Complex Analysis

Preliminary Exam

May 2009

Instructions:

Do each of the following problems. Show all relevant steps which lead to your solutions.

Notation:

$\mathbb{C}$ denotes the complex plane. $\mathbb{C}_\infty$ denotes the extended complex plane, i.e., $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

For $z \in \mathbb{C}$, $\Re z$ and $\Im z$ denote the real and imaginary parts of z, respectively.

$\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$, i.e., $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

$B(a, r)$ denotes the open disk in $\mathbb{C}$ centered at a of radius r, i.e., $B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$.

$\mathbb{U}$ denotes the upper half-plane in $\mathbb{C}$, i.e., $\mathbb{U} = \{z \in \mathbb{C} : \Im z > 0\}$.

For a region $G \subset \mathbb{C}$, let $\mathcal{A}(G) = \{f : f \text{ is analytic on } G\}$.

1. Show that if the power series $\sum_{n=0}^{\infty} a_n z^n$ converges on the disk $B(0, R)$, then the power series $\sum_{n=0}^{\infty} n a_n z^{n-1}$ converges on the disk $B(0, R)$.

   Note: this result is used to prove the fact that when a function f is defined by a power series on a disk $B(0, R)$, then f has a derivative f' and f' has a power series representation and the power series representation for f' converges on the same disk $B(0, R)$.

2. State and prove the Casorati-Weierstrass Theorem.

3. Use the residue theorem to evaluate the integral $\displaystyle\int_0^\infty \frac{\sin x}{x(1 + x^2)}\, dx$. Verify each step.

4. Suppose that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is meromorphic on $\mathbb{C}_\infty$. Prove if 0 is not in the range of f, then f is constant.

5. Let $G_1 = B(0, 1) \setminus \overline{B(1 + i, 1)}$ and $G_2 = \{z : |\Re z| < 1\}$. Find a conformal map f from G_1 to G_2 such that $f(0) = 0$.

6. Suppose that $f \in \mathcal{A}(B(0, R + 1))$ and that there exists a constant M such that $|f(z)| \le M$ for all $z \in \partial B(0, R)$. For $0 < \rho < R$, find an upper bound, in terms of M, ρ, R and n, for $|f^{(n)}(z)|$ for $z \in \overline{B(0, \rho)}$.

7. Let $f \in \mathcal{A}(\mathbb{C})$. Suppose that for every unbounded sequence $\{z_n\}$ that the sequence $\{f(z_n)\}$ is also unbounded. Prove that f is a polynomial.

8. Find the domain of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{z - 1}{z + 1}\right)^n$, i.e., the maximal region G on which the series converges uniformly on compact subsets. Justify your work.

9. Give an explicit example of a function $f \in \mathcal{A}(\mathbb{C} \setminus \{1\})$ which has the property that the range $f(\mathbb{D}) \subset \mathbb{D}$ and the range $f(\mathbb{C} \setminus \{1\})$ is dense in $\mathbb{C}$.

10. Use the definition to prove that the function $u(x, y) = 2x - 3y - \log(x^2 + y^2)$ is harmonic in the left half-plane $H_- = \{z = x + iy : x < 0\}$. Then find an analytic function $f(z)$ such that $\Im f(z) = u(x, y)$ for all $z = x + iy \in H_-$ and $f(-1) = 1 + 2i$.