Complex Analysis

Preliminary Examination

May 2008

Answer all questions completely. Calculators may not be used.
Notation: $\mathbb{D} = \{z : |z| < 1\}$, $H(G) = \{f : f \text{ is analytic on the region } G\}$.

1. State and prove Hurwitz's Theorem.

2. Find a conformal map from the region $\{re^{i\theta} : 0 < r < 1, \frac{-\pi}{6} < \theta < \frac{\pi}{6}\}$ onto the region $\{re^{i\theta} : 0 < r < \infty, 0 < \theta < \frac{\pi}{6}\}$.

3. Does there exist a function $f \in H(\mathbb{D})$ such that $\lim\limits_{|z|\to 1} |f(z)| = \infty$? Why or why not?

4. Prove that if f is nonconstant, continuous on all of $\mathbb{C}$ and analytic on $\mathbb{C} \setminus \{x + iy : y = 0\}$, then f must be unbounded.

5. Prove that if w is a root of the polynomial $5z^4 + z^3 + z^2 - 7z + 14$ then $|w| < 2$.

6. Let $\{f_n\}$ be a sequence in $H(G)$ and there exists $M > 0$ so that for all $z \in G$ and all $n \in \mathbb{N}$, $|f_n(z)| < M$. Suppose $\{f_n\}$ converges pointwise to f on G. Show that $\{f_n\}$ converges to f in $H(G)$.

7. For each of the following, give an example or prove that no such example exists.

   A. A harmonic function on a region G which does not have a harmonic conjugate on G.

   B. An entire function with a pole at infinity.

   C. A Möbius transformation with exactly 3 distinct fixed points.

8. Evaluate $\displaystyle\int_\gamma \frac{e^z}{z(z+2)(z-1)^2}\, dz$, where γ is the closed rectifiable curve indicated below.