

Do all 8 problems. Notation:  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ .

1. Evaluate  $\int_{-\infty}^{\infty} \frac{\cos(2x)}{1+x^2} dx$ .
2. Find a one-to-one conformal map  $f$  of the half-strip  $\{x+iy : 0 < y < 4, x > 0\}$  onto the upper half-plane  $\{x+iy : y > 0\}$ .
3. Prove the Argument Principle: Let  $f$  be meromorphic in  $G$  with poles  $p_1, p_2, \dots, p_m$  and zeros  $z_1, z_2, \dots, z_n$ , counted according to multiplicity. If  $\gamma$  is a closed rectifiable curve in  $G$  with  $\gamma \approx 0$  and not passing through  $p_1, p_2, \dots, p_m$  or  $z_1, z_2, \dots, z_n$ , then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{k=1}^n n(\gamma; z_k) - \sum_{j=1}^m n(\gamma; p_j).$$

4. A function  $w : \mathbb{D} \rightarrow \mathbb{C}$  is called **planar harmonic** if it can be written as  $w = u + iv$ , where  $u$  and  $v$  are harmonic (but not necessarily harmonic conjugates). Prove that  $w : \mathbb{D} \rightarrow \mathbb{C}$  is planar harmonic if and only if it can be written in the form  $w = f + \bar{g}$  where  $f$  and  $g$  are analytic on  $\mathbb{D}$ .
5. Let  $f$  be an analytic function mapping  $\mathbb{D}$  onto a subset of the right half-plane  $\{x+iy : x > 0\}$  with  $f(0) = 1$ . Show that  $|f'(0)| \leq 2$ .
6. Show that the series  $\sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$  converges for  $z \in \mathbb{D}$ . Show that the limit function is analytic in  $\mathbb{D}$ .
7. Prove that  $1+z-az^n$  has a root inside  $\{z : |z| < 2\}$  for all  $|a| > 2$  and  $n > 2$ .
8. Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be entire. Suppose there exist discs  $D_1$  and  $D_2$  such that  $f(\mathbb{C} \setminus D_1) \subset D_2$ . Prove  $f$  must be constant.