Complex Analysis **Preliminary Examination** May 2012

Answer all 8 **questions.** Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, $B(a;r) = \{z \in \mathbb{C} \mid |z - a| < r\}$, $H(G) = \{f : G \to \mathbb{C} \mid f \text{ is analytic on the domain } G\}$, $[a,b] =$ the line segment connecting a and b.

1. Let $f, g \in H(\mathbb{C})$ with $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that there exists $c \in \mathbb{C}$ such that $f = cg$.

2. Evaluate $\displaystyle\int_0^\infty \frac{x^p}{1+x^2}\,dx$ for $-1 < p < 1$.

3. Find a one-to-one conformal map f of $\mathbb{D} \setminus \left[\frac{1}{2}, 1\right]$ onto $B\left(0; \frac{1}{2}\right)$ such that real numbers are mapped to real numbers.

4. Suppose $f = u + iv \in H(G)$. Prove that if $v(z) = e^{u(z)}$ for all $z \in G$, then f must be constant.

5. Find the number of zeros of $f(z) = z^5 - 20z^4 + 5z^3 - z^2 + 50z - 17$ inside the annulus $\{z \in \mathbb{C} \mid 1 < |z| < 5\}$. Justify your answer.

6. Suppose the sequence $\{f_n\}$ converges to f in $H(\mathbb{D})$, and for each n, the function f_n has exactly one zero in $\mathbb{D}$.
   (a) Show that either f is identically zero on $\mathbb{D}$ or else has at most one zero in $\mathbb{D}$.
   (b) Give an example of a sequence $\{f_n\}$ as above, which converges to f in $H(\mathbb{D})$, but where the limit function has no zeros in $\mathbb{D}$.

7. Find the Laurent expansion centered at $z = 0$ of $f(z) = \dfrac{z+2}{z^2 - z - 2}$ in the following annuli:
   (a) $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$
   (b) $\{z \in \mathbb{C} \mid 2 < |z| < \infty\}$

8. State the Riemann Mapping Theorem and prove the uniqueness part.