Complex Analysis **Preliminary Examination** August 2004

Do all 8 problems. Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

1. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(2x)}{1+x^2}\, dx$.

2. Find a one-to-one conformal map f of the half-strip $\{x+iy : 0 < y < 4, x > 0\}$ onto the upper half-plane $\{x+iy : y > 0\}$.

3. Prove the Argument Principle: Let f be meromorphic in G with poles $p_1, p_2, \ldots, p_m$ and zeros $z_1, z_2, \ldots, z_n$, counted according to multiplicity. If γ is a closed rectifiable curve in G with $\gamma \approx 0$ and not passing through $p_1, p_2, \ldots, p_m$ or $z_1, z_2, \ldots, z_n$, then
$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j).$$

4. A function $w : \mathbb{D} \to \mathbb{C}$ is called **planar harmonic** if it can be written as $w = u + iv$, where u and v are harmonic (but not necessarily harmonic conjugates). Prove that $w : \mathbb{D} \to \mathbb{C}$ is planar harmonic if and only if it can be written in the form $w = f + \overline{g}$ where f and g are analytic on $\mathbb{D}$.

5. Let f be an analytic function mapping $\mathbb{D}$ onto a subset of the right half-plane $\{x+iy : x > 0\}$ with $f(0) = 1$. Show that $|f'(0)| \leq 2$.

6. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$ converges for $z \in \mathbb{D}$. Show that the limit function is analytic in $\mathbb{D}$.

7. Prove that $1 + z - az^n$ has a root inside $\{z : |z| < 2\}$ for all $|a| > 2$ and $n > 2$.

8. Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Suppose there exist discs D_1 and D_2 such that $f(\mathbb{C} \setminus D_1) \subset D_2$. Prove f must be constant.