Complex Analysis

# Preliminary Exam

August 2007

**Instructions**:

Do each of the following problems. Show all relevant steps which lead to your solutions.

**Notation**:

$\mathbb{C}$ denotes the complex plane.

For $z \in \mathbb{C}$, $\Re(z)$ and $\Im(z)$ denote the real and imaginay parts of $z$, respectively.

$\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$, i.e., $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

$\mathbb{U}$ denotes the upper half-plane in $\mathbb{C}$, i.e., $\mathbb{U} = \{z \in \mathbb{C} : \Im(z) > 0\}$.

For a region $G \subset \mathbb{C}$, let $\mathcal{A}(G) = \{f : f \text{ is analytic on } G\}$.

1. Use the Cauchy-Riemann equations to find a harmonic conjugate for the function $u(x, y) = e^{3x} \sin(3y) - 4x$.

2. The function $f(z) = \dfrac{1}{(1-z)^2} + \dfrac{1}{3-z}$ can be expanded in a series of the form $\displaystyle\sum_{n=-\infty}^{\infty} a_n z^n$ in several different ways.
   (a) How many such expansions are there?
   (b) Find the maximal regions of the complex plane in which each of them valid?
   (c) Find each such expansion.

3. Use residue calculus to evaluate the integral $\displaystyle\int_0^{\infty} \frac{x^2+1}{x^4+1}\, dx$. Verify each step.

4. Consider $\tan z = \displaystyle\sum_{n=0}^{\infty} a_n z^n$.
   (a) Find the first three non-vanishing terms in the series expansion.
   (b) Find the radius of convergence of the series.

5. Let $A = \{z : 0 < |z| < 1\}$ and $f \in \mathcal{A}(A)$. Suppose that $\displaystyle\int_{|z|=r} f(z) z^k\, dz = 0$ for $k = 0, 1, 2, \cdots$, for $0 < r < 1$. Prove that $f$ has a removable singularity at $z = 0$.

6. Suppose $T = \dfrac{az+b}{cz+d}$ has exactly two (distinct) fixed points $z_1$, $z_2$. Prove that $T'(z_1)T'(z_2) = 1$.

7. For a region $G \subset \mathbb{C}$, let $f \in \mathcal{A}(G)$, $\{f_n\} \subset \mathcal{A}(G)$. Prove that if $\{f_n\}$ converges to $f$ in the topology of uniform convergence on compact subsets of $G$, then $\{f_n'\}$ converges to $f'$ (in the topology of uniform convergence on compact subsets of $G$).

8. Let $f$ be an entire function and let $a$ and $b$ be distinct points in $\mathbb{C}$.
   (a) Let $\gamma$ be a simply closed curve in $\mathbb{C}$ which encloses both $a$ and $b$. Evaluate the integral $\displaystyle\int_{\gamma} \frac{f(z)}{(z-a)(z-b)}\, dz$.
   (b) State Liouville's theorem and use the above result to prove it.

9. Suppose that $f \in \mathcal{A}(\mathbb{U})$ and $\Im(f(z)) \geq 0$ for $z \in \mathbb{U}$. Prove that $\left|\dfrac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}}\right| \leq \left|\dfrac{z - z_0}{z - \overline{z_0}}\right|$ for $z, z_0 \in \mathbb{U}$.