Complex Analysis

Preliminary Examination

August 2008

Answer all questions completely. Calculators may not be used.
Notation: $\mathbb{D} = \{z : |z| < 1\}$, $H(G) = \{f : f \text{ is analytic on the region } G\}$.

1. State and prove the Fundamental Theorem of Algebra.

2. Suppose $f \in H(\mathbb{D})$, f is non-constant and $\lim_{|z|\to 1} |f(z)| = 1$. Show that $f(\mathbb{D}) \subset \mathbb{D}$.

3. Classify and find the residues at the isolated sigularities of the function
$$\frac{1}{z} - \frac{1}{\sin z}.$$

4. Suppose $|a_0| > |a_1| > \cdots > |a_n| > 0$. Show that the polynomial $na_0 + a_1 z + a_2 z^2 + \ldots a_n z^n$ has no zeros in $\mathbb{D}$.

5. Find all conformal maps from $\mathbb{H} = \{x + iy : y > 0\}$ onto $\mathbb{S} = \{x + iy : |y| < \pi\}$.

6. Let $S_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$, $n = 1, 2, \ldots$. Show that for all $r > 0$ there exists $N \in \mathbb{N}$ so that $n \geq N$ and $|z| < r$ implies $S_n(z) \neq 0$.

7. Use the Residue Theorem to evaluate $\displaystyle\int_0^\infty \frac{3x^2 + 1}{(x^2 + 1)(x^2 + 4)}\, dx$. Justify all steps.

8. Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic and has two distinct fixed points (ie, there exist $z_1, z_2 \in \mathbb{D}, z_1 \neq z_2$ with $f(z_1) = z_1$ and $f(z_2) = z_2$. Show that f must be the identity map.