Complex Analysis **Preliminary Examination** May 2004

Do all 9 problems. Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

1. Find a one-to-one conformal map f of $\mathbb{D} \setminus (-1, 0]$ onto $\mathbb{D}$ such that $f(1/2) = 0$.
2. Give an explicit formula for a one-to-one conformal map defined on $\mathbb{D}$ whose range is dense in $\mathbb{C}$.
3. State and prove Rouche's Theorem.
4. Let g be a rational function with $|g(z)| = 1$ whenever $|z| = 1$. Prove that

$$g(z) = \frac{b_1(z)}{b_2(z)},$$

where b_1 and b_2 are finite products of the form

$$e^{i\theta} \prod_{j=1}^{n} \frac{\alpha_j - z}{1 - \overline{\alpha_j} z},$$

where $|\alpha_j| < 1, j = 1, 2, \ldots n$, and $\theta \in \mathbb{R}$.

5. Suppose f is entire and $\lim_{z \to \infty} \frac{f(z)}{z} = 0$. Prove that f must be a constant function.
6. Let f be a one-to-one, analytic map of $\mathbb{D}$ into $\mathbb{D}$. Prove that

$$|f'(z)| \leq \frac{1}{1 - |z|^2},$$

for all $z \in \mathbb{D}$.

7. Let $h(z) = z + \frac{z^2}{2}$. Find the area of $h(\mathbb{D})$.
8. Let $\Omega \subset \mathbb{C}$ be a simply connected region and $u : \Omega \to \mathbb{R}$ a harmonic function. Prove that there exists $v : \Omega \to \mathbb{R}$ such that $u + iv$ is analytic on Ω.
9. Let

$$p_n(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}.$$

Prove that for every $R > 0$ there exists $N > 0$ such that for all $n \geq N$ the zeros of p_n belong to the set $\{z : |z| > R\}$.