Complex Analysis **Preliminary Examimation** August 2005

Do all problems. Present adequate work to justify your answers.

Notation: $B(0;r) = \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{D} = B(0;1)$, $\operatorname{ann}(a;\alpha,\beta) = \{z \in \mathbb{C} : \alpha < |z-a| < \beta\}$

1. Let G be the region in the first quadrant bounded by the line segment $[0,1]$ and the arc of the circle which passes through 0 and 1 and which is tangent to the line $\mathrm{Re}z = \mathrm{Im}z$ at $z = 0$. Construct a one-to-one, conformal map of G onto $\mathbb{D}$.

2. Consider the rational function $f(z) = \dfrac{z^2 - 2z}{z(1-z)(2-z)^2}$.

   a) Classify all of the singularities of f, including the singularity at ∞.
   b) Classify all of the singularities of $f(z^2)$, including the singularity at ∞.
   c) Find the Laurent expansion of f on the annulus $\operatorname{ann}(0;1,2)$.

3. State and prove Louiville's Theorem.

4. Let $\mathbb{D}' = \mathbb{D} \setminus \{0\}$ and let $\mathcal{A}(\mathbb{D}')$ denote the set of analytic functions on $\mathbb{D}'$. Let $\{f_n\} \subset \mathcal{A}(\mathbb{D}')$ and $f \in \mathcal{A}(\mathbb{D}')$ such that f_n converges to f in the topology of local uniform convergence on compacta. Let $\displaystyle\sum_{k=-\infty}^{\infty} a_k^{(n)} z^k$ be the Laurent series expansion of f_n on $\mathbb{D}'$ and $\displaystyle\sum_{k=-\infty}^{\infty} a_k z^k$ be the Laurent series expansion of f on $\mathbb{D}'$. Prove for each k that the sequence $\{a_k^{(n)}\}$ converges to a_k as $n \to \infty$.

5. Let f be analytic on $B(0;10)$ such that for $z \in \partial B(0;1)$ that $\mathrm{Im}f(z) = \mathrm{Im}\ z$. Find a representation for f if $f(0) = 1$.

6. Show for $\alpha > 1$ that $\alpha z^3 e^z = 1$ has exactly three roots in $B(0;2)$.

7. For f analytic on $\mathbb{C}$ we say that ζ is an *attractive fixed point* of f if ζ is a fixed point of f and if there exists a $\delta > 0$ such that $|f(z) - \zeta| < |z - \zeta|$ for $0 < |z - \zeta| < \delta$. Let $f(z) = z^2 - (2 - \frac{1}{2}i)z$. Find the attractive fixed points of f.

8. Let G be a region in $\mathbb{C}$ and let f be analytic on G. Suppose there exists $\overline{B(a;r)} \subset G$ such that for $z \in \partial B(a;r)$, $|f(z)| = 1$. If $\inf\limits_{z \in G} |f(z)| > 0$, show that f is constant.

9. Prove that the function $f(z) = \dfrac{1}{z^2}$ cannot be uniformally approximated by polynomials on the annulus $\operatorname{ann}(0;1,2)$.

10. Let G be a region in $\mathbb{C}$ and let $\mathcal{A}(G)$ denote the set of analytic functions on G. For any subset $\mathcal{F}$ of $\mathcal{A}(G)$ let $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$. If $\mathcal{F}$ is a normal subset of $\mathcal{A}(G)$, prove that $\mathcal{F}'$ is also normal.