Complex Variables
Preliminary Exam
August 2014

Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ = the complex plane; $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ = the extended complex plane; $\mathcal{A}(G) = \{f : f \text{ is analytic on a region } G \subset \mathbb{C}\}$; $\mathcal{H}(G) = \{u : u \text{ is harmonic on a region } G \subset \mathbb{C}\}$; $B(a; b) = \{z : |z - a| < b\}$; $\mathbb{D} = B(0; 1) = \{z : |z| < 1\}$; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. Let $RHP = \{z : \Re(z) > 0\}$. Let $D = \mathbb{D} \cap RHP$. Find a one-to-one conformal map f of $\mathbb{D}$ onto D such that $f(1/2) = 1/2$.

2. Find all $f \in \mathcal{A}(\mathbb{C})$ such that $|f(z)| < 10|z|^{3/2}$ for all z such that $|z| > 2$.

3. A function f is said to be *complex harmonic* on a region G if there exist two functions $u, v \in \mathcal{H}(G)$ such that $f = u + iv$ on G. Let f be complex harmonic on a region G. Prove that if $|f|$ is constant on G, then f is constant on G.

4. Let f be a polynomial. Suppose that $\displaystyle\int_{\partial\mathbb{D}} f(z)\bar{z}^j dz = 0$ for $j = 1, 2, 3, \cdots$. Prove that $f \equiv 0$.

5. Let $f \in \mathcal{A}(\mathbb{D})$ satisfy $f(\mathbb{D}) \subset \mathbb{D}$. Prove that if f has two distinct fixed points in $\mathbb{D}$, then $f(z) \equiv z$.

6. Prove that if the power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ converges on the disk $B(0; R)$, where $R > 0$, then the power series $\displaystyle\sum_{n=0}^{\infty} n a_n z^{n-1}$ converges on the disk $B(0; R)$.

7. Locate and classify the isolated singularities (including any potential isolated singularity at the point at infinity) for each of the following functions by type and order, where applicable.

(**a**) $\dfrac{z}{(1 - z^2)^2}$ (**b**) $\dfrac{z}{\sin z}$ (**c**) $z^2 \cos \dfrac{1}{z}$

8. Let $f \in \mathcal{A}(\mathbb{C})$. Suppose that $|f(z)| = 1$ for $|z| = 1$. Show that there exists an integer n and a constant λ with $|\lambda| = 1$ such that $f(z) = \lambda z^n$.