Complex Variables
Preliminary Exam
August 2010

Directions: Do all of the following ten problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. Describe and then graph the following subsets of the complex plane: **(a)** $\{z : |z+i| + |z-i| = 4\}$, **(b)** $\{z : \Re(z(1-i)) < \sqrt{2}\}$, **(c)** $\{z : \frac{\pi}{4} < \arg(z+i) < \frac{\pi}{2}\}$.
2. **(a)** Prove that any analytic function f on $\mathbb{D}$ has a primitive on $\mathbb{D}$.
   (b) Give an example of a domain G and an analytic function g on G which does not have a primitive on G.
3. Determine if there exists an analytic function f on the disk $D_2 := \{z : |z| < 2\}$ such that
$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^3}$$
   for all positive integers n.
4. Let f be an entire function such that $|\Re(f(z))| + |\Im(f(z))| \geq 1$ for each $z \in \mathbb{C}$. Prove that f is constant.
5. Let f be an analytic function on an open set containing the closed unit disk $\overline{\mathbb{D}}$. Prove that if $z \in \mathbb{D}$, then
$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{\sin(w-z)}\, dw,$$
   where γ traverses the unit circle once in the positive direction.
6. Let
$$f(z) = \frac{e^{\frac{1}{z-1}}}{e^z - 1}.$$
   Locate and classify all the singularities of f (including any singularity at $z = \infty$) as isolated or non-isolated. Further, classify the isolated singularities by type (removable, pole, essential). Calculate the residues of f at its poles.
7. Suppose that a is a real number greater than 1. Use the Residue Calculus to evaluate the integral
$$\int_0^{2\pi} \frac{d\theta}{a + \sin\theta}.$$
8. **(a)** State any version of Rouché's Theorem.
   (b) Determine how many poles, counting multiplicity, the rational function
$$R(z) = \frac{1 - z^2}{z^5 - 6z^4 + z^3 + 2z - 1}$$
   has in the unit disk $\mathbb{D}$.
9. Find a conformal mapping $w = f(z)$ from the upper half-plane $\mathbb{H} := \{z : \Im(z) > 0\}$ onto the exterior of the ellipse $L := \{w : |w-1| + |w+1| = 4\}$ such that $f(1+i) = \infty$.
10. **(a)** State the Weierstrass Factorization Theorem for an entire function.
    (b) Construct an entire function that has second order zeros at the points $z_n = n^2$, $n \geq 0$, and no other zeros.