Complex Variables
Preliminary Exam
August 2013

Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $\mathbb{Z}$ — the set of integers; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. Let $z_1, z_2, \ldots, z_n$ be $n \geq 3$ distinct points in the complex plane which do not lie on a straight line. Prove that if
$$z_1 + z_2 + \cdots + z_n = 0$$
then no closed half-plane, the boundary of which is a straight line passing through the origin, can contain all of the points $z_1, z_2, \ldots, z_n$.

2. **(a)** Let $f(z)$ be analytic in $\mathbb{D}$ such that $f(1/n) = f(-1/n)$ for all integers $n \geq 2$. Prove that $f(z)$ is an even function.
(b) Let $g(z)$ be analytic in the domain $D := \{z : 0 < |z| < 1\}$. Suppose that $g(z) = g(2z)$ for all z such that $|z| < 1/2$. Prove that $g(z)$ is constant.

3. Let
$$f(z) = \frac{\pi}{\sinh(\pi z)} + e^{1/z^2} + \frac{2z}{1+z^2}.$$
Locate and classify all the singularities of $f(z)$ (including any singularity at $z = \infty$) as isolated or non-isolated. Further, classify the isolated singularities by type (removable, pole, essential). Calculate the residues of $f(z)$ at its poles.

4. Let $f(z)$ be analytic in the strip $S := \{z : |\Re(z)| < \pi/4\}$. Suppose that $f(0) = 0$ and $|f(z)| < 1$ for all $z \in S$. Prove that $|f(z)| \leq |\tan z|$ for all $z \in S$.

5. Let
$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}.$$
Prove that for every $\rho > 0$ there is positive integer N such that for all $n \geq N$ all zeros of $f_n(z)$ belong to the disk $\{z : |z| < \rho\}$.

6. Use a Residue Theorem to show that
$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx = \pi\left(\sqrt{2}-1\right).$$

7. Find the number of solutions of the equation
$$z^4 + 2z^3 + 3z^2 + z + 2 = 0$$
in the right half-plane and in the first quadrant.

8. **(a)** State the Weierstrass Product Theorem.
(b) Give an example of a function $f(z)$ analytic on $\mathbb{C}$, which has zeros at the points $z_n = \frac{2n-1}{2}$, $n \in \mathbb{Z}$ and no other zeros.