Complex Analysis — Preliminary Exam — Aug 2009

Instructions:

$\mathbb{C}$ denotes the complex plane. $\mathbb{C}_\infty$ denotes the extended complex plane, i.e., $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.
For $z \in \mathbb{C}$, $\Re z$ and $\Im z$ denote the real and imaginary parts of z, respectively.
$\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$, i.e., $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
$B(a, r)$ denotes the open disk in $\mathbb{C}$ centered at a of radius r, i.e., $B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$.
$\mathbb{U}$ denotes the upper half-plane in $\mathbb{C}$, i.e., $\mathbb{U} = \{z \in \mathbb{C} : \Im z > 0\}$.
For a region $G \subset \mathbb{C}$, let $\mathcal{A}(G) = \{f : f \text{ is analytic on } G\}$.

1. Find the Laurent expansion of $f(z) = \dfrac{1}{z^2(1-z)^2}$ on the annulus

   a. $0 < |z| < 1$ b. $1 < |z| < \infty$

2. Use the residue theorem to evaluate

   a. $\displaystyle\int_\gamma \frac{z(z+1)}{\sin(z+1)}dz$ where $\gamma(t) = 3e^{it}, 0 \le t \le 2\pi$ b. $\displaystyle\int_\gamma \frac{z(z+1)}{\sin(z+1)}dz$ where $\gamma(t) = 5e^{it}, 0 \le t \le 2\pi$

3. Prove that if G is a simply connected region in $\mathbb{C}$ and if $f \in \mathcal{A}(G)$ such that f has no zeros on G, then there exists a $g \in \mathcal{A}(G)$ such that g is a branch of logarithm for f.

4. Find the image of the quarter disk, $\Omega = \{z \in \mathbb{D} : \Re z > 0, \Im z > 0\}$ under the map $w = g(z) = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$. Prove that g is one-to-one on Ω.

5. Let $f, g \in \mathcal{A}(\overline{\mathbb{D}})$. Suppose that $\dfrac{f(z)}{g(z)} > 0$ for $z \in \partial\mathbb{D}$. Show that f, g have the same number of zeros in $\mathbb{D}$.

6. Suppose that $f \in \mathcal{A}(G)$, where G is a region which contains 0. Suppose that $\left|f(\frac{1}{n})\right| \le e^{-n}$ for all positive integers n. Prove that $f \equiv 0$ on G.

7. Give an explicit example of a function $f \in \mathcal{A}(\mathbb{D})$ which is one-to-one on $\mathbb{D}$ such that the range $f(\mathbb{D})$ is dense in $\mathbb{C}$.

8. Consider the function $f(z) = \sqrt{7 - z^2}$, where the branch of square root is chosen so that $f(1) > 0$. Determine the radius of convergence of the MacLauren series for f.

9. Let $f \in \mathcal{A}(\mathbb{C})$. For any point $\zeta \in \mathbb{C}$, let $\sum_{n=0}^\infty a_n(\zeta)(z - \zeta)^n$ denote the Taylor's series representation for f centered at ζ. Suppose for each such ζ that $a_7(\zeta) = 0$. Prove that f is a polynomial.

10. The operators $\dfrac{\partial}{\partial z}$ and $\dfrac{\partial}{\partial \bar{z}}$ are defined as follows:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Let $f(z)$ be a nonzero analytic function in a domain $D \subset \mathbb{C}$. Find $\dfrac{\partial}{\partial \bar{z}} f(z)$. Then prove the following:

$$\frac{\partial}{\partial z}\left(|f(z)|\right) = \frac{1}{2}|f(z)|\,\frac{f'(z)}{f(z)},$$