Complex Analysis **Preliminary Examination** August 2012

Answer all 8 **questions.** Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, $B(a;r) = \{z \in \mathbb{C} \mid |z - a| < r\}$, $H(G) = \{f : G \to \mathbb{C} \mid f \text{ is analytic on the domain } G\}$, $[a,b]$ = the line segment connecting a and b.

1. For $0 < r < 1$, let f_r be the one-to-one, analytic map of the slit half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\} \setminus [0, r]$ onto the unit disc $\mathbb{D}$ with $f_r(1) = 0$ and $f_r'(1) > 0$. Prove that $d(r) = f_r'(1)$ is a strictly increasing function of r, for $0 < r < 1$.

2. State and prove the Argument Principle.

3. Prove or give a counterexample to each of the following:
   (a) Suppose G is simply connected, f is analytic on G, and $f'(z) \neq 0$ for all $z \in G$. Then f is one-to-one on G.
   (b) Let $f \in H(\mathbb{D} \setminus \{0\})$ be such that f has a pole at 0. Then there exists $M > 0$ so that for all $|w| > M$, there exists $z \in \mathbb{D}$ so that $w = f(z)$.

4. Suppose $f \in H(B(0;5))$ and f maps the closed annulus $\{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ into $\mathbb{D}$. Prove that the restriction of f to $B(0;2)$ has exactly one fixed point. (A function f has a fixed point at a if $f(a) = a$.)

5. Find all entire functions f for which $|f(z)| \leq |z|^2$ if $|z| \leq 1$ and $|f(z)| \leq |z|^3$ if $|z| \geq 1$.

6. Define convergence in the space $H(\mathbb{D})$. Let $f_n(z) = \displaystyle\sum_{k=1}^{n} \frac{z^k}{k^2(1+z^k)}$ for $n = 1, 2, 3, \ldots$.
   Show that the sequence $\{f_n\}$ converges in $H(\mathbb{D})$.

7. Does there exist a closed rectifiable curve γ in $\mathbb{C} \setminus \{0, 1\}$ that satisfies the condition $\displaystyle\frac{1}{2\pi i}\int_\gamma \frac{5z-3}{z(z-1)}\,dz = 1$? Either sketch such a curve and explain why it satisfies the given condition or prove such a curve does not exist.

8. Find a one-to-one conformal map f of $\mathbb{D}$ onto the region G (shown in the figure below) which contains the point $z = 0$ and is bounded by arcs of the following three circles:
   (a) $\{z \in \mathbb{C} \mid |z| = 1\}$
   (b) $\{z \in \mathbb{C} \mid |z - (-1+i)| = 1\}$
   (c) $\{z \in \mathbb{C} \mid |z - (-1-i)| = 1\}$.