1)\alpha_1$. Hence, by Theorem 2.4, there is a solution $z(t)$ of $\left(p(t)z^\Delta\right)^\Delta + kf_y(t,\alpha_1)z^\sigma = 0$ with
$0 < \alpha_1 \leq z(t) \leq u_\beta(t) \leq \beta$ on $[T,\infty)_{\mathbb{T}}$. By Theorem 2.1, it follows that

$$\int^\infty kP(t,a)f_y(t,\alpha_1)\Delta t < \infty.$$

Since $\dfrac{P(\sigma(t),a)}{P(t,a)}$ is bounded, we have

$$\int^\infty P(\sigma(t),a)f_y(t,\alpha_1)\Delta t < \infty,$$

which is the desired contradiction.

Conversely, let $0 < \alpha < \alpha_0$ be such that

$$\int^\infty P(\sigma(t),a)f_y(t,\alpha)\Delta t < \infty$$

for $a \in \mathbb{T}$ with $a < t$, and let

$$M = \max\{g(v) : 0 \leq v \leq \alpha\}.$$

Choose $T \geq 0$ such that

$$\int_T^\infty [P(\sigma(s),a) - P(T,a)]f_y(s,\alpha)\Delta s < \frac{1}{M} \quad \text{and} \quad \int_T^\infty f_y(s,\alpha)\Delta s < \frac{1}{M}.$$

We shall now define a sequence of functions on $[T,\infty)_{\mathbb{T}}$ in the following manner:

Let $y_0(t) = \alpha$ on $[T,\infty)_{\mathbb{T}}$. Now for $t \geq T$

$$\begin{aligned}0 &\leq \int_t^\infty [P(\sigma(s),a) - P(t,a)]f(s,\alpha)g(0)\Delta s\\ &= \int_t^\infty [P(\sigma(s),a) - P(t,a)][f(s,\alpha) - f(s,0)]g(0)\Delta s\\ &= \alpha\int_t^\infty [P(\sigma(s),a) - P(t,a)]f_y(s,\eta(s))g(0)\Delta s, \quad \eta(s) \in (0,\alpha)\\ &\leq \alpha M\int_t^\infty [P(\sigma(s),a) - P(t,a)]f_y(s,\alpha)\Delta s\\ &\leq \alpha M\int_T^\infty [P(\sigma(s),a) - P(T,a)]f_y(s,\alpha)\Delta s\\ &< \alpha.\end{aligned}$$

By defining $y_1(t) := \alpha - \int_t^\infty [P(\sigma(s),a) - P(t,a)] f(s,\alpha) g(0)\Delta s,\ t \geq T$, we have $0 \leq y_1(t) < \alpha$. Differentiating y_1, we obtain

$$y_1^\Delta(t) = \frac{1}{p(t)} \int_t^\infty f(s,\alpha) g(0)\Delta s.$$

It follows that for $t \geq T$

$$\begin{aligned} p(t)y_1^\Delta(t) &= \int_t^\infty f(s,\alpha)g(0)\Delta s \\ &\leq \alpha M \int_t^\infty f_y(s,\eta(s))\Delta s, \quad \eta(s) \in (0,\alpha) \\ &\leq \alpha M \int_T^\infty f_y(s,\eta(s))\Delta s \\ &< \alpha. \end{aligned}$$

Proceeding inductively, we define for all $m \geq 1$

$$y_{m+1}(t) := \alpha - \int_t^\infty [P(\sigma(s),a) - P(t,a)] f(s, y_m^\sigma(s)) g(p(s)y_m^\Delta(s))\Delta s, \quad t \geq T, \quad (9)$$

and obtain $0 \leq y_m(t),\ p(t)y_m^\Delta(t) < \alpha$ for all $m \geq 1$.

We claim that $\{y_m(t)\}_{m=0}^\infty$ is equicontinuous on $[T, T_N]_{\mathbb{T}}$ for any fixed $N \geq 1$. Since $p(t)y_m^\Delta(t) < \alpha$ for all $m \geq 1$ on $[T, T_N]_{\mathbb{T}}$, we have

$$0 \leq |y_m^\Delta(t)| \leq \frac{\alpha}{p(t)} \leq \alpha P_N \quad \text{for all } t \in [T, T_N]_{\mathbb{T}},$$

where $P_N = \max\left\{\frac{1}{p(t)} : t \in [T, T_N]_{\mathbb{T}}\right\}$. Consequently,

$$|y_m(t) - y_m(s)| = \left|\int_s^t y_m^\Delta(u)\Delta u\right| \leq \alpha P_N |t-s| < \epsilon_1$$

for all $t, s \in [T, T_N]_{\mathbb{T}}$ provided $|t - s| < \delta_1 = \dfrac{\epsilon_1}{\alpha P_N}$. Hence the claim holds. The Ascoli-Arzela theorem along with a diagonalization argument yields a uniformly convergent subsequence $\{y_{m_k}(t)\}$ on compact subintervals $[T, T_N]_{\mathbb{T}}$ of $[T, \infty)_{\mathbb{T}}$. Let

$$u_\alpha(t) := \lim_{k\to\infty} y_{m_k}(t),$$

for $t \in [T, \infty)_{\mathbb{T}}$. Furthermore, from the definition of y_{m+1} given by (9), one can show that

$$y_{m+1}^{\Delta\Delta}(t) = -\frac{1}{p^\sigma(t)} \left[p^\Delta(t) y_{m+1}^\Delta(t) + f(t, y_m^\sigma(t)) g(p(t)y_m^\Delta(t))\right]$$

for all $m \geq 1$. Taking the absolute value of both sides, we find that

$$\left|y_{m+1}^{\Delta\Delta}(t)\right| \leq P_N[\alpha P_N |p^\Delta(t)| + M|f(t, y_m^\sigma(t))|]$$

on $[T, T_N]_{\mathbb{T}}$. Since f is right-dense continuous, it is regulated. It follows that f is bounded on $[T, T_N]_{\mathbb{T}}$, and so there is a constant $F_N > 0$ such that $|f(t, y_m^\sigma(t))| \leq F_N$ for all $t \in [T, T_N]_{\mathbb{T}}$ and all $m \geq N$. Also, since $p^\Delta(t)$ is bounded, we have

$$\left|y_{m+1}^{\Delta\Delta}(t)\right| \leq P_N[\alpha P_N P + M F_N] =: L_N$$

on $[T, T_N]_{\mathbb{T}}$ for all $m \geq N$, where $P > 0$ is such that $|p^{\Delta}(t)| \leq P$ on $(0, \infty)_{\mathbb{T}}$. Consequently,

$$|y_m^{\Delta}(t) - y_m^{\Delta}(s)| = \left| \int_s^t y_m^{\Delta\Delta}(u)\Delta u \right| \leq L_N|t-s| < \epsilon_2$$

for all $t, s \in [T, T_N]_{\mathbb{T}}$ provided $|t-s| < \delta_2 = \dfrac{\epsilon_2}{L_N}$. Therefore $\left\{y_m^{\Delta}(t)\right\}$ is equicontinuous on $[T, T_N]_{\mathbb{T}}$. Using the Ascoli-Arzela theorem along with a diagonalization argument, we obtain a subsequence $\left\{y_{m_k}^{\Delta}\right\}$ that converges uniformly on compact subintervals $[T, T_N]_{\mathbb{T}}$ of $[T, \infty)_{\mathbb{T}}$. It follows that $y_{m_k}^{\Delta}$ converges uniformly to $u_\alpha^{\Delta}(t)$ on compact subintervals of $[T, \infty)_{\mathbb{T}}$, thereby yielding

$$\lim_{k\to\infty} f(t, y_{m_k}^{\sigma})g(p(t)y_{m_k}^{\Delta}(t)) = f(t, u_\alpha^{\sigma})g(p(t)y_\alpha^{\Delta}(t))$$

uniformly on compact subintervals of $[T, \infty)_{\mathbb{T}}$. Replacing m in equation (9) by m_k and letting $k \to \infty$, we obtain

$$u_\alpha(t) = \alpha - \int_t^\infty [P(\sigma(s), a) - P(t, a)]f(s, u_\alpha^{\sigma}(s))g(p(s)u_\alpha^{\Delta}(s))\Delta s$$

on $[T, \infty)_{\mathbb{T}}$. It follows that $u_\alpha(t)$ is a solution of (1). As $\lim\limits_{t\to\infty} u_\alpha(t) = \alpha$, the proof is complete. $\square$

We continue with an example that shows how Theorem 3.1 can be applied.

Example 3.2. Consider the dynamic equation

$$\left(ty^{\Delta}(t)\right)^{\Delta} + \frac{y^2(\sigma(t))}{t^2}\left|ty^{\Delta}(t)\right| = 0 \tag{10}$$

on $\mathbb{T} = [1, \infty)_{\mathbb{Z}}$. Here $p(t) = t$, $f(t, y) = \dfrac{y^2}{t^2}$, and $g(v) = |v|$. Let $\alpha_0 > 0$ be given.

We have that $p : \mathbb{T} \to \mathbb{T}$ is right-dense continuous, delta-differentiable, and has a bounded derivative. Also,

$$\int_1^\infty \frac{\Delta}{t} = \lim_{b\to\infty} \sum_{t=1}^{b} \frac{1}{t} = \infty.$$

Hence (A_4) holds.

With the choice of f, (A_1) holds immediately. Then $f_y(t, y) = \dfrac{2y}{t^2} \geq 0$ and $f_y(t, y)$ is strictly increasing for $t \in \mathbb{T}$ and $y \geq 0$ since $f_{yy}(t, y) = \dfrac{2}{t^2} > 0$. Thus (A_2) holds. Furthermore, (A_0) and (A_3) hold.

Note that for any $a \in \mathbb{T}$

$$\frac{P(\sigma(t), a)}{P(t, a)} = \frac{\sum_{s=a}^{t+1} \frac{1}{s}}{\sum_{s=a}^{t} \frac{1}{s}} = 1 + \frac{\frac{1}{t+1}}{\sum_{s=a}^{t} \frac{1}{s}} = 1 + \frac{1}{t+1} < 2, \quad t \in \mathbb{T}.$$

We now have that all assumptions of Theorem 3.1 are satisfied. Additionally, for all $\alpha > 0$, we have

$$\begin{aligned}\int_1^\infty P(\sigma(t),a) f_y(t,\alpha)\,\Delta t &= \lim_{b\to\infty}\sum_{t=1}^{b}\left(\sum_{s=a}^{t+1}\frac{1}{s}\right)\frac{2\alpha}{t^2}\\ &< 2\alpha\lim_{b\to\infty} S(b)\sum_{t=1}^{b}\frac{1}{t^2}\\ &< \infty,\end{aligned}$$

where $S(b) := \displaystyle\sum_{s=a}^{b+1}\frac{1}{s} < \ln(b+1)$. Since condition (ii) of Theorem 3.1 holds, we conclude that there is a solution satisfying (i) for all $0 < \alpha < \alpha_0$.

As other examples of equations to which Theorem 3.1 applies, we have

$$\big(p(t)x^\Delta\big)^\Delta + xe^{t(x-\alpha_0)}(1 + p(t)x^\Delta) = 0 \tag{11}$$

and

$$\big(p(t)x^\Delta\big)^\Delta + xe^{t(x^2-\alpha_0^2)+cp(t)x^\Delta}(1 + (p(t)x^\Delta)^2) = 0 \tag{12}$$

where c is an arbitrary real number. Then for $0 < \alpha < \alpha_0$, there is a solution $u_\alpha(t)$ of (11) with $\lim\limits_{t\to\infty} u_\alpha(t) = \alpha$, and for $0 < |\alpha| < \alpha_0$ there is a solution $y_\alpha(t)$ of (11) with $\lim\limits_{t\to\infty} y_\alpha(t) = \alpha$.

In [19] it is shown that $y'' + a(t)y^{2n+1} = 0,\ n \geq 0$, where $a(t) \geq 0$ for $t \geq 0$ and $g(v) = 1$ for all v, has solutions for which

$$\lim_{t\to\infty}\frac{y(t)}{t} = \alpha > 0 \tag{13}$$

if and only if

$$\int^\infty t^{2n+1}a(t)dt < \infty. \tag{14}$$

We will show that an analogous form of (14) is sufficient for (13) when considering the dynamic equation (1), and (13) is sufficient for an analogous form of (14) provided $f(t,y)$ satisfies the following additional condition:

(A_5) There exist positive real numbers c and λ such that $\liminf\limits_{v\to\infty}\dfrac{f(t,v)}{vf_v(t,cv)} \geq \lambda > 0$, for all sufficiently large t.

Note that in the case of $y'' + a(t)y^{2n+1} = 0$, c and λ may be any positive real numbers with $\lambda c^{2n} \leq 1/(2n+1)$. We first establish the following result.

Lemma 3.3. *Assume (A_0)-(A_4) hold and let there exist a real number $\beta > 0$ with*

$$\int^\infty P(\sigma(t),a)f_y(t,\beta P(\sigma(t),T))\Delta t < \infty.$$

Then there exist solutions of $\big(p(t)y^\Delta\big)^\Delta + f(t,y^\sigma(t))g(p(t)y^\Delta) = 0$, say $y(t)$, such that $\lim\limits_{t\to\infty}\dfrac{y(t)}{P(t,a)}$ exists and is positive.

Proof. Let $T > 0$ be such that

$$\int_T^\infty P(\sigma(t),a)f_y(t,\beta P(\sigma(t),T))\Delta t < \frac{1}{2M},$$

where $M = \max\{g(v) : 0 \le v \le \beta\}$. We define a solution of (1) by

$$p(T)u(T) = 0, \quad p(T)u^\Delta(T) = \beta,$$

and we assert that the solution satisfies $p(t)u^\Delta(t) \ge \dfrac{\beta}{2}$ for $t \ge T$. Assume, on the contrary, that there is a $\delta > 0$ with $\delta < \beta/2$ and a $t_1 > T$ with $0 < p(t_1)u^\Delta(t_1) \le \delta$ and $p(t)u^\Delta(t) \ge \delta$ on $[T, t_1)_{\mathbb{T}}$. Then for $T \le t \le t_1$ we have

$$p(T)u^\Delta(T) = p(t)u^\Delta(t) + \int_T^t f(s, u^\sigma(s))g(p(s)u^\Delta(s))\Delta s. \tag{15}$$

Since $\left(p(t)u^\Delta(t)\right)^\Delta \le 0$ on $[T, t_1)_{\mathbb{T}}$, we have

$$p(t)u^\Delta(t) \le \beta \text{ on } (T, t_1)_{\mathbb{T}} \tag{16}$$

and

$$u(t) \le \beta \int_T^t \frac{1}{p(s)}\Delta s = \beta P(t, T) \le \beta P(t, a) \text{ on } [T, t_1)_{\mathbb{T}}. \tag{17}$$

Applying the Mean Value Theorem to (15) and the monotonicity of f_y, we have

$$\begin{aligned}
\beta = p(T)u^\Delta(T) &= p(t)u^\Delta(t) + \int_T^t f(s, u^\sigma(s))g(p(s)u^\Delta(s))\Delta s \\
&\le p(t)u^\Delta(t) + M\int_T^t f(s, u^\sigma(s))\Delta s \\
&= p(t)u^\Delta(t) + M\int_T^t [f(s, u^\sigma(s)) - f(s, 0)]\Delta s \\
&= p(t)u^\Delta(t) + M\int_T^t u^\sigma(s) f_u(s, \eta(s))\Delta s, \quad 0 < \eta(s) < u^\sigma(s) \\
&\le p(t)u^\Delta(t) + M\beta\int_T^t P(\sigma(s), a) f_u(s, u^\sigma(s))\Delta s \\
&\le p(t)u^\Delta(t) + M\beta\int_T^t P(\sigma(s), a) f_u(s, \beta P(\sigma(s), T))\Delta s \\
&< p(t)u^\Delta(t) + M\beta\frac{1}{2M} \\
&= p(t)u^\Delta(t) + \frac{\beta}{2}.
\end{aligned}$$

Hence, $p(t_1)u^\Delta(t_1) > \dfrac{\beta}{2}$, which is a contradiction. Thus, $p(t)u^\Delta(t) \ge \dfrac{\beta}{2}$ on $[T, \infty)_{\mathbb{T}}$, and so $p(t)u(t) > 0$ on $[T, \infty)_{\mathbb{T}}$.

Since $(p(t)u^\Delta(t))^\Delta \le 0$ on $[T, \infty)_{\mathbb{T}}$, $p(t)u^\Delta(t)$ decreases on $[T, \infty)_{\mathbb{T}}$. Hence $L := \lim_{t\to\infty} p(t)u^\Delta(t)$ exists and is positive where $\beta/2 \le L < \beta$. By L'Hôpital's Rule [8, Theorem 1.120], the following limit

$$\lim_{t\to\infty} \frac{u(t)}{P(t, a)} = \lim_{t\to\infty} \frac{u(t)}{1/p(t)} = \lim_{t\to\infty} p(t)u^\Delta(t)$$

exists and is positive. □

We now show (13) is sufficient for an analogous form of (14) provided (A_5) holds.

Theorem 3.4. *Assume conditions (A_0)-(A_5) hold. Then,* (1) *has a solution, say* $y(t)$*, such that* $\lim_{t\to\infty} \dfrac{y(t)}{P(t,a)}$ *exists and is positive, if and only if*

$$\int^{\infty} P(\sigma(t),a) f_y(t, \beta P(\sigma(t),a))\Delta t < \infty \text{ for some } \beta > 0.$$

Proof. Let $\alpha > 0$ and let y be solution of (1) with

$$\lim_{t\to\infty} \frac{y(t)}{P(t,a)} = \alpha.$$

Let $T \geq 0$ be such that $y(t) \geq \dfrac{\alpha P(t,a)}{2}$ for $t \geq T$ and let

$$m := \min\{g(v) : 0 \leq v \leq p(T)y^{\Delta}(T)\}.$$

By condition (A_5), there is a $T_1 \geq T$ such that

$$\begin{aligned} f(t, y^{\sigma}(t)) \geq \lambda y^{\sigma}(t) f_y(t, c y^{\sigma}(t)) &\geq \lambda \frac{\alpha P(\sigma(t),a)}{2} f_y\left(t, \frac{\alpha c P(\sigma(t),a)}{2}\right) \\ &= k P(\sigma(t),a) f_y\left(t, \frac{\alpha c P(\sigma(t),a)}{2}\right) \end{aligned}$$

for $t \geq T_1$, where $k = \dfrac{\lambda\alpha}{2}$. Since $0 < p(t)y^{\Delta}(t) \leq p(T)y^{\Delta}(T)$ for $t \geq T$, we have

$$f(t, y^{\sigma}(t)) g(p(t)y^{\Delta}(t)) \geq mkP(\sigma(t),a) f_y\left(t, \frac{c\alpha P(\sigma(t),a)}{2}\right), \quad t \geq T_1.$$

Therefore,

$$\begin{aligned} p(T_1)y^{\Delta}(T_1) &= p(t)y^{\Delta}(t) + \int_{T_1}^{t} f(s, y^{\sigma}(s)) g(p(s)y^{\Delta}(s))\Delta s \\ &\geq p(t)y^{\Delta}(t) + mk\int_{T_1}^{t} P(\sigma(s),a) f_y\left(s, \frac{c\alpha P(\sigma(s),a)}{2}\right). \end{aligned}$$

Since $\lim_{t\to\infty} p(t)y^{\Delta}(t) > 0$,

$$\int_{T_1}^{\infty} P(\sigma(s),a) f_y\left(s, \frac{c\alpha P(\sigma(s),a)}{2}\right) < \infty,$$

and this proves the theorem. □

Example 3.5. Consider the dynamic equation

$$\left(ty^{\Delta}(t)\right)^{\Delta} + \frac{y^2(\sigma(t))}{t^2}\left|ty^{\Delta}(t)\right| = 0 \tag{10}$$

on $\mathbb{T} = [1,\infty)_{\mathbb{Z}}$. As shown in Example 3.2, conditions (A_0)-(A_4) hold. If we choose $c = \frac{1}{4}$ and $\lambda = 1$, then

$$\liminf_{v\to\infty} \frac{f(t,v)}{v f_v(t,cv)} = \liminf_{v\to\infty} \frac{t^{-2}v^2}{\frac{1}{2}t^{-2}v^2} = 2 > 1 = \lambda > 0.$$

Hence all assumptions of Theorem 3.4 are satisfied. Furthermore, for $\beta = \frac{1}{2}$,

$$\int_1^{\infty} P(\sigma(t),a) f_y(t, \beta P(\sigma(t),a))\Delta t = \lim_{b\to\infty} \sum_{t=1}^{b} \left[\left(\sum_{s=a}^{t+1} \frac{1}{s}\right)\left(\frac{1}{t^2}\sum_{r=a}^{t+1}\frac{1}{r}\right)\right] < \infty.$$

Therefore, we conclude that there is a solution y of (10) such that $\lim_{t\to\infty} \dfrac{y(t)}{P(t,a)}$ exists and is positive.

4. **Conclusion and future directions.** In this paper, we have obtained necessary and sufficient conditions for the existence of a bounded nonoscillatory solution with prescribed limit at infinity and a nonoscillatory solution whose derivative has positive limit at infinity to

$$\big(p(t)y^{\Delta}(t)\big)^{\Delta} + f(t, y^{\sigma})g(p(t)y^{\Delta}) = 0.$$

These results were attained using the method of upper and lower solutions and applying the Mean Value Theorem and L'Hôpital's Rule.

In the future, we plan to consider the possibility of including the delay $\tau : \mathbb{T} \to \mathbb{T}$, where $t \le \tau(t) \le \sigma(t)$ and $\lim_{t\to\infty} \tau(t) = \infty$ in the case $\int^{\infty} \dfrac{\Delta t}{p(t)} = \infty$.

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