Complex Analysis

Preliminary Exam

August 2011

Instructions:

Do each of the following problems. Show all relevant steps which lead to your solutions.

Notation:

$\mathbb{C}$ denotes the complex plane. $\mathbb{C}_\infty$ denotes the extended complex plane, i.e., $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

For $z \in \mathbb{C}$, $\Re z$ and $\Im z$ denote the real and imaginary parts of z, respectively.

$\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$, i.e., $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

$B(a,r)$ denotes the open disk in $\mathbb{C}$ centered at a of radius r, i.e., $B(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$.

$\operatorname{ann}(a;\alpha,\beta)$ denotes the open annulus in $\mathbb{C}$ centered at a of inner radius α and outer radius β, i.e., $\operatorname{ann}(a;\alpha,\beta) = \{z \in \mathbb{C} :\ \alpha < |z-a| < \beta\}$.

$\mathbb{U}$ denotes the upper half-plane in $\mathbb{C}$, i.e., $\mathbb{U} = \{z \in \mathbb{C} : \Im z > 0\}$.

For a region $G \subset \mathbb{C}$, let $\mathcal{A}(G) = \{f : f \text{ is analytic on } G\}$.

For a connected set $F \subset \mathbb{C}$, let $\mathcal{C}(F) = \{f \mid f : F \to \mathbb{C} \text{ and } f \text{ is continuous on } F\}$.

1. (a) Find all points $z \in \mathbb{C}$ where the function $f(z) = z\,\Re z$ is differentiable.
   (b) Find all points $z \in \mathbb{C}$ where the function $f(z) = z\,\Re z$ is analytic.
   (c) At each point $z \in \mathbb{C}$ where the function $f(z) = z\,\Re z$ is differentiable, find $f'(z)$.
2. Find an analytic function f such that $\Re\{f(z)\} = e^y \sin x$ for $z = x + iy$.
3. State and prove the Residue Theorem.
4. For $0 < a < 1, 0 < b < \pi$, evaluate the integral $\displaystyle\int_0^\infty \frac{x^{-a}}{x^2 + 2x\cos b + 1}dx$. Justify your steps.
5. Let a, b be non-zero real numbers and let n be a positive integer. Consider the roots of the equation
$$z^{2n} + az^{2n-1} + b^2 = 0.$$
   If n is even, prove that the number of roots with positive real part is n. However, if n is odd, prove that the number of roots with positive real part is $n-1$ if $a > 0$ and $n+1$ if $a < 0$. Hint: Consider the argument principle.
6. Let $L = \{z : |\Re\, z| = 1\}$. Let f be an entire function such that $f(\mathbb{C}) \subset \mathbb{C} \setminus L$. Prove that f is constant.
7. Find a one-to-one analytic map $w = f(z)$ from $\mathbb{D}$ onto $\left\{w : |\Im\, w| < \dfrac{\pi}{2}\right\} \setminus \{w : w \geq 0\}$.
8. Suppose $f \in \mathcal{A}(\mathbb{D})$ and satisfies $f(\mathbb{D}) \subset \mathbb{D}$ and $f(0) = 0$. Determine whether
$$F(z) \equiv \sum_{k=1}^{\infty} f(z^k) = f(z) + f(z^2) + f(z^3) + \cdots \in \mathcal{A}(\mathbb{D}).$$